Math 161 Modern Geometry Homework 1

Submit your answers to the following questions at the discussion class on Thursday 13th April.

Guidelines:

- You should write legibly.
- You should write down your proof carefully and clearly state how you start the proof (direct proof, proof by contrapositive, by contradiction etc.) and write complete sentences.
- You can discuss homework with your classmates. But please clearly state who you work with on your homework. The cooperation stops at exchanging ideas. You must write the solutions on your own and not copy from your friend's homework.

1. In the game of Nim, two players are given several piles of coins, each pile having a finite number of coins. On each turn a player picks a pile and removes as many coins as they want from that pile, as long as they remove at least one coin. The player who takes the last coin wins. Suppose that there are two piles, where one pile has more coins than the other. Prove that the first player to move can always win the game.

2. Consider a system where we have children in a classroom choosing different flavors of ice cream. Suppose we have the following axioms:

   A1 There are exactly five flavors of ice cream: vanilla, chocolate, strawberry, cookie dough, and bubble gum.

   A2 Given any two different flavors, there is exactly one child who likes these two flavors.

   A3 Every child likes exactly two flavors of ice cream.

   (a) How many children are in the classroom? Prove your assertion.

   (b) Prove that any pair of children likes at most one common flavor.

   (c) Prove that for each flavor, there are exactly four children who like that flavor.

3. Consider an axiomatic system that consists of elements in a set S and a set P of pairings of elements (a, b) that satisfy the following axioms:

   A1 If (a, b) is in P, then (b, a) is not in P.

   A2 If (a, b) is in P and (b, c) is in P, then (a, c) is in P.

   (a) Let $S = \{1, 2, 3, 4\}$ and $P = \{(1, 2), (2, 3), (1, 3)\}$. Is this a model for the axiomatic system? Why/why not?

   (b) Now let S be the set of real numbers and let P consist of all pairs (x, y) where $x < y$. Is this a model for the system? Explain.

   (c) Finally, use the resuls of (a) and (b) to argue that the axiomatic system with sets S and P is not complete. I.e., think of another independent axiom that could be added to the axioms A1 and A2 for which S and P in part (a) is a model, but for which S and P from part (b) is not a model.

4. Show how the algebraic identity $(a+b)^2 = a^2 + 2ab + b^2$ can be established geometrically. Can you make your argument work even if a or b are ≤ 0?

5. (Euclidean Algorithm) The Elements is most known for its development of geometry, but the 13 books also contain significant non-geometric material. In Book VII of the Elements, Euclid explores basic number theory. This book starts out with one of the most useful algorithms in all of mathematics, the Euclidean Algorithm. This algorithm computes the greatest common divisor (gcd) of two integers. A common divisor of two integers is an integer that divides both. For example, 4 is a common divisor of 16 and 24. The greatest common divisor is the largest common divisor. The Euclidean Algorithm works as follows:

   - Given two positive integers a and b, assume $a \geq b$.
   - Compute the quotient q_1 and remainder r_1 of dividing a by b. That is, find integers q_1, r_1 with $a = q_1 b + r_1$ and $0 \leq r_1 < b$. If $r_1 = 0$, we stop and b is the $gcd(a, b)$.
   - Otherwise, we find the quotient of dividing b by r_1, that is, find q_2, r_2 with $b = q_2 r_1 + r_2$ and $0 \leq r_2 < r_1$. If $r_2 = 0$, we stop and r_1 is the $gcd(a, b)$.
   - Otherwise, we iterate this process of dividing each new divisor (r_1, r_2, etc.) by the last remainder (r_2, r_3, etc.), until we finally reach a new remainder of 0. The last non-zero remainder is then the gcd of a, b.

   Use this algorithm to show that the gcd(36, 123) = 3. Finally, prove that this algorithm works, that is, prove that it does find the gcd of any two positive integers.