$L(\mathbb{R},\mu)$ IS UNIQUE

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Abstract. Under various appropriate hypotheses it is shown that there is only one determinacy model of the form $L(\mathbb{R},\mu)$ in which μ is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. This gives a positive answer to a question asked by W.H. Woodin in 1983.

1. Introduction

This paper deals with several set theories, most of which include $\mathrm{ZF}+\mathrm{DC}$. Here, ZF is Zermelo-Fraenkel set theory and DC is the Dependent Choice principle. One such theory is ZFC, which is $\mathrm{ZF}+\mathrm{AC}$, where AC is the Axiom of Choice. Another example is

$$\mathrm{ZFC}+\text{There exists a measurable cardinal.}$$

By definition, an uncountable cardinal κ is *measurable* if and only if there is a non-principal κ-complete and normal ultrafilter on $\mathcal{P}(\kappa)$. In ZFC, this is equivalent to the existence of transitive class M and an elementary embedding $j : V \to M$ with critical point κ. The proof of this equivalence uses an ultrapower construction and Łos' Theorem, which in turn uses AC. The existence of a measurable cardinal is an example of a large cardinal axiom. Another example is the existence of a supercompact cardinal. An uncountable cardinal κ is S-*supercompact* if and only if there is a κ-complete ultrafilter on $\mathcal{P}_\kappa(S)$ which is fine and normal. We say κ is *supercompact* if and only if it is S-supercompact for every non-empty set S. In ZFC, this is equivalent to, for every cardinal λ, there exists a transitive class M with ${}^\lambda M \subseteq M$ and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Again we emphasize that AC is used to prove this equivalence. Clearly, in ZFC if κ is supercompact then κ is measurable and the set of measurable cardinals is unbounded in κ. This can be used to show that the consistency of the theory

$$\mathrm{ZFC}+\text{There is a measurable cardinal}$$

is a theorem of the theory

$$\mathrm{ZFC}+\text{There is a supercompact cardinal.}$$

In other words the second theory has greater consistency strength than the first. It is an empirical fact that large cardinal axioms line up this way.

Another important aspect of this paper is inner model theory, which we describe in brief to suit our purposes. The constructible universe, L, is the minimal transitive proper class model of ZFC. Gödel proved this fact in ZF. For any set S, we construct a transitive proper class $L[S]$ by setting $L_{\alpha+1}[S]$ to be the family of subsets of $L_\alpha[S]$ that are definable over the structure

$$(L_\alpha[S], \in, S \cap L_\alpha[S])$$

and taking unions at limits. If $\mathcal{U}$ is a normal measure on $\mathcal{P}(\kappa)$ and

$$\overline{\mathcal{U}} = \mathcal{U} \cap L[\mathcal{U}],$$

then

$$\overline{\mathcal{U}} \in L[\overline{\mathcal{U}}] = L[\mathcal{U}]$$

and

$$L[\overline{\mathcal{U}}] \models \text{ZFC} + \overline{\mathcal{U}} \text{ is a normal measure on } \mathcal{P}(\kappa).$$

This is a theorem of Solovay; see [1]. Extending this, Kunen (cf. [3]) proved the following uniqueness result.

Theorem 1 (Kunen)**.** *Assume* ZFC*. Let κ be an ordinal. Assume that for $i < 2$,*

$$L[\mathcal{U}_i] \models \text{ZFC} + \mathcal{U}_i \textit{ is a normal measure on } \kappa.$$

Then $L[\mathcal{U}_0] = L[\mathcal{U}_1]$.

There is more to Kunen's result that we are suppressing for this introduction. For several decades inner model theory has strived to extend such results to more powerful large cardinals. In spite of great progress, supercompact cardinals remain beyond our reach so far in the context of ZFC. Roughly, this paper is on analogs of Kunen's theorem for $\text{ZF} + \text{DC}$ + There exists an $\mathbb{R}$-supercompact cardinal.

In order to continue, we must discuss determinacy principles. If S is a set, then AD_S says that, for every game of length ω in which two players alternate choosing members of S, one or the other player has a winning strategy. The instances relevant here are AD_ω, more commonly called AD or the Axiom of Determinacy, and $\text{AD}_\mathbb{R}$. It is an easy well known result that AC implies AD fails. In other words $\text{ZFC} + \text{AD}$ is inconsistent. However, the consistency of the theory

$$\text{ZF} + \text{DC} + \text{AD}$$

is a theorem of the theory

$$\text{ZFC} + \text{There is a supercompact cardinal}$$

by results of Solovay (for DC) and Woodin (for AD). In fact, Woodin showed in ZFC that if there is a supercompact cardinal, then $L(\mathbb{R})$ is a model of AD. Unfortunately, the reader must distinguish between types of parentheses. Here $L(\mathbb{R})$ is constructed by setting $L_0(\mathbb{R}) = \mathbb{R}$ Here $L(\mathbb{R})$ is constructed by setting $L_0(\mathbb{R}) = \mathbb{R}$ (we identify $\mathbb{R}$ with HC in this case) and $L_{\alpha+1}(\mathbb{R})$ to be the family of sets definable over $(L_\alpha(\mathbb{R}), \in)$, and taking unions at limits. Woodin reduced the hypothesis of this result to a large cardinal axiom strictly between measurablility and supercompactness. In fact, he showed that the existence of a certain countable structure called $\mathcal{M}_\omega^\sharp$ suffices. Woodin also showed that the consistency of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_\mathbb{R}$$

is a theorem of the theory

$$\text{ZFC} + \text{There is a supercompact cardinal.}$$

As we said, in ZFC, from a supercompact cardinal, we get inner models of determinacy principles. Next we mention some of the landmark results in the other direction. Solovay proved that, under $\mathrm{ZF} + \mathrm{AD}$, the club filter on ω_1 is a normal measure and is the unique such measure. He also proved that under $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$, ω_1 is $\mathbb{R}$-supercompact as witnessed by the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$. We remind the reader that C is a club subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ if there is $\pi : {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that $\sigma \in C$ if and only if σ is closed under π.

The theory

$$\mathrm{ZF} + \mathrm{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact}$$

is the focus of this paper. We start to discuss this theory in further detail. For this we must define another kind of model which is built by a combination of our two types of relative constructibility. Suppose μ is a collection of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. By $L(\mathbb{R},\mu)$, we mean "throw in $\mathbb{R}$ at the bottom" and "use μ as a predicate". That is, define $L_0(\mathbb{R},\mu) = \mathrm{HC}$ and $L_{\alpha+1}(\mathbb{R},\mu)$ to be the collection of sets definable over the structure

$$(L_\alpha(\mathbb{R},\mu), \in, \mu \cap L_\alpha(\mathbb{R},\mu)),$$

and take unions at limits. Notice that μ might not belong to $L(\mathbb{R},\mu)$ but $\mu \cap L(\mathbb{R},\mu)$ does and

$$L(\mathbb{R},\mu) = L(\mathbb{R},\mu \cap L(\mathbb{R},\mu))$$

We usually think of $L(\mathbb{R},\mu)$ as a structure in which the extra symbol $\dot{\mu}$ is interpreted as $\mu \cap L(\mathbb{R},\mu)$. It is immediate from Solovay's theorem and other well-known facts that $L(\mathbb{R},\mathcal{C})$ is a model of

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact}$$

as witnessed by $\dot{\mu}^{L(\mathbb{R},\mathcal{C})} = \mathcal{C} \cap L(\mathbb{R},\mathcal{C})$.

Following Solovay, Woodin began the analysis of models of the form $L(\mathbb{R},\mu)$ and obtained the following uniqueness result, the proof of which can be found in [17].

Theorem 2 (Woodin)**.** *Suppose* $\mathrm{AD}_\mathbb{R}$ *holds. Then the club filter is the unique* $\mathbb{R}$*-supercompact measure on* ω_1*.*

Motivated by this result and Kunen's theorem on the uniqueness of $L[\mathcal{U}]$, Woodin asked the following question (cf. [17]).

Question 3 (Woodin, 1983)**.** *Assume* $\mathrm{ZF} + \mathrm{DC}_\mathbb{R} + \mathrm{AD}$. *Is there at most one model of the form* $L(\mathbb{R},\mu)$ *that satisfies* $\mathrm{AD}+$ ω_1 *is* $\mathbb{R}$*-supercompact?*

One of our two main results is that the answer is yes. In fact, the unique model has the form $L(\mathbb{R}, \mathcal{C})$, where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$. For future reference:

Theorem 4. *Assume* $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}$*. Then, there is at most one model of the form* $L(\mathbb{R}, \mu)$ *that satisfies* AD + ω_1 *is* $\mathbb{R}$*-supercompact. Moreover if such a model exists, then* $L(\mathbb{R}, \mathcal{C})$ *is the unique one.*

The "moreover" part of the theorem is analogous to another result about measurable cardinals. Namely, if $\kappa = \omega_1$ and $0^\dagger$ exists, then there is exactly one model of the form $L[\mathcal{U}]$ in which $\mathcal{U} \cap L[\mathcal{U}]$ is a normal measure on κ, namely take $\mathcal{U}$ to be the club filter on κ.

It is also natural ask Woodin's question with the assumption ZFC instead of $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}$. We do not know the answer to the modified question, but under ZFC together with a technical large cardinal hypothesis we obtain a positive answer, our second main result.

