Math 161 Modern Geometry Homework Answers 1

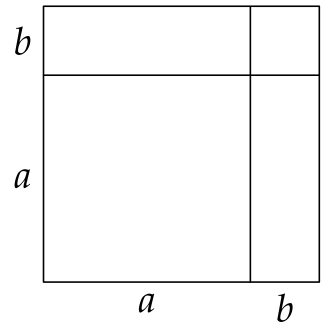
1. Given that the initial situation has two piles with *different* numbers of coins, the strategy of the first player will be to even the piles out. Thus the first move by the first player is to take coins from the larger pile so that both piles have the same number of coins remaining.
   The second player much take some coins. If they remove all the coins in one pile, then the first player immediately wins by taking the remaining pile. Thus the second player is obliged to take some, but not all, of the coins in one of the piles. The first player responds by evening the piles out again, and so the sequence repeats.
   The total number of coins is a decreasing sequence of positive integers. In extremis, the first player will remove all the coins except for leaving one in each pile. The second player is forced to take one, leaving a single coin which the first player takes, and thus wins.

   More mathematically. Let the number of coins in the two piles be x_n and y_n after player two has played n times. Without loss of generality, we may assume that $x_0 > y_0 \geq 1$, and we may always relabel the piles after each pair of moves in such a way that $x_n \geq y_n$. We claim that in fact $x_n > y_n$ We prove by induction. The base case is true by assumption. Now assume that $x_n > y_n$ for some fixed $n \in \mathbb{N}$. Player one removes $x_n - y_n$ coins from the larger pile, leaving two piles of y_n coins. Player two must remove at least one coin from one of the equal piles and so we are left with one pile being smaller than the other: Thus $x_{n+1} = y_n > y_{n+1}$.
   Now, $(y_n) \subseteq \mathbb{N}_0$ is a decreasing sequence and must therefore eventually equal zero. At this point player one removes all the coins in the non-empty pile and wins.

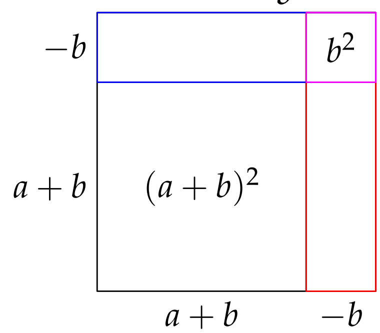
2. (a) By A2 and A3, there is exactly one child for every pair of distinct flavors of ice-cream. By A1 we see that there are exactly as many children as there are ways to choose two flavors from 5. This is the binomial coefficient $\binom{5}{2} = \frac{5!}{3!2!} = 10$. Thus there are 10 children.
   (b) Suppose that a pair of children liked at least two common flavors, A and B. Since A and B are different, this contradicts axiom A2.
   (c) Consider the set of all children who like flavor x. By A3, each of these children likes exactly one other flavor of ice-cream. By A2, each other flavor of ice-cream is liked by one of the children who likes flavor x. There are therefore exactly as many children liking x as there are flavors of ice-cream *other* than x. By A1, this number is 4.

3. (a) We check each of the axioms.
   A1 $(1,2)$, $(2,3)$, and $(1,3)$ are all in P, but $(2,1)$, $(3,2)$, and $(3,1)$ are not in P. Thus A1 is satisfied.
   A2 There is only one pair to check. $(1,2),(2,3) \in P$ and $(1,3) \in P$. Thus A2 is satisfied.
   This is therefore a model.
   (b) This is also a model. Again we check the axioms.
   A1 $(x,y) \in P \implies x < y \implies y \not< x \implies (y,x) \notin P$.
   A2 $(x,y),(y,z) \in P \implies x < y$ and $y < z \implies x < z \implies (x,z) \in P$.
   (c) For example: A3 $S \subseteq \mathbb{N}$ is an axiom satisfied by (a), but not by (b).

4. The essential picture is on the right. You should be confident that such a picture is constructible from a horizontal line segment separated into a piece of length a and another of length b.
Now simply compare areas.



If a or b is zero, there is nothing to prove. If both are negative, then the picture can be relabeled using $-a$ and $-b$ and there is no problem.



The only challenge is if *one* of a, b is negative and the other positive. Suppose WLOG that $b < 0 < a$. In the second picture, the rectangle made of the red and pink regions has area $-ab$, as does the rectangle made of the blue and pink regions. Summing the areas $(a+b)^2 - 2ab$ counts the pink square *twice*. Thus

$$(a+b)^2 - 2ab = a^2 + b^2.$$

Rearranging gives the result.