Math 161 Homework 4 Solution

(1) Consider the stereographic projection described in class.

(a) Consider the circle $(x-5)^2+(y-3)^2=1$ on the xy-plane (recall this is identified with the complex numbers). To what point (X,Y,Z) on the unit sphere is the center of this circle mapped by the stereographic projection?

(b) Consider the plane $-7X+2Y+\frac{3}{2}Z=\frac{5}{2}$ in $\mathbb{R}^3$. Show that this plane intersects the unit sphere (**Hint:** compute the distance between $(0,0,0)$ and this plane). Let the intersection be the circle (c). Compute the coordinates of the center and the radius of the corresponding circle on the xy-plane by the stereographic projection (i.e. compute the equation of the image of the circle (c) under the stereographic projection).

Proof. (a): The center of the circle in question is $(5,3)$ or as a complex number: $5+3i$. By the formula derived in lectures, the point (X,Y,Z) is given by the formula: $X=\frac{2(5)}{5^2+3^2+1}=\frac{2}{7}$, $Y=\frac{2(3)}{5^2+3^2+1}=\frac{6}{35}$, $Z=\frac{5^2+3^2-1}{5^2+3^2+1}=\frac{33}{35}$.

(b): To show the plane in the hypothesis intersects the unit sphere, we need to see the distance between the origin and the plane is <1. The distance formula (discussed in class) gives: $\frac{|5/2|}{\sqrt{(-7)^2+2^2+(3/2)^2}}<1$.

Now note that $5/2-3/2=1$ (i.e. $d-c=1$ as in lecture). We have $a=-7, b=2, c=3/2, d=5/2$. We conclude from our calculation that the equation of the circle on the complex plane which is the image of (c) under the stereographic projection is: $(x-(-7))^2+(y-2)^2=(-7)^2+2^2-2(3/2)-1=49$. So the circle has radius 7 and center $(-7,2)$. □

(2) Consider points in the plane as ordered pairs (x,y) and consider the function f on the plane defined by $f(x,y)=(kx+a,ky+b)$ where k,a,b are fixed real constants and $k\neq 0$. Is f a transformation? Is f an isometry?

Proof. f is a transformation. First, we show f is one-to-one. Let $(x,y)\neq(v,w)$ be two distinct points. We assume $x\neq v$ (the case $y\neq w$ is similar). Then since k,a are fixed and $k\neq 0$, $kx+a\neq kv+a$ (why? if equality occurs, then $kx+a-(kv+a)=0$; this gives $k(x-v)=0$, but $k\neq 0$, hence $x-v=0$, contradicting $x\neq v$). So $f(x,y)\neq f(v,w)$. So f is one-to-one. Now suppose (v,w) is an arbitrary point, we need to find (x,y) such that $(kx+a,ky+b)=(v,w)$. Since $k\neq 0$, we easily get $x=\frac{v-a}{k}$ and $y=\frac{w-b}{k}$. This shows f is onto.

Now rewrite the definition of f a little bit, we get:

$$f(\begin{bmatrix}x\\y\end{bmatrix})=k(\begin{bmatrix}x\\y\end{bmatrix})+\begin{bmatrix}a\\b\end{bmatrix}.$$

First, suppose $k=1$, then f is simply the translation by vector $\begin{bmatrix}a\\b\end{bmatrix}$, so f is an isometry.

Now suppose $k \neq 1$. Then f first scales the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by a factor of $k \neq 1$ and then translate via vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Since $k \neq 1$, clearly f does not preserve lengths. □

(3) Show that the matrix for the reflection map about the line through the origin that is inclined at the angle θ to the positive x-axis is

$$M_{2\theta} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Proof. Let l be the line in the question and f be the corresponding reflection. Let $P = (x, y)$, where $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$, be an arbitrary point. Let $P' = f(P)$. (In the following, you should draw a picture to make it easy to follow the proof). Assume without loss of generality that $\varphi < \theta$.

The angle produced by l and OP is $\theta - \varphi$. Hence $\angle P'OX = \theta + (\theta - \varphi) = 2\theta - \varphi$, where $\angle P'OX$ is the angle OP' created with the x-axis.