Theorem 5. *Assume* ZFC *and suppose that* $\mathcal{M}^\sharp_{\omega^2}$ *exists.Then*

(1) $L(\mathbb{R}, \mathcal{C}) \models$ AD + $\mathcal{C}$ *is an* $\mathbb{R}$*-supercompact measure, and*
(2) *if* $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ *is such that*
$$L(\mathbb{R}, \mu) \models \text{AD} + \omega_1 \textit{ is } \mathbb{R}\textit{-supercompact,}$$
then $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$.

The meaning of $\mathcal{M}^\sharp_{\omega^2}$ and the sense in which it is iterable will be discussed in Section 2. The large cardinal hypothesis that $\mathcal{M}^\sharp_{\omega^2}$ exists is slightly stronger than the consistency strength of the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We do not know how to do without this mild "extra" assumption but conjecture that it is possible and have some partial results in this direction that will be mentioned.

Conjecture 6. Assume ZFC. Then there is at most one model of the form $L(\mathbb{R}, \mu)$ that satisfies AD+ ω_1 is $\mathbb{R}$-supercompact.

Theorems 4 and 5 will come at the end of a series of uniqueness results with varying hypotheses, which we label propositions. One of these, the following, is the main result of Section 2, which strengthens Theorem 5 in that there is no stationarity assumptions for the members of μ.

Proposition 7. Assume ZFC and suppose that $\mathcal{M}^\sharp_{\omega^2}$ exists. Then

(1) $L(\mathbb{R}, \mathcal{C}) \models$ AD + $\mathcal{C}$ is an $\mathbb{R}$-supercompact measure, and
(2) if $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ is such that for any $A \in \mu$, A is stationary and
$$L(\mathbb{R}, \mu) \models \text{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact,}$$
then $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$.

To read Section 2 the reader should be familiar with the technique of iterating mice to make reals generic. Section 7 of [11] is the main background needed.

In Section 3, we prove Theorem 5. For this, we use the HOD analysis of the models $L(\mathbb{R},\mu)$ satisfying $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact, to show that on a Turing cone of reals x,

$$\mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})} = \mathrm{HOD}_x^{L(\mathbb{R},\mu)}.$$

See [9] for the HOD analysis in $L(\mathbb{R})$; other good sources on this subject are [13] and [15]. The meaning of ordinal definability in $L(\mathbb{R},\mu)$ is different from the usual notion in that the language for the definitions includes the predicate $\dot{\mu}$ which is interpreted as $\mu \cap L(\mathbb{R},\mu)$.

Finally, in Section 4, we use Theorem 5 and its proof to show Theorem 4 after first proving yet another approximation, namely:

Proposition 8. Assume $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$. Then, there is at most one model of the form $L(\mathbb{R},\mu)$ that satisfies $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact. Moreover if such model exists then $L(\mathbb{R},\mathcal{C})$ is the unique such model, where $\mathcal{C}$ is the club filter.

The determinacy principle AD^+ is a technical strengthening of AD. It is the conjunction of $\mathrm{DC}_{\mathbb{R}}+\mathrm{AD}$ and the statement:

> If $\varphi(v)$ is a Σ_1 formula, x is a real and $L(\mathcal{P}(\mathbb{R})) \vDash \varphi[x]$, then there are ordinals ξ and γ less than the largest Suslin cardinal such that $L_\gamma(\mathcal{P}(\mathbb{R}) \restriction \xi) \vDash \varphi[x]$.

Recall that $\mathcal{P}(\mathbb{R}) \restriction \xi$ is the collection of $A \subseteq \mathbb{R}$ such that the Wadge rank of A is less than ξ. In all known models of AD, AD^+ holds. It is not known whether AD implies AD^+.

In addition to the prerequisites mentioned earlier, for Section 4, the reader should be familiar with certain concepts of descriptive inner model theory. For example, [5] is a good source.

2. The ZFC case under the stationarity assumption

In this section, we prove Proposition 7. Assume its hypotheses. Recall that $\mathcal{M}^\sharp_{\omega^2}$ is the unique, active, sound mouse projecting to ω, with ω^2-many Woodin cardinals all whose initial segments are ω^2-small. See [11] for a detailed exposition. Part of what it means to be a mouse is that $\mathcal{M}^\sharp_{\omega^2}$ has an $(\omega,\omega_1,\omega_1+1)$-iteration strategy, which happens to be unique; we call it Σ. We now make an additional assumption about Σ that we will eliminate when we finish the proof of Proposition 7 at the end of this section. Let $\kappa = (2^{\mathfrak{c}})^+$. Assume that Σ is coded by a κ-universally Baire set of reals. In other words, there are trees T and U such that:

- $p[T]$ codes Σ and $p[U]$ codes its complement.
- If $\mathbb{P} \in V_\kappa$, then $V^{\mathbb{P}}$ satisfies that $p[T]$ codes an $(\omega,\omega_1,\omega_1+1)$-iteration strategy on $\mathcal{M}^\sharp_{\omega^2}$ and $p[U]$ codes its complement.

Fix such trees T and U. We will abuse notation by using Σ to refer to the strategy coded by $p[T]$ in any small generic extension of V.

If P is a countable Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, then Σ can also be considered an $(\omega, \omega_1, \omega_1+1)$-iteration strategy on P. If Q is a Σ-iterate of such a P and there is no dropping on the branch from P to Q, then we write $P \stackrel{\Sigma}{\to} Q$ for the branch embedding.

We need to review a certain construction that plays a role in the proof of Proposition 7. Consider an arbitrary model M of enough set theory that has ω^2-many Woodin cardinals. Let δ^M_α be the α-th Woodin cardinal of M. Also, let $\lambda^M_\beta = \sup\{\delta^M_\alpha \mid \alpha \in \beta\}$. Suppose G is an M-generic filter on $\mathrm{col}(\omega, <\lambda^M_{\omega^2})$. Let $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{M[G\restriction\alpha]}$ and $\mathbb{R}^* = \bigcup_{\alpha<\omega i} \mathbb{R}^{M[G\restriction\alpha]}$. In $M[G]$, define *the tail filter*, $\mathcal{F}$, on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$ as follows: for $A \subseteq P_{\omega_1}(\mathbb{R}^*)$

$$A \in \mathcal{F} \text{ if and only if } \exists n \in \omega\, \forall m \geq n\ (\sigma_m \in A)$$

The fact we will use is that $L(\mathbb{R}^*, \mathcal{F}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. This is proved in [15].

Eventually, we will find a Σ-iterate, M, and an M-generic filter G on $\mathrm{col}(\omega, <\lambda^M_{\omega^2})$ such that the associated $\mathbb{R}^*$ is $\mathbb{R}$ and the corresponding tail filter contains the club filter, $\mathcal{C}$. Towards this, the following gets us started.

Lemma 9. *Suppose that γ is a cardinal such that $\gamma \geq 2^{\mathfrak{c}^+}$. Let X_0 and X_1 be countable elementary substructures of H_γ such that $\mathbb{R} \cap X_0 \in X_1$ and $T, U \in X_0$. Then there is an iteration tree $\mathcal{T}$ on $\mathcal{M}^\sharp_{\omega^2}$ of successor length $\zeta+1$ such that $\mathcal{T}\restriction\alpha \in X_0$ for all $\alpha < \zeta$ and $\mathcal{T} \in X_1$, and there exists $G \in X_1$ such that G is $M^{\mathcal{T}}_\zeta$-generic on $col(\omega, <\lambda^{M^{\mathcal{T}}_\zeta}_\omega)$ and the associated set of symmetric reals is $\mathbb{R} \cap X_0$.*

Proof. Given the assumptions above note that $\mathbb{R} \cap X_0 \in X_1$, so there is $\langle x_i \mid i \in \omega\rangle$ an enumeration of $\mathbb{R} \cap X_0$ in X_1. Now let $\mathcal{T}_0$ be the iteration tree on $\mathcal{M}^\sharp_{\omega^2}$ according to Σ, with last model P_0, such that $i : \mathcal{M}^\sharp_{\omega^2} \to P_0$ exists and x_0 is generic for $\mathbb{B}^{P_0}_{\delta_0}$, the extender algebra at $\delta^{P_0}_0$. Note $\mathcal{M}^\sharp_{\omega^2} \in X_0$ and has a unique strategy, hence $\mathcal{T}_0$ belongs to X_0 and is countable there. We continue iterating $P_0 \to P_1$ in the interval (δ_0, δ_1), say via $\mathcal{T}_1$, to make the next real x_1, generic for the extender algebra at $\delta^{P_1}_1$. Note that in this case both x_0 and x_1 are set generic over P_1 for posets in $V^{P_1}_{\lambda^{P_1}_\omega}$. Continuing in this fashion we get Σ-iteration trees $\mathcal{T}_n$ with branch embeddings $P_{n-1} \to P_n$ such that x_n is P_n-generic for the extender algebra at $\delta^{P_n}_n$. Also every x_i for $i < n$ is set generic over P_n.

In X_1, define $\mathcal{T}$ to be the concatenation of the $\mathcal{T}_n$. Now $\mathcal{T}$ has a unique cofinal branch b and letting $P = M^{\mathcal{T}}_b$ there is a P-generic filter G for $\mathrm{col}(\omega, <\lambda^P_\omega)$ in X_1 such that the associated set of symmetric reals is $\mathbb{R} \cap X_0$. This is by a well known argument given in Lemma 3.1.5 of [4]. $\square$

Note that in the proof of the Lemma 9 we used the interval $(\lambda_0^M, \lambda_\omega^M)$, where $M = \mathcal{M}^\sharp_{\omega^2}$ but we could have used any Σ-iterate M of $\mathcal{M}^\sharp_{\omega^2}$ with $M \in X_0$ and any interval $(\lambda_{\omega i}^M, \lambda_{\omega(i+1)}^M)$ to obtain the same result.

