# Math 161 Modern Geometry Homework Questions 6

(1) (a) Find the hyperbolic line in the Poincare disk model on which lie the points $(1/2, 0)$ and $(0, 1/4)$.
(b) Use your answer to find the hyperbolic distance between the points in part (a).

*Proof.*
Substitute both points into the equation $x^2 + y^2 + ax + by + 1 = 0$. We obtain

$$\begin{cases} \frac{1}{4} + \frac{1}{2}a + 1 = 0 \\ \frac{1}{16} + \frac{1}{4}b + 1 = 0 \end{cases} \implies a = -\frac{5}{2},\ b = -\frac{17}{4}$$

The hyperbolic line is therefore the arc of the circle

$$x^2 + y^2 - \frac{5}{2}x - \frac{17}{4}y + 1 = 0 \iff \left(x - \frac{5}{4}\right)^2 + \left(y - \frac{17}{8}\right)^2 = \left(\frac{5\sqrt{13}}{8}\right)^2$$

To find the distance, we find the co-ordinates of the intersections $R, S$ of the two circles:

$$\begin{cases} x^2 + y^2 - \frac{5}{2}x - \frac{17}{4}y + 1 = 0 \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} 2 = \frac{5}{2}x + \frac{17}{4}y \\ x^2 + y^2 = 1 \end{cases}$$

$$\iff \begin{cases} 8 = 10x + 17y \\ x^2 + y^2 = 1 \end{cases}$$

Substituting the first equation in the second and solving the quadratic, we obtain

$$x = \frac{5(16 \pm 17\sqrt{13})}{389} \approx -0.56865,\ 0.97996$$

Solving for $y$, we obtain

$$R = \left(\frac{5(16 - 17\sqrt{13})}{389}, \frac{2(68 + 25\sqrt{13})}{389}\right) \approx (-0.5822, 0.8131)$$

$$S = \left(\frac{5(16 + 17\sqrt{13})}{389}, \frac{2(68 - 25\sqrt{13})}{389}\right) \approx (0.9935, -0.1138)$$

The hyperbolic distance $d(P, Q)$ is then (enjoy ...)

$$|\ln \tfrac{|PR||QS|}{|PS||QR|}| \approx 1.25$$

If you want to investigate the web, you'll find an alternative expression for the distance which is easier to compute directly:

$$d(P, Q) = \operatorname{arccosh}\left(1 + \frac{(|\overrightarrow{OP} - \overrightarrow{OQ}|)^2}{\left(1 - |\overrightarrow{OP}|^2\right)\left(1 - |\overrightarrow{OQ}|^2\right)}\right) = \operatorname{arccosh}\frac{17}{9} = \ln\frac{17 + 4\sqrt{13}}{9} = 1.25029\ldots$$

□

(2) Let $O$ be the origin and $P$ be a point in the Poincare disk. Let $r$ be the Euclidean distance between $O$ and $P$. Show that the hyperbolic distance between $O$ and $P$, $d = 2\tanh^{-1}(r)$ or equivalently, $r = \tanh(d/2)$.

*Proof.* Let $R$ and $S$ be the Omega points of the line $OP$. Notice $OP$ is a straight line because $O$ is the origin. Here $O$ is between $R$ and $P$ and $P$ is between $O$ and $S$. So we have

$$RP = 1 + r,\ PS = 1 - r,\ OR = OS = 1.$$

The hyperbolic distance between $O$ and $P$ is:

$$|\ln \frac{|OS||PR|}{|OR||PS|} = \ln \frac{1+r}{1-r}.$$

Recall $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $\tanh^{-1}(x) = 1/2 \ln \frac{1+x}{1-x}$. The above equation gives us what we want. □

(3) Show that if $\ell$ and $m$ are limiting parallel lines, then they cannot have a common perpendicular.

