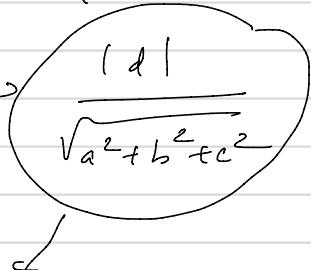


Theorem: Circles in  $\mathbb{H}^2$  map to circles or lines in  $\mathbb{C}$  (line being a circle through  $\infty$ ). 30

Proof: Every circle is the intersection of a plane

$$ax + by + c^2 = d \text{ with the sphere.}$$

nonempty intersection  $\Leftrightarrow$



$$|d| < 1 \Leftrightarrow d^2 < a^2 + b^2 + c^2.$$

distance from  $(a, b, c)$  to the plane

Suppose  $z = x + iy$  is projected onto said plane in the sphere.

$$\text{Then: } dax + dy + c(x^2 + y^2 - 1) = d(x^2 + y^2 + 1)$$

$$\Leftrightarrow (d - c)(x^2 + y^2) - 2ax - 2by + (c + d) = 0.$$

3 cases:

1)  $c = d = 0 \Rightarrow ax + by = 0 \Rightarrow$  all lines through origin

2)  $c = d \neq 0$  normalize: assume  $c = d = 1$

$\Rightarrow ax + by = 1$ : all straight lines  
not passing thru origin.

3)  $c \neq d \Rightarrow$  dividing through by  $d - c$ , we have  $d - c = 1$

$$\Rightarrow x^2 + y^2 - 2ax - 2by + 2c + 1 = 0 \quad \begin{matrix} \\ d = c + 1 \end{matrix}$$

$$\Leftrightarrow (x-a)^2 + (y-b)^2 = a^2 + b^2 - 2c - 1$$

$$d^2 < a^2 + b^2 + c^2 \Rightarrow (c+1)^2 < a^2 + b^2 + c^2 \Rightarrow a^2 + b^2 - 2c - 1 > 0$$

So this is an  $\mathbb{H}^2$  of a circle in  $\mathbb{C}$

can choose any  $a, b$  + in fact any radius  $b$   
varying  $c$ . <sup>center</sup>