Lemma 10. *In $V^{col(\omega,2^{\mathfrak{c}})}$, there is a correct iterate P of $\mathcal{M}^\sharp_{\omega^2}$ and a P-generic filter G for $col(\omega, < \lambda^P_{\omega^2})$ such that if $\mathcal{F}$ is the associated tail filter, then $\mathcal{C}^V$ is contained in $\mathcal{F}$.*

Proof. In the statement of the lemma, we are writing $\mathcal{C}^V$ for the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ as computed in V. In $V^{\mathrm{col}(\omega,(2^{\mathfrak{c}}))}$, we let $\langle X_i | i \in \omega\rangle$ be a chain of countable elementary substructures of $H^V_{\kappa^+}$ such that $\bigcup_{i\in\omega} X_i \supseteq \mathcal{P}(\mathbb{R})^V$ and if $\sigma_i = X_i \cap \mathbb{R}$, then $\sigma_i \in X_{i+1}$ and σ_i is countable in V. We may assume that $\mathcal{M}^\sharp_{\omega^2}$, T and U are in X_0. We construct an iteration of the form $\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \to \cdots \to P_i \to P_{i+1} \cdots \to P$ by recursion using Lemma 9 so that the iteration $P_{i-1} \to P_i$ is done in the interval $(\lambda_{\omega(i-1)}, \lambda_{\omega i})$ and makes σ_{i-1} the set of symmetric reals associated a P_i-generic on $\mathrm{col}(\omega, < \lambda^{P_i}_{\omega i})$. Let P be the direct limit of the P_i, by a variant of Lemma 3.1.5 in [4] there is a P-generic filter G for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{P[G\restriction\alpha]}$. Note that the set of symmetric reals associated to G and P is $\mathbb{R}^V$. Let $\mathcal{F}$ be the corresponding tail filter. Consider any $A \in \mathcal{C}^V$. Let $\pi \in V$ be such that $\pi : \mathbb{R}^{<\omega} \to \mathbb{R}$ and its closure points belong to A. Then there is an $n \in \omega$ such that $\pi \in X_n$. So for all $m \geq n$, $\pi \in X_m$ and σ_m is closed under π, thus $A \in \mathcal{F}$. $\square$

The two key facts in the proof of Lemma 10 are that if A is an element of $\mathcal{C}^V$, then there is an $i \in \omega$ such that $A \in X_i$, and that every X_i is closed under Σ. This motivates the following definition.

Definition 11. Suppose N is a set model of some set theory, such that $\mathcal{P}(\mathbb{R})^N$ is countable. Then we say $\langle X_i \,|\, i \in \omega\rangle$ is a *good resolution* of N if for all $i \in \omega$, we have $X_i \prec N$ and $\mathbb{R} \cap X_i \in X_{i+1}$, and $\bigcup_{i\in\omega} X_i \supset \mathcal{P}(\mathbb{R})^N$.

Note that in the proof of Lemma 10 instead of $H_{(2^{\mathfrak{c}})^+}$ we could have used any N that is a model of enough set theory and $\mathcal{P}(\mathbb{R})^N$ is countable in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$. We give an example of such a situation in the following lemma.

Lemma 12. *Suppose that A is stationary in $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then it is forced by $col(\omega, 2^{\mathfrak{c}})$ that there is a Σ-iterate P of $\mathcal{M}^\sharp_{\omega^2}$, and P-generic filter G for $col(\omega, < \lambda^P)$ such that A belongs to the tail filter associated to G and P.*

Proof. By homogeneity it is enough to find a generic filter for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$ with the desired property. Consider $\mathbb{P}_A$, the forcing poset whose conditions are countable, closed, increasing sequences from A. In other, words $p = \langle \sigma_\alpha \,|\, \alpha < \beta\rangle$ is a condition in $\mathbb{P}_A$ if

- for all $\alpha < \beta$, we have that σ_α belongs to A,
- for every α and α' in β if $\alpha < \alpha'$ then $\sigma_\alpha \subseteq \sigma_{\alpha'}$, and
- if $\alpha < \beta$ is a limit ordinal, then $\sigma_\alpha = \bigcup_{i\in\alpha} \sigma_i$.

We say $p \leq q$ if p end-extends q. It is easy to see that this poset shoots a club through A. Also, the usual argument will show that this forcing is (ω_1, ∞)-distributive, so in particular it does not add any new reals. Let h be V-generic for $\mathbb{P}_A$. Then $\mathbb{R}^{V[h]} = \mathbb{R}$ and, as the forcing has size continuum, we have that $2^{\mathfrak{c}}$ is the same ordinal in V and $V[h]$. Applying Lemma 10 in $V[h]$, if G' is $V[h]$-generic for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$, then in $V[h][G']$ the conclusion of the lemma holds as $A \in \mathcal{C}^{V[h]}$. Finally, note that there is a V-generic filter G for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$ such that $V[G] = V[h][G']$. $\square$

Suppose that in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$ there are two Σ-iterates P and Q of $\mathcal{M}^\sharp_{\omega^2}$ and generic filters G and H for $\mathrm{col}(\omega, <\lambda^P_{\omega^2})$ and $\mathrm{col}(\omega, <\lambda^Q_{\omega^2})$ respectively such that the set of symmetric reals of $P[G]$ and $Q[H]$ is precisely $\mathbb{R}^V$. Let $\mathcal{E}$ and $\mathcal{F}$ be the tail filters associated to P, G and Q, H respectiveley. We will show that if this is the case then $L(\mathbb{R}, \mathcal{E}) = L(\mathbb{R}, \mathcal{F})$.

Lemma 13. *In $V^{col(\omega,2^{\mathfrak{c}})}$, let N_1 and N_2 be ransitive sets containing T and U that model a reasonable amount of* ZFC *such that $\mathbb{R}^{N_i} = \mathbb{R}^V$ for $i = 1, 2$. Let $\langle X^1_i | i \in \omega\rangle$ and $\langle X^2_i | i \in \omega\rangle$ be good resolutions of N_1 and N_2 respectively and $\mathcal{F}^1$ and $\mathcal{F}^2$ be the the associated tail filters. Then $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}^2)$.*

In practice N_1 would be $H^V_{\kappa^+}$ and N_2 would be $H^{V[h]}_{\kappa^+}$ for some small forcing V-generic filter h.

Proof. Let $\sigma^1_j = X^1_j \cap \mathbb{R}$ and similarly $\sigma^2_j = X^2_j \cap \mathbb{R}$. Iterate $\mathcal{M}^\sharp_{\omega^2}$ inductively as follows. Let $\sigma_0 = \sigma^1_0$ and let $\mathcal{M}^\sharp_{\omega^2} \to P_0$ be the iteration to make σ_0 generic on the first ω-many Woodins. Note that σ_0 can be coded as a single real, so there is i_1 such that $\sigma_0 \in X^2_{i_1}$ and thus the iteration $M^\sharp_\omega \to P_0$ is actually in $X^2_{i_1}$. Define $\sigma_1 = \sigma^2_{i_1}$ and iterate $P_0 \to P_1$ on the second ω-many Woodins to make σ_1 generic. There is i_2 such that $\sigma_1 \in X^1_{i_2}$. Let $\sigma_2 = \sigma^1_{i_2}$ and continue the iteration in this fashion. We get an iteration $\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \cdots \to P_i \to P_{i+1} \to \cdots \to P$ and a P-generic filter G for $\mathrm{col}(\omega, <\lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{P[G\restriction\alpha]}$. Let $\mathcal{F}$ be the associated tail filter. Note also that for any $i \in \omega$ there are $j > i$ and $k > i$, and natural numbers m and n such that $\sigma^1_j = \sigma_m$ and $\sigma^2_k = \sigma_n$.
Claim: $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}) = L(\mathbb{R}, \mathcal{F}^2)$.

Proof of the Claim. We have that $\mathcal{F}^1$ is an ultrafilter relative to sets in $L(\mathbb{R}, \mathcal{F}^1)$, and similarly $\mathcal{F}$ is an ultrafilter in $L(\mathbb{R}, \mathcal{F})$. Now by an induction on $\alpha \in \mathrm{ON}$, we see that $L_\alpha(\mathbb{R}, \mathcal{F}^1) = L_\alpha(\mathbb{R}, \mathcal{F})$ and $\mathcal{F} \cap L_\alpha(\mathbb{R}, \mathcal{F}^1) = \mathcal{F} \cap L_\alpha(\mathbb{R}, \mathcal{F})$, which would give the desired claim. Limit stages are clear.

Now if the induction hypotheses hold at α, it is clear that $L_{\alpha+1}(\mathbb{R},\mathcal{F}^1) = L_{\alpha+1}(\mathbb{R},\mathcal{F})$. Given $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R}) \cap \mathcal{F}^1$, a set in $L_{\alpha+1}(\mathbb{R})[\mathcal{F}^1]$, we have that either A or its complement is in $\mathcal{F}$. But A contains a tail of the σ_i^1, hence by construction its complement cannot contain a tail of σ_i, which means $A \in \mathcal{F}$. The other direction is similar, so the induction hypotheses hold at $\alpha + 1$. □

Clearly the claim completes the proof of the Lemma 13. □

For simplicity we will refer to the unique model in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$ coming from constructions a la Lemma 10 as $L(\mathbb{R},\mathcal{F})$. Note that by the homogeneity of the collapse, $L(\mathbb{R},\mathcal{F})$ is definable from T and U in V, as is $\mathcal{F} \cap V$. We refer to $\mathcal{F} \cap V$ as $\mathcal{F}$ when there is no ambiguity.