And so: $X = r\cos(2\theta - \varphi)$ and $Y = r\sin(2\theta - \varphi)$. Expanding we get: $X = r\cos(2\theta - \varphi) = \cos 2\theta\ r\cos(\varphi) + \sin 2\theta\ r\sin 2\theta = x\cos 2\theta + y\sin 2\theta$ and $Y = r\sin(2\theta - \varphi) = \sin 2\theta\ r\cos\varphi - \cos 2\theta\ r\sin 2\theta = x\sin 2\theta - y\cos 2\theta$. □

(4) Let f be the composition of the reflection through the line $y = x$, followed by a rotation by $\pi/3$, and followed by a reflection through the y-axis. Identify f (i.e. determine whether f is a rotation or a reflection).

Proof. The first map: reflection through $y = x$ is $\mu_{\pi/4}$ (the angle between $y = x$ and the positive x-axis is $\pi/4$); so by the previous exercise, its matrix is

$$M_{\pi/2} = \begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) \\ \sin(\pi/2) & -\cos(\pi/2) \end{bmatrix}.$$

. The second map: rotation by $\pi/3$, is $\rho_{\pi/3}$. So its matrix is

$$R_{\pi/3} = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}.$$

Similarly, the last map has matrix:

$$M_{\pi} = \begin{bmatrix} \cos(\pi) & \sin(\pi) \\ \sin(\pi) & -\cos(\pi) \end{bmatrix}.$$

Now the product (convince yourself of this by carrying out the actual multiplication)

$$M_{\pi} R_{\pi/3} M_{\pi/2} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}.$$

So f is rotation by $\pi/6$ about the origin. □

(5) We saw in class that every isometry can be thought of as a function $f_{A,\mathbf{c}}$: $\mathbf{x} \mapsto \mathbf{Ax} + \mathbf{c}$ where A is an orthogonal matrix and $\mathbf{c}$ is a constant vector. That is, every isometry is a combination of a rotation/reflection (multiplying by A) and a translation (adding $\mathbf{c}$). A rotation/reflection would have $\mathbf{c} = \mathbf{0}$, while a pure translation would have $A = I$ (the identity matrix).

(a) Prove that composition works as follows $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{AB,A\mathbf{d}+\mathbf{c}}$. Thus the composition of any two isometries is an isometry.
(b) What is the inverse of the isometry $f_{A,\mathbf{c}}$? That is, if $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{I,\mathbf{0}}$, where I is the identity matrix, then what are $B, \mathbf{d}$?
(c) Compute the composition $f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f^{-1}_{A,\mathbf{c}}$,. *You should obtain a pure translation. This shows that translations form a normal subgroup of the group of isometries.*

Proof.
(a) Evaluate the composition on a vector $\mathbf{x}$:

$$\begin{aligned} f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}}(\mathbf{x}) &= f_{A,\mathbf{c}}\left(f_{B,\mathbf{d}}(\mathbf{x})\right) = f_{A,\mathbf{c}}\left(B\mathbf{x} + \mathbf{d}\right) \\ &= A(B\mathbf{x} + \mathbf{d}) + \mathbf{c} = (\mathbf{AB})\mathbf{x} + (\mathbf{Ad} + \mathbf{c}) \\ &= f_{AB,A\mathbf{d}+\mathbf{c}}(\mathbf{x}) \end{aligned}$$

(b) If $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{I,\mathbf{0}}$, then

$$\begin{cases} AB = I \\ A\mathbf{d} + \mathbf{c} = \mathbf{0} \end{cases} \implies B = A^{-1}, \quad \mathbf{d} = -\mathbf{A^{-1}c}$$

Thus $f^{-1}_{A,\mathbf{c}} = f_{A^{-1},-A^{-1}\mathbf{c}}$
(c) Just compute:

$$\begin{aligned} f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f^{-1}_{A,\mathbf{c}} &= f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f_{A^{-1},-A^{-1}\mathbf{c}} = f_{AI,A\mathbf{d}+\mathbf{c}} \circ f_{A^{-1},-A^{-1}\mathbf{c}} \\ &= f_{A,A\mathbf{d}+\mathbf{c}} \circ f_{A^{-1},-A^{-1}\mathbf{c}} = f_{AA^{-1},A(-A^{-1}\mathbf{c})+(\mathbf{Ad}+\mathbf{c})} \\ &= f_{I,A\mathbf{d}} \end{aligned}$$

□