Theorem: Circles in unit sphere map to circles or lines in $\mathbb{C}$ (line being a circle through ∞).

Proof: Every circle is the intersection of a plane

$$ax + by + cz = d$$

with the sphere.

nonempty intersection $\iff \frac{|d|}{\sqrt{a^2+b^2+c^2}} < 1 \iff d^2 < a^2+b^2+c^2$.

distance from $(0,0,0)$ to the plane

Sps $z = x+iy$ is projected onto said plane in the sphere.

Then: $2ax + 2by + c(x^2+y^2-1) = d(x^2+y^2+1)$

$\iff (d-c)(x^2+y^2) - 2ax - 2by + (c+d) = 0$.

3 cases:

1) $c = d = 0$ $\rightsquigarrow$ $ax+by=0$ all lines through origin
2) $c = d \neq 0$ Normalize: assume $c = d = 1$

$\to ax+by = 1$: all straight lines not passing thru origin.

3) $c \neq d$: diving through by $d-c$, WMA: $d-c = 1$
$\hookrightarrow d = c+1$

$\Rightarrow x^2+y^2 - 2ax - 2by + 2c+1 = 0$

$\iff (x-a)^2 + (y-b)^2 = a^2+b^2 - 2c - 1$

$d^2 < a^2+b^2+c^2 \iff (c+1)^2 < a^2+b^2+c^2 \iff a^2+b^2-2c-1 > 0$

So this is an eqn of a circle in $\mathbb{C}$

can choose any a, b (center) + in fact any radius by varying c.