Lemma 14. *Let $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, and suppose that μ contains only stationary sets. Then $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F})$.*

Proof. First we will show again inductively that $L_\alpha(\mathbb{R},\mathcal{F}) = L_\alpha(\mathbb{R},\mu)$ and $\mathcal{F} \restriction L_\alpha(\mathbb{R},\mathcal{F}) = \mu \restriction L_\alpha(\mathbb{R},\mathcal{F})$. As in the proof of the claim in the last theorem, we only need to take care of the successor steps. Now given $A \in \mathcal{F} \cap L_{\alpha+1}(\mathbb{R},\mathcal{F})$, by induction either A or its complement is in μ. For contradiction suppose $A \notin \mu$. Then $A^c \in \mu$, so A^c is stationary, applying Lemmas 12 and 13 giving $A^c \in \mathcal{F}$, which is a contradiction. □

Lemma 15. *$\mathcal{C} \cap L(\mathbb{R},\mathcal{F}) = \mathcal{F} \cap L(\mathbb{R},\mathcal{F})$.*

Proof. Otherwise by Lemma 10, there is $A \in \mathcal{F} \cap L(\mathbb{R},\mathcal{F})$ that does not contain a club, which means that A^c is stationary so by Lemma 12 and 13 we have $A^c \in \mathcal{F}$, which gives a contradiction. □

To summarized we have seen that $L(\mathbb{R},\mathcal{C})$ is the unique model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact under the hypotheses of Proposition 7 and the additional assumption that Σ is $(2^{\mathfrak{c}})^+$-universally Baire. Our final step is to eliminate this extra assumption.

Assume that $\mathcal{M}^\sharp_{\omega^2}$ exists and Σ is an $(\omega,\omega_1,\omega_1+1)$-iteration strategy but not necessarily universally Baire. Suppose that μ is as in the statement of the proposition. Let γ be such that V_γ reflects enough set theory, and let $N \prec V_\gamma$ be countable such that Σ and μ are in N. Let H be the transitive collapse of N and $\pi : H \to N$ be the uncollapsing map. Define $\bar{\mu} = \pi^{-1}(\mu)$ and $\overline{\Sigma} = \pi^{-1}(\Sigma)$.

Let us review how the universally Bairness of Σ was used in the proofs of the earlier lemmas. The key points are that Σ canonically extends to a strategy in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$ and $\mathcal{P}(\mathbb{R})^V$ is countable in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$. The relationship between H and V is similar enough to the relationship between V and $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$ to obtain the following in V without assuming Σ universally Baire. There is a countable iterate P of $\mathcal{M}^\sharp_{\omega^2}$ and a P-generic filter K for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ such that $\mathbb{R}^H$ is the set of symmetric reals of $P[K]$. Moreover, if $\overline{\mathcal{F}}$ is the associated tail filter, then $\overline{\mathcal{F}} \cap H$ belongs to H and in H,

$L(\mathbb{R}^H, \overline{\mathcal{F}}) = L(\mathbb{R}^H, \overline{\mu}) = L(\mathbb{R}^H, \mathcal{C}^H)$, and the three filters are the same on the common model. By elementarity and the choice of γ, $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{C})$ and the two filters agree on the common model. This completes the proof of Proposition 7.

3. The general ZFC case

Assume $\mathcal{M}^{\sharp}_{\omega^2}$ exists. In the last section we saw that if μ consists only of stationary sets and $L(\mathbb{R}, \mu)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact, then $\mu \cap L(\mathbb{R}, \mu) = \mathcal{C} \cap L(\mathbb{R}, \mu)$. Let us give an example that illustrates there is more to do. Let $\mathcal{S}^V$ be the collection of stationary subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$ in V. By Proposition 7 we have that $L(\mathbb{R}, \mathcal{S}^V)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact. Let $A \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ be a stationary set whose complement is also stationary and let h be a V-generic filter for the poset that shoots a club through A^c (as in the proof of Theorem 12). Applying Proposition 7 in $V[h]$, $L(\mathbb{R}, \mathcal{S}^{V[h]})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact. We would like to conclude that $L(\mathbb{R}, \mathcal{S}^V) = L(\mathbb{R}, \mathcal{S}^{V[h]})$ but it does not follow from Proposition 7 applied in $V[h]$ because $A \in \mathcal{S}^V$ but A is nonstationary.

Notice that the proof given in the last section relies heavily on the fact that if $A \in \mu$, then one can shoot a club through A without adding reals. Without this available to us we need a different idea. We use Woodin's Analysis of HOD in order to prove Theorem 5. The HOD Analysis for structures of the form $L(\mathbb{R}, \mu)$ was done in [15], however we will use a variant closer to the exposition of [9]. We start by doing the analysis for $L(\mathbb{R}, \mathcal{C})$ and then generalize to $L(\mathbb{R}, \mu)$. We first give some useful definitions and lemmas. We will work, as in the last section, with $\mathcal{M}^{\sharp}_{\omega^2}$ and its strategy Σ, as well as with trees T and U that witness that Σ is $(2^{\mathfrak{c}})^+$-universally Baire. Ultimateley, the universally Baire assumption on Σ will be eliminated using the same ideas from last section.

Definition 16. Given μ, a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$, such that $L(\mathbb{R}, \mu) \models$ AD + ω_1 is $\mathbb{R}$-supercompact, we use the following notation:

- $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}(\mathbb{R})^{L(\mathbb{R}, \mu)}$
- $\boldsymbol{\delta}^2_1(\mu) = \boldsymbol{\delta}^{2\,L(\mathbb{R}, \mu)}_1$
- $\Theta(\mu) = \Theta^{L(\mathbb{R}, \mu)}$

The following lemma says that the power sets of the reals of such models line up with that of $L(\mathbb{R}, \mathcal{C})$.

Lemma 17. *Suppose that $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ is such that $L(\mathbb{R}, \mu) \models$ AD + ω_1 is $\mathbb{R}$-supercompact. Then either $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_{\mathcal{C}}(\mathbb{R})$ or $\mathcal{P}_{\mathcal{C}}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$.*

Proof. Suppose neither $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_{\mathcal{C}}(\mathbb{R})$ nor $\mathcal{P}_{\mathcal{C}}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$. Let $\Gamma = \mathcal{P}_{\mathcal{C}}(\mathbb{R}) \cap \mathcal{P}_\mu(\mathbb{R})$. By Theorem 3.7.1 of [16] $L(\mathbb{R}, \Gamma) \models \mathrm{AD}_{\mathbb{R}}$. Hence by a theorem of Solovay mentioned in the introduction, if ν is the club filter defined in $L(\mathbb{R}, \Gamma)$, then $L(\mathbb{R}, \nu) \models$ AD + ω_1 is $\mathbb{R}$-supercompact. Moreover, we have that ν is a subset of $\mathcal{C}$, so by the proof of Lemma 13 we have $L(\mathbb{R},\nu) = L(\mathbb{R},\mathcal{C})$, which readily gives a contradiction. $\square$

We will need the notion of the *Envelope* of a point-class. For a complete exposition of this subject the reader may consult Chapter 3 of [16]. We will mostly be interested in envelopes of point-classes of the form $\Sigma_1^{Lp(\mathbb{R})|\gamma}$ where Lp is the *lower part* operator (see Chapter 3 of [5]). We recall the definitions below.

Definition 18. For a set X we have the following.

- Given a mouse $\mathcal{M}$ on X we say that $\mathcal{M}$ is *countably iterable* if for any $\bar{\mathcal{M}}$ countable and elementary embeddable into $\mathcal{M}$, we have that $\bar{\mathcal{M}}$ is $\omega_1 + 1$ iterable.
- $Lp(X)$ is the union of all countably iterable and sound X-mice that project to X.

Definition 19. Suppose that γ is an admissible ordinal of $Lp(\mathbb{R})$. Let $\Gamma = \Sigma_1^{Lp(\mathbb{R})|\gamma}$. For $A \subseteq \mathbb{R}$

- We say $A \in \mathrm{OD}^{<\gamma}$ if there is $\alpha < \gamma$ such that A is $\mathrm{OD}^{Lp(\mathbb{R})|\alpha}$.
- We say $A \in Env(\Gamma)$ if for every $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ there is $A' \in \mathrm{OD}^{<\gamma}$ such that $A \cap \sigma = A' \cap \sigma$.

We also note that the definition of the envelope can be relativized to any real x. Recall that $\mathbf{Env}(\Gamma)$, the boldface envelope, is $\bigcup_{x\in\mathbb{R}} Env(\Gamma(x))$. The notion of the envelope is particularly useful when analyzing the Σ_1-gaps and the pattern of scales in the structure $Lp(\mathbb{R})$ (see [7], [10] and [6]).

We turn now to prove that for any μ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, we have that $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\mathcal{C}(\mathbb{R})$.