*Proof.* Suppose that $\ell$ and $m$ had a common perpendicular $PQ$, where $P$ lies on $\ell$. Then the angle of parallelism of $P$ with $m$ is $90°$. A contradiction to the exterior angle theorem for Omega triangles (or to Problem 3). □

(4) Show that two hyperbolic lines cannot have more than one common perpendicular.

*Proof.* Suppose that two hyperbolic lines had two common perpendiculars. These together with segments of the original lines would form a rectangle. Contradiction. □

(5) Let $PQ\Omega$ be an Omega-triangle. Prove that the sum of the angles $\angle PQ\Omega$ and $\angle QP\Omega$ is less than $180°$.

*Proof.* Let $\alpha$ be the exterior angle to angle $\angle PQ\Omega$. Note that $m(\alpha) + m(\angle PQ\Omega) = 180°$. By the exterior angle theorem for Omega triangles, $m(\alpha) > m(\angle QP\Omega)$. This means $m(\angle QP\Omega) + m(\angle PQ\Omega) < 180°$. □

(6) Suppose that an Omegra triangle is drawn with vertices at $O = (0, 0)$, $\Omega = (1, 0)$ and $P = (0, h)$ where $h > 0$. Prove that the hyperbolic line $P\Omega$ is an arc of a circle with equation $(x - 1)^2 + (y - k)^2 = k^2$ for some $k > 0$.

*Proof.* Since $\Omega = (1, 0)$, it follows that the hyperbolic line intersects the unit circle at right-angles at $\Omega$ (in other words, the tangent to this circle must be horizontal since the tangent to the unit circle at $\Omega$ is vertical), and so its center (as a Euclidean circle) must lie directly above $\Omega$ at some point $(1, k)$. The radius of this circle is clearly $k$, whence it has equation $(x - 1)^2 + (y - k)^2 = k^2$. □

(7) Prove that any hyperbolic line in the Poincare disk model of hyperbolic geometry is either a straight line, or an arc of a circle of the form $x^2 + y^2 + ax + by + 1 = 0$ with $a^2 + b^2 > 4$. Conversely, prove that any such arc is a hyperbolic line.

*Proof.* If a hyperbolic line goes through the center of the Poincare disk then it is a diameter: a straight line. Otherwise it is the arc of a circle intersecting the unit circle orthogonally. If the circle centered at $C$, radius $r$, defines a hyperbolic line, then the triangle $\triangle OPC$ is right-angled at $P$ (here $P$ is a point of intersection of the two circles). Applying Pythagoras' gives the distance of $C$ from the origin: $\sqrt{1+r^2}$. If $\theta$ is the polar angle of $C$ with the positive $x$-axis, then $C$ has co-ordiantes

$$C = \left(\sqrt{1+r^2}\cos\theta, \sqrt{1+r^2}\sin\theta\right)$$

The equation of the hyperbolic line is then

$$\left(x - \sqrt{1+r^2}\cos\theta\right)^2 + \left(y - \sqrt{1+r^2}\sin\theta\right)^2 = r^2$$

Rearranging this, we obtain

$$x^2 + y^2 - 2\sqrt{1+r^2}\cos\theta x - 2\sqrt{1+r^2}\sin\theta y + 1 = 0$$

Thus $a = -2\sqrt{1+r^2}\cos\theta$ and $b = -2\sqrt{1+r^2}\sin\theta$ in our description, where the center $C = (-\frac{a}{2}, -\frac{b}{2})$. Moreover, $a^2+b^2 = 4(1+r^2) > 4$ if $r > 0$.

Conversely, if $a, b$ are such that $a^2 + b^2 > 4$, then define $R := \sqrt{a^2+b^2}$, whence there is a unique $\theta \in [0, 2\pi)$ for which $a = R\cos\theta$ and $b = R\sin\theta$. Since $R > 2$, there is a unique $r > 0$ for which $R = \sqrt{1+r^2}$. Now, by our earlier discussion, $x^2+y^2+ax+by+1 = 0$ is the equation of an orthogonal circle to $x^2+y^2 = 1$.

□