Lemma 20. *Suppose that μ is a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu)$ satisfies $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $L(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\mu)$ have the same sets of reals.*

Proof. For contradiction suppose that this is not the case. Without loss of generality we may assume that μ and $\mathcal{C}$ measure some subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ differently, as otherwise the lemma would follow trivially. By Lemma 17 we have the following two cases.

Case 1: $\mathcal{P}_\mu(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mathcal{C}(\mathbb{R})$.
In this case without loss we will assume that μ is such that $\mathcal{P}_\mu(\mathbb{R})$ is minimal. In other words, given any other $\nu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, then $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$.

By $(\mathbb{R},\mu)^\sharp$, we mean the theory of the reals and indiscernibles of $L(\mathbb{R},\mu)$ in a language with predicates for membership and μ. Let B belong to $\mathcal{P}_\mathcal{C}(\mathbb{R})$ but not to $\mathcal{P}_\mu(\mathbb{R})$. Then $(\mathbb{R},\mu)^\sharp = \oplus_{n\in\omega}\mathcal{T}_n^\mu$, where each $\mathcal{T}_n^\mu$ is Wadge reducible to B. Since there is a real x that codes all these reductions, $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},\mathcal{C})$. Also, by results of [15] we have that $L_{\boldsymbol{\delta}_1^2(\mathcal{C})}(\mathbb{R},\mathcal{C}) \prec_1$

$L(\mathbb{R},\mathcal{C})$, hence there is such a sharp in $L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Let $\bar{\mu}$ be such that $(\mathbb{R},\bar{\mu})^\sharp \in L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\bar{\mu}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Claim: In $L(\mathbb{R},\mathcal{C})$, $\mathcal{M}^\sharp_{\omega^2}$ exists and is ω_1+1-iterable.

Proof of the claim. Let us work in $L(\mathbb{R},\mathcal{C})$. First, by results of [15] we have that $\mathcal{P}_C(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ and $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$. Note that if M is an $\mathbb{R}$-mouse in $L(\mathbb{R},\bar{\mu})$ projecting to $\mathbb{R}$, then there is a set of reals in $\mathcal{P}_{\bar{\mu}}(\mathbb{R})$ coding it. Thus $M \in L_{\boldsymbol{\delta}^2_1(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Also, if M is countably iterable in $L(\mathbb{R},\bar{\mu})$, by definition, given any countable hull of M, call it $\bar{M}$, we have that $\bar{M}$ is iterable in $L(\mathbb{R},\bar{\mu})$, and as $\mathbb{R} \subset L(\mathbb{R},\mathcal{C})$ then $\bar{M}$ is ω_1-iterable in $L(\mathbb{R},\mathcal{C})$. But ω_1 in measurable in $L(\mathbb{R},\mathcal{C})$ hence $\bar{M}$ is iterable in $L(\mathbb{R},\mathcal{C})$. This gives us that:

$$Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})} \lhd (Lp(\mathbb{R})|\boldsymbol{\delta}^2_1(\mathcal{C}))^{L(\mathbb{R},\mathcal{C})}.$$

Because of this we have that $\boldsymbol{\delta}^2_1(\bar{\mu})$ starts a Σ_1-gap in $Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$. Let

$$\Gamma = \Sigma_1^{Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}}.$$

We claim that $\mathbf{Env}(\Gamma) = P_{\bar{\mu}}(\mathbb{R})$, where the envelope is as defined in $L(\mathbb{R},\mathcal{C})$.

For this note that by results of [10], we have $\mathbf{Env}(\Gamma) = P(\mathbb{R})^{Lp(\mathbb{R})|\gamma}$, where γ is the largest ordinal such that $Lp(\mathbb{R})|\boldsymbol{\delta}^2_1(\bar{\mu}) \prec_1 Lp(\mathbb{R})|\gamma$. Note that $\gamma \geq \Theta^{L(\mathbb{R},\bar{\mu})}$, but this inequality could be strict. However, since $[\boldsymbol{\delta}^2_1(\bar{\mu}),\gamma]$ is a Σ_1-gap, we have that $Lp(\mathbb{R})|(\gamma+1)$ is the first initial segment of $L(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ that has a subset of the reals not in $Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$, in fact $(\mathbb{R},\bar{\mu})^\sharp \in Lp(\mathbb{R})|(\gamma+1)$. Thus $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})|\gamma$ and so $\mathbf{Env}(\Gamma) = \mathcal{P}_{\bar{\mu}}(\mathbb{R})$, as wanted.

Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Since $\vec{B}$ is countable, there exists a real x such that each element in $\vec{B}$ is $\mathrm{OD}_x^{L(\mathbb{R},\bar{\mu})}$. Let us define $\mathcal{M}$ to be $\mathcal{M}_{\infty,\bar{\mu}}(x)$, the direct limit relative to x done in the HOD Analysis of $L(\mathbb{R},\bar{\mu})$ using the techniques of Section 3 in [15]. Then $\mathcal{M}$ has ω^2-many Woodin cardinals and term captures every B in $\vec{B}$. Let τ_B be the standard term in $\mathcal{M}$ capturing B. Let us define $\mathcal{N} = Hull^{\mathcal{M}}(\{\tau_B \mid B \in \vec{B}\})$. Hence $\mathcal{N}$ is a mouse that captures all the elements of a self-justifying system. Thus, by a theorem of Woodin, the strategy that picks realizable branches into $\mathcal{M}$ and moves these term relations correctly is an iteration strategy for $\mathcal{N}$ (see [5]). In other words, $\mathcal{N}$ is ω_1-iterable in $L(\mathbb{R},\mathcal{C})$, and hence $\mathcal{N}^\sharp$ exists and is ω_1-iterable in $L(\mathbb{R},\mathcal{C})$. Therefore $\mathcal{M}^\sharp_{\omega^2}$ exists and it is ω_1 (and hence ω_1+1) iterable.

□

We claim that if ν is the club filter in $L(\mathbb{R},\mathcal{C})$, then $L(\mathbb{R},\nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. For this, let γ be such that $L_\gamma(\mathbb{R},\mathcal{C})$ reflects enough set theory. By results of [15] we have that DC holds in $L(\mathbb{R},\mathcal{C})$. Therefore, there is a countable set N, such that $N \prec L_\gamma(\mathbb{R},\mathcal{C})$ and $\mathcal{M}^\sharp_{\omega^2}$ and its strategy are in N. Let $\bar{N}$ be the transitive collapse of N, then the proof of Proposition 7 will imply that $\bar{N}$ believes that "if ν is its club filter, then

$L(\mathbb{R},\nu)\models$ AD $+\ \omega_1$ is $\mathbb{R}$-supercompact". By elementarily and the choice of γ, we get that $L(\mathbb{R},\mathcal{C})$ believes this. Also, $\nu\subseteq\mathcal{C}$ and by an induction on the constructive hierarchy, as in the proof of Lemma 13, we get $L(\mathbb{R},\nu)=L(\mathbb{R},\mathcal{C})$. So $L(\mathbb{R},\mathcal{C})=L(\mathcal{P}_{\mathcal{C}}(\mathbb{R}))$ and applying Theorem 9.103 of [18] we get that $L(\mathbb{R},\mathcal{C})\models \mathrm{AD}_{\mathbb{R}}$, but by [15] $L(\mathbb{R},\mathcal{C})\models\Theta=\theta_0$, which is a contradiction.

Case 2: $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mu(\mathbb{R})$.
Using the same argument as in Case 1, we have that $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$ is contained in $L_{\boldsymbol{\delta^2_1}(\mu)}(\mathbb{R},\mu)$. By Theorem 1.2 of [15] we have that $\mu\cap L_{\boldsymbol{\delta^2_1}(\mu)}(\mathbb{R},\mu)$ is a subset of the club filter of $L(\mathbb{R},\mu)$. So if $A\in\mathcal{P}_{\mathcal{C}}(\mathbb{R})\cap\mu$ then A contains a club in V, hence $\mathcal{C}\cap L(\mathbb{R},\mathcal{C})\subseteq\mu$, a contradiction. □

Lemma 20 implies that any two models, $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$, satisfying AD $+\ \omega_1$ is $\mathbb{R}$-supercompact would have same Θ and so they would also share the same $\boldsymbol{\delta^2_1}$. This will justify referring to $\Theta(\mu)$ and $\boldsymbol{\delta^2_1}(\mu)$ simply as Θ and $\boldsymbol{\delta^2_1}$ respectively.

We start the outline of the HOD Analysis by defining the standard notions. This means, we will define in V a direct limit recovering $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$ and then define in $L(\mathbb{R},\mathcal{C})$ its corresponding covering system.

Definition 21. We say P is a δ_0-*bounded* and Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$ if $\mathcal{M}^\sharp_{\omega^2}\overset{\Sigma}{\to}P$ via a tree in which all extenders used have critical point below the image of $\delta_0^{\mathcal{M}^\sharp_{\omega^2}}$.

Let

$$\mathcal{D}^+=\{P\,|\,\mathcal{M}^\sharp_{\omega^2}\overset{\Sigma}{\to}P \text{ via a } \delta_0 \text{ bounded countable tree}\}$$

and for P and Q in $\mathcal{D}^+$ say $P\preceq Q$, if P iterates to Q via Σ in a δ_0-bounded way, in which case we let $\pi_{P,Q}$ be the corresponding unique embedding given by Σ. Note that $(\mathcal{D}^+,\preceq,\pi_{Q,P})$ is a directed system, as by the Dodd-Jensen property of Σ the embeddings commute. Define M^+_∞ to be the direct limit of $(\mathcal{D}^+,\preceq,\pi_{P,Q})$ with the last extender iterated away ON-many times. Also, let $\pi_{Q,\infty}$ be the natural map from Q to M^+_∞.

As in the case of $L(\mathbb{R})$, we will prove that in fact $L[M^+_\infty,\Sigma\restriction X]=\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$, where $\vec{\mathcal{T}}$ is in X if $\vec{\mathcal{T}}\in M^+_\infty|(\lambda^{M^+_\infty}_{\omega^2})$ and is a full finite and δ_0-bounded stack; for these definitions see [13]. We start with definition of suitability.

Let us work from now on in $L(\mathbb{R},\mathcal{C})$ and let $\Gamma=\mathcal{P}_{\mathcal{C}}(\mathbb{R})$.

Definition 22. Let $\alpha\in\omega^2$. Let P be a pre-mouse. We say P is α-*suitable* if the following hold.

(1) $P\models$ "There are α-many Woodin Cardinals" . We will let $\langle\delta^P_i\rangle_{i\in\alpha}$ be the enumeration of these in increasing order.
(2) For any $\eta<o(P)$ a cut-point of P, we have $Lp(P|\eta)\trianglelefteq P$
(3) For any $\eta<o(P)$ not in the sequence $\langle\delta^P_i\rangle_{i\in\alpha}$ then $Lp(P|\eta)\models$ "η is not Woodin".

(4) If $\lambda = \sup_{i\in\alpha} \delta_i^P$, then $o(P) = \sup_{n\in\omega}(\lambda^{+n})^P$.

We will say P is *suitable* if there is $\alpha < \omega^2$ such that P is α-suitable, and we define $\alpha(P) = \alpha$.

Definition 23. We recall the definition for an iteration tree to be Γ-*guided*. Namely, for a normal tree $\mathcal{T}$ on a suitable mouse P, we say $\mathcal{T}$ is Γ-guided or simply guided if and only if for all limit $\eta < \mathrm{lh}(\mathcal{T})$, we have that $Q([0,\eta]_T, \mathcal{T}\restriction\eta)$ exists and is an initial segment of $Lp(\mathcal{M}(\mathcal{T}\restriction\eta))$. We say that $\mathcal{T}$ is *maximal* if $Lp(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin, otherwise we say $\mathcal{T}$ is *short*.

Definition 24. Consider an α-suitable pre-mouse P and $A \subseteq \mathbb{R}$. We say that P captures A at δ_i^P if there is a name τ for a set of reals, such that whenever $g \subseteq \mathrm{col}(\omega, \delta_i^P)$ is generic over P, we have $\tau_g \cap \mathbb{R} = A \cap \mathbb{R}$. We say that P captures A if for any $i < \alpha(P)$, P captures A at δ_i^P.

Note that given $A \subset \mathbb{R}$ and a suitable P capturing A at δ_i^P then there is a unique standard term doing so in the sense of [9]. We will call this term $\tau^P_{A,i}$.

We need to define a notion of iterability that is downwards absolute to $L(\mathbb{R}, \mathcal{C})$ and is strong enough so that one can compare suitable mice (note the connection with [9]). Suitable mice are allowed to have countably many Woodin cardinals as opposed to just finitely many in the traditional case of $L(\mathbb{R})$, that is why we need a stronger form of iterability that we describe below.

We will define a slight modification of Definition 1.8. from [8]. A suitable P is said to be *weakly* (ω, ω^2)-iterable* if player II has a winning strategy for the game in which I and II alternate moves for ω^2 many rounds as follows. The game starts by letting $P_0 = P$ and at round α player I plays a countable normal, guided, putative iteration tree $\mathcal{T}_\alpha$ on P_α and II can either accept I's move, in case the last model of $\mathcal{T}_\alpha$ is welfounded, or play a maximal well-founded branch, b_α, on $\mathcal{T}_\alpha$ such that if $\mathcal{T}$ is short then $Q(b_\alpha, \mathcal{T})$ exists and it is an initial segment of $Lp(\mathcal{M}(\mathcal{T}))$. We then let $P_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}_{b_\alpha}$ and I can play round $\alpha + 1$. If I and II have played for all $\beta < \alpha$ and α is limit then:

- If there is $i < \alpha(P)$ such that for infinitely many β, we have that $\mathcal{T}_\beta$ is a tree based on $P_\beta|\delta_i^{P_\beta}$ then he looses.
- Otherwise it is II's responsibility to ensure that if b is the unique cofinal branch of the concatenation of the trees played so far, then b is well-founded (otherwise he loses). In this case we let P_α be the direct limit of the models.

The α-th round starts with I playing a putative normal guided tree on P_α. After the ω^2 rounds have been played the only condition for II is that the direct limit along the main branch is well-founded. We illustrate the weak* game, $\mathcal{WG}^*(P, \omega^2)$ game as follows:

Player	0		1	...		ω	...
I	$\mathcal{T}_0$ on P_0		$\mathcal{T}_1$ on $\mathcal{M}^{\mathcal{T}_0}_{b_0}$		$\left.\right\} P_\omega$	$\mathcal{T}_\omega$	
II		b_0		b_1			b_ω

Note that if P and Q are suitable pairs such that II has a winning strategies τ_P and τ_Q for $\mathcal{WG}^*(P,\omega^2)$ and $\mathcal{WG}^*(Q,\omega^2)$ respectiveley. Then we can form $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$-guided iteration trees $\mathcal{T}_P$ and $\mathcal{T}_Q$ using the extenders that cause the "least" disagreement, and using τ_P and τ_Q when a maximal tree arises in this comparison. Since each P and Q have $< \omega^2$-many Woodin cardinals, this comparison succeeds. Note also that the end model of this comparison is still weakly* (ω,ω^2)-iterable.

If P is a suitable mouse capturing some $A \subseteq \mathbb{R}$ it will be desirable that "good" iterations of P maintain the suitability condition and move the terms capturing A correctly.

Definition 25 (A-iterations)**.** For a suitable P capturing A a set of reals, we have the following definitions.

(1) We say P is A-iterable in case II has a winning strategy for the game $\mathcal{WG}^*(P,\omega^2)$ such that whenever $\vec{\mathcal{T}}$ is a stack given by the concatenation of the trees and branches played by I and II in a game according to the strategy, and the main branch b played by the strategy is such that $i^{\vec{\mathcal{T}}}_b : P \to \mathcal{M}^{\vec{\mathcal{T}}}_b$ exists, then $\mathcal{M}^{\vec{\mathcal{T}}}_b$ is suitable and for any $i < \alpha(P)$ we have that $i^{\vec{\mathcal{T}}}_b(\tau^P_{A,i}) = \tau^{\mathcal{M}^{\vec{\mathcal{T}}}_b}_{A,i}$. We will call such strategy an A-*strategy* for P.

(2) We will call $\vec{\mathcal{T}}$, a stack on P, an A-*iteration* if it is the result of concatenating the trees and branches of a run in $\mathcal{WG}^*(P,\omega^2)$, according to an A-strategy.

(3) We will say Q is an A-*iterate* of P if there is an A-iteration $\vec{\mathcal{T}}$ on P with last model Q and $\pi : P \to Q$ given by this stack exists.

(4) For $\mathcal{T}$ a normal guided tree on P of successor length $\eta+1$, such that $\mathcal{T}\restriction\eta$ is maximal, we let $\mathcal{T}^- = \mathcal{T}\restriction\eta$. In other words $\mathcal{T}^-$ is $\mathcal{T}$ without the last branch.

Also, given a family $\vec{A}$ of finite subsets of the reals the definition of $\vec{A}$-iterability generalizes in the following way. We say P is $\vec{A}$-*iterable* in case there exist a winning strategy in $\mathcal{WG}(P,\omega^2)$ that is simultaneously witness A-iterability for any A in the sequence $\vec{A}$.

So far we do not know whether there are A-iterable suitable mice but as in the case of $L(\mathbb{R})$ these exist when $\mathcal{M}^\sharp_{\omega^2}$ is present. The following pair of lemmas are essentially in Chapter 3 of [9] so we omit their proofs.

Lemma 26. *Suppose $A \subseteq \mathbb{R}$ is definable in $L(\mathbb{R},\mathcal{C})$ from indiscernibles, then any suitable initial segment of a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$ is A-iterable.*

The idea in the proof of this last lemma is the following. Note that if N is a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, by Lemma 10 given δ a Woodin cardinal of N, we can iterate $N \stackrel{\Sigma}{\to} K$ above δ to make $L(\mathbb{R},\mathcal{C})$ realizable as the model given by the symmetric reals and the tail filter of a symmetric collapse over K. Then one can define truth in $L(\mathbb{R},\mathcal{C})$ in K using the homogeneity of the collapse. The fullness of N and the existence of an elementary embedding $j : N \to K$ will give us that the relevant terms are actually in the suitable initial segments of N. This lemma readily implies the following Corollaries.

Corollary 27. *Suppose A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals. Then for any $\alpha < \omega^2$ there is an α-suitable P that is A-iterable.*

Lemma 28 (Comparison)**.** *Suppose that P is A-iterable and Q is B-iterable. Then there is an $A \oplus B$-iterable suitable mouse R, an A-iteration from P to a suitable initial segment of R and a B iteration from Q to suitable initial segment of R.*

The point in the proof of Corollary 27 and Lemma 28 is that given a counterexample in $L(\mathbb{R},\mathcal{C})$ to a statement of the form "for all $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ sets of reals" then one can by minimizing the counterexample, find a definable one, but then the suitable initial segments of $\mathcal{M}^\sharp_{\omega^2}$ will witness that there is no definable counterexample.

It is clear that our notion of A-iterability is downwards absolute to $L(\mathbb{R},\mathcal{C})$. We would like to define our covering system using pairs (P, A), where A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and P is an A-iterable mouse. However, it could be the case that for such a P, there are two different A-iterations $\pi : P \to Q$ and $\sigma : P \to Q$, and this would be a clear problem in building a directed limit. For this reason we need to work with relevant hulls and a stronger notion of iterability. We define below these concepts.

Definition 29. For P an A-iterable mouse, we let

1. $\gamma^P_{A,i} = \sup(Hull^P(\tau^P_{A,i}) \cap \delta^P_0)$.
2. $\gamma^P_A = \sup_{i\in\alpha(P)} \gamma^P_{A,i}$.
3. $\xi^P_A = \gamma_A^{P|(\delta_0^{+\omega})^P}$.[1]
4. $H(P, A) = Hull^P(\xi^P_A \cup \{\tau^P_{A,i} \mid i < \alpha(P)\})$
5. $P^- = P|(\delta_0^{+\omega})^P$

Using the usual "zipper argument" we get the following lemma.

Lemma 30. *Let $\mathcal{T}$ be a tree of limit length on P, a suitable pre-mouse. Suppose further that there are branches b and c such that $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are A-iterations and $\mathcal{M}^\mathcal{T}_b$ and $\mathcal{M}^\mathcal{T}_c$ are A-iterable. Then $i^\mathcal{T}_b \restriction \gamma^P_A = i^\mathcal{T}_c \restriction \gamma^P_A$ and so $i^\mathcal{T}_b \restriction H(P, A) = i^\mathcal{T}_c \restriction H(P, A)$.*

A key point here is that if P is an A-iterable mouse we could potentially have two A-iterations associated to two different trees on P leading to the same end model Q, so Lemma 30 would not apply. Hence we define the

[1] Note that $P|(\delta_0^{+\omega})^P$ is 1-suitable and A iterable .

notion of strong iterability in the natural way and prove the existence of strongly iterable mice

Definition 31. For $A \subseteq \mathbb{R}$ and P a suitable A-iterable mouse we say P is *strongly A-iterable* if whenever $i : P \to Q$ and $j : P \to Q$ are two A-iterations then $i \restriction H(P, A) = j \restriction H(P, A)$.

Note again that when proving that for any $A \subseteq \mathbb{R}$ which is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ there is a strongly A-iterable mouse it is sufficient to prove that for any definable set A there is a strongly A-iterable mouse. The following lemma in contrast to most of what we have been discussed so far, is not a straightforward generalization of the HOD Analysis in $L(\mathbb{R})$. The reason of this is the extra complexity in the iteration games considered. We give a detailed proof for the existence of strongly A-iterable mice.

Lemma 32. *Let A be an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and let P be A-iterable. Then there is an A-iterate of P that is strongly A-iterable.*

Proof. By the discussion above we may without loss assume that A is definable in $L(\mathbb{R}, \mathcal{C})$. Given an A-iterable mouse P, by comparison we can A-iterate P to Q, an initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}$. We claim that Q is as wanted.

Suppose $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are A-iteration stacks on Q with the same las model R. We want to show that the embeddings given by $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ agree on $H(Q, A)$. We will actually show that both embeddings agree with embeddings given by Σ on $H(Q, A)$. Here we have to be an extra bit more careful than in the analogous situation of $L(\mathbb{R})$, because our iteration games can have more rounds and at limits stages it is not straightforward how to proceed, we will show next the details of how to overcome this difficulty.

For this we look inductively at the trees in the stack $\vec{\mathcal{T}} = \langle \mathcal{T}_i \mid i \in \alpha \rangle$. Let Q_i (for $i \in \alpha$) be the model starting round i in the weak* game. We will construct trees $\mathcal{S}_i$ inductively such that $\vec{\mathcal{S}} = \langle \mathcal{S} \mid i \in \alpha \rangle$ is according Σ and has the property that the embedding given by $\vec{\mathcal{S}}$ agrees with $i^{\vec{\mathcal{T}}}$ on γ^Q_A. We will assume with no loss of generality that every tree $\mathcal{T}_i$ for $i \in \alpha$ in based on a window of the form $(\delta^{Q_i}_{k_i}, \delta^{Q_i}_{k_i+1})$.

Starting with $\mathcal{T}_0$, let us define $\mathcal{S}_0$ as follows. First suppose that $\mathcal{T}_0$ is based on ${Q_0}^-$, and if it is according to Σ we let $\mathcal{S}_0 = \mathcal{T}_0$. Otherwise $\mathcal{T}_0$ is a maximal tree with a last branch b. Recall that $\mathcal{T}_0^-$ denotes the maximal part of $\mathcal{T}_0$. Let c be the branch given by Σ through $\mathcal{T}_0^-$, and note that c respects A by Lemma 26. Let $\mathcal{S}_0$ be $\mathcal{T}_0^- \frown c$. Also, by Lemma 30, we have that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ^Q_A. Recall that Q_1 is the last model of $\mathcal{T}_0$, and let $\bar{Q}_1$ be the last model of $\mathcal{S}_0$, hence by fullness and maximality of $\mathcal{T}_0^-$, we get $Q_1^- = \bar{Q}_1^-$.

If $\mathcal{T}_0$ is above δ_0 then let $\mathcal{S}_0 = \emptyset$, $i^{\mathcal{S}_0} = \mathrm{id}$ and $\bar{Q}_1 = Q_0$. Here we get also get trivially that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ^Q_A and $Q_1^- = \bar{Q}_1^-$.

Let us consider then $\mathcal{T}_1$. If it is based on ${Q_1}^-$ we can regard it as a tree on $\bar{Q}_1$ and then we can again use Σ to get $\mathcal{S}_1$ on $\bar{Q}_1$ such that $i^{\mathcal{T}_1}$ and $i^{\mathcal{S}_1}$ agree on $\xi_A^{Q_1} = \xi_A^{\bar{Q}_1}$. Again, by fullness we get that if $\bar{Q}_2$ is the last model of $\mathcal{S}_1$, then $Q_2^- = \bar{Q}_2^-$.

Otherwise we just let $\mathcal{S}_1 = \emptyset$ and the desired agreement is maintained so far.

Note that by the rules of the weak* game one has that $\mathcal{T}_i$ can be based on ${Q_i}^-$ only for finitely many $i \in \omega$. Hence $\bar{Q}_\omega$ agrees with Q_ω up to their common 1-suitable initial segment and the embedding on the $\mathcal{T}$-side agrees with the one given by the $\mathcal{S}$-side up to ξ_A^Q.

We proceed inductively in this fashion. At successors simply use Σ if the tree is based below the least Woodin cardinal, and otherwise define the corresponding tree in the $\mathcal{S}$-side as empty.

After α-many steps in this induction we will have that $\bar{Q}_\alpha$ is a Σ-iterate of Q. Let $\bar{\sigma}$ be the branch embedding. Then we have that $\bar{Q}_\alpha$ agrees with $Q_\alpha = R$ up to their common 1-suitable initial segment, and that $\bar{\sigma} \restriction \xi_A^Q = i^{\vec{\mathcal{T}}} \restriction \xi_A^Q$.

Similarly for $\vec{\mathcal{U}}$ one can get the analogous construction. So, we get that $i^{\vec{\mathcal{U}}}$ agrees with $\sigma' : Q \to Q'_\alpha$, an embedding given by Σ. Furthermore R, $\bar{Q}_\alpha$ and Q'_α agree up to their 1-suitable initial segment, and so since δ_0^R is a cut-point of both $\bar{Q}_\alpha$ and Q'_α by the Dodd Jensen property of Σ we can conclude that $\bar{\sigma}$ and σ' agree up to $\delta_0^{Q_0}$, and so $i^{\vec{\mathcal{T}}}$ and $i^{\vec{\mathcal{U}}}$ agree up to ξ_A^Q. Hence they agree on $H(Q, A)$. This concludes the proof that Q is strongly A-iterable. $\square$

Our covering system in $L(\mathbb{R}, \mathcal{C})$ will be

$$\mathcal{D}^- = \{H(P, \vec{A}) \,|\, P \text{ is strongly } \vec{A}\text{-iterable and } \vec{A} \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}\}.$$

Also we let $(P, \vec{A}) \preceq (Q, \vec{B})$ if Q is an A iterate of P and $\vec{A} \subseteq \vec{B}$. We let $\sigma_{(P,\vec{A}),(Q,\vec{B})}$ be the unique embedding from $H(P, \vec{A})$ to $H(Q, \vec{B})$ given by an (any) $\vec{A}$-iteration from P to Q. The following results show that the suitable initial segments of correct iterates of $\mathcal{M}^\sharp_{\omega^2}$ together with the theories of indiscieribles for $L(\mathbb{R}, \mathcal{C})$ are dense in $\mathcal{D}^-$.

Let

$$M_\infty = \lim(\mathcal{D}^-, \preceq, \sigma_{(P,A),(Q,B)})$$

and let us define $\sigma_{(P,A),\infty}$ the natural embedding from $H(P, A)$ to this direct limit.

Let $\mathcal{T}_n^{\mathcal{C}}$ be the theory of n-many indiscernibles with real parameters of $L(\mathbb{R}, \mathcal{C})$ (coded as a subset of $\mathbb{R}$). Lemma 5.6 and Lemma 5.9 in [9] give the following results, we omit their proofs as they are word by word the same, except that we use $\mathcal{M}^\sharp_{\omega^2}$ and $L(\mathbb{R}, \mathcal{C})$ instead of $\mathcal{M}_\omega$ and $L(\mathbb{R})$ (the key, again, is that one can realize $L(\mathbb{R}, \mathcal{C})$ as the derived model of an iterate of $\mathcal{M}^\sharp_{\omega^2}$).

Lemma 33. *Suppose P is a suitable initial segment of a Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$, then $\delta_0^P = \sup\{\xi^P_{\mathcal{T}^{\mathcal{C}}_n} \mid n \in \omega\}$.*

Lemma 34. *Assume A is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$, and P is strongly A-iterable suitable mouse. Then there is R, a suitable initial segment of a Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$, and a natural n such that $(P,A) \preceq (R, A \oplus \mathcal{T}^{\mathcal{C}}_n)$ and moreover $\tau^R_A \in H(R,\mathcal{T}^{\mathcal{C}}_n)$*[2]*.*

Let us pause for a moment and discuss the general $L(\mathbb{R},\mu)$ case. The lemma above will also be valid in this context by an application of Σ_1-reflection.

Lemma 35. *Suppose $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact and let A be $\mathrm{OD}^{L(\mathbb{R},\mu)}$. Given P a strongly A-iterable suitable mouse and B an $\mathrm{OD}^{L(\mathbb{R},\mu)}$ set of reals, with $A \leq_W B$, there is R a suitable and $A \oplus B$-iterable mouse, such that $(P,A) \preceq (R, A \oplus B)$ and moreover $\tau^R_A \in H(R,B)$.*

Proof. Otherwise fix A and B a counterexample to the statement. Fix γ large enough such that $L_\gamma(\mathbb{R},\mu) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ and A and B are ordinal definable over $L_\gamma(\mathbb{R},\mu)$, but $L_\gamma(\mathbb{R},\mu)$ has no R and A-iteration of P witnessing the conclusion of the Lemma. This Σ_1 statement about γ can then be reflected below $\boldsymbol{\delta}^2_1$. Hence there is such a $\gamma < \boldsymbol{\delta}^2_1$. But then $L_\gamma(\mathbb{R},\mathcal{C}) = L_\gamma(\mathbb{R},\mu)$ since below $\boldsymbol{\delta}^2_1$ both μ and $\mathcal{C}$ are just the club filter. We get then that there are A and B counterexamples of the statement in $L_\gamma(\mathbb{R},\mathcal{C})$ (and moreover OD in this structure). But then we can get the desired R and A-iteration in $L(\mathbb{R},\mathcal{C})$ and, by closure of γ, an A-iteration of P leading to R can be computed in $L_\gamma(\mathbb{R},\mu)$, so A and B cannot be the counterexample of $L_\gamma(\mathbb{R},\mathcal{C})$, contradiction. □

The lemmas above allow us to compute the direct limit of $\mathcal{D}^-$ just by looking at suitable initial segments of Σ-iterates of $\mathcal{M}^{\sharp}_{\omega^2}$ with corresponding theories of indiscernibles. We then have the following agreement.

Theorem 36. $M_\infty = M^+_\infty | \lambda^{M^+_\infty}_{\omega^2}$

Proof. We define a map $i : M_\infty \to M^+_\infty$ that is surjective below $\lambda^{M^+_\infty}_{\omega^2}$ and respects the membership relation as follows. For $x \in M_\infty$ there is a natural n and a suitable initial segment of a correct iterate of $\mathcal{M}^{\sharp}_{\omega^2}$, say P, such that x is in the range of $\sigma_{(P,A\oplus\mathcal{T}^{\mathcal{C}}_n),\infty}$, and there is $z \in H(P,\mathcal{T}^{\mathcal{C}}_n)$ such that $\sigma_{(P,A\oplus\mathcal{T}^{\mathcal{C}}_n),\infty}(z) = x$. Now we have an iteration $\mathcal{M}^{\sharp}_{\omega^2} \xrightarrow{\Sigma} \mathcal{N}$ such that P is a suitable initial segment of $\mathcal{N}$. Note that $\mathcal{N}$ might not be a δ_0-bounded iterate of $\mathcal{M}^{\sharp}_{\omega^2}$. We can however split the iteration from $\mathcal{M}^{\sharp}_{\omega^2}$ to $\mathcal{N}$ in a δ_0-bounded part and the rest. Namely, there is $\mathcal{N}^*$ such that $\mathcal{M}^{\sharp}_{\omega^2} \xrightarrow{\Sigma} \mathcal{N}^*$ into a δ_0-bounded way, and $\mathcal{N}^* \xrightarrow{\Sigma} \mathcal{N}$. Note that this second iteration does

[2] Here we actually mean that for any $i < \alpha(R)$ we have $\tau^R_{i,A} \in H(R,\mathcal{T}^{\mathcal{C}}_n)$ so that $H(R, A \oplus \mathcal{T}^{\mathcal{C}}_n) = H(R,\mathcal{T}^{\mathcal{C}}_n)$.

not move $\delta_0^{\mathcal{N}^*}$ (because all its extenders have critical point above the first Woodin cardinal). This implies that z is in the range of $\pi_{\mathcal{N}^*,\mathcal{N}}$, let $\bar{z}$ be its pre-image. Then we define $i(x) = \pi_{\mathcal{N}^*,\infty}(\bar{z})$, it is routine to show that i is well defined (see Theorem 5.10. of [9]). Now Lemma 34 gives us the surjectivity as follows: Let $x \in M_\infty^+|\lambda_{\omega^2}^{M_\infty^+}$ so there is $z \in \mathcal{N}$ a correct iterate of $\mathcal{M}_{\omega^2}^\sharp$ such that $\pi_{\mathcal{N},\infty}(z) = x$. Let P be a suitable initial segment of $\mathcal{N}$ such that $z \in P$. Now because $\mathcal{N}$ is an δ_0 bounded iterate of $\mathcal{M}_{\omega^2}^\sharp$ we have that z is definable from ordinals less than $\delta_0^{\mathcal{N}}$ and indiscernibles, but this is easily computable from $\mathcal{T}_n^{\mathcal{C}}$ for a suitable n (again this follows essentially by Corollary 5.7 of [9]). Because $\xi_{\mathcal{T}_n^{\mathcal{C}}}^P$ is unbounded in $\delta_0^{\mathcal{N}}$ we conclude that $z \in H(P, \mathcal{T}_n^{\mathcal{C}})$ for a sufficiently large n. This readily implies x is in the range of i as wanted. □

Let us work for a moment in $V^{\mathrm{col}(\omega,\mathbb{R})}$. Here we have that M_∞ is a countable Σ-iterate of $\mathcal{M}_{\omega^2}$. Also if G is M_∞^+-generic for $\mathrm{col}(\omega, < \lambda_{\omega^2}^{M_\infty})$, and $\mathbb{R}^*$ and $\mathcal{F}$ are the symmetric reals and associated tail filter, then we have that $L(\mathbb{R}^*, \mathcal{F})$ is model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Following the notation and the content of Chapter 6 from [9], for every n we can define, $\mathcal{T}_n^{\mathcal{C}^*}$, an $\mathrm{OD}^{L(\mathbb{R}^*,\mathcal{F})}$ set, by pieces as follows. For $(P, \mathcal{T}_n^{\mathcal{C}})$, an element of $\mathcal{D}^-$, and for $i < o(P)$ let $\tau_{\mathcal{T}_n^{\mathcal{C}},i}^* = \sigma_{(P,\mathcal{T}_n^{\mathcal{C}}),\infty}(\tau_{\mathcal{T}_n^{\mathcal{C}},i}^P)$ and $\mathcal{T}_n^{\mathcal{C}^*} = \bigcup_{i\in\omega^2} (\tau_{\mathcal{T}_n^{\mathcal{C}},i}^*)_{G\restriction\delta_i^{M_\infty}}$. We also have that any suitable initial segment of M_∞ is strongly $\mathcal{T}_n^{\mathcal{C}^*}$-iterable (in $V^{\mathrm{col}(\omega,\mathbb{R})}$ as witnessed by Σ and in $L(\mathbb{R}^*, \mathcal{F})$ by absoluteness). Similarly we define A^*, an ordinal definable in $L(\mathbb{R}, \mathcal{F})$ set of reals, for each A in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$. Recall that M_∞^- is the 1-suitable initial segment of M_∞. We summarize the discussion above in the following lemmas.

Lemma 37. *For any set of reals A which is OD in $L(\mathbb{R}, \mathcal{C})$ we have that A^* is OD in $L(\mathbb{R}^*, \mathcal{F})$. Moreover for any such A, M_∞^- is strongly A^*-iterable in the sense of $L(\mathbb{R}^*, \mathcal{F})$.*

Proof. This follows exactly as in the case of $L(\mathbb{R})$ so we omit details. These proofs can be essentially be found in Chapter 6: Claims 1,2 and 3 of [9]. □

Recall that X is the set of finite full stacks on M_∞^- in $M_\infty|(\lambda_{\omega^2}^{M_\infty})$. We then have that when computing the correct branches through $\vec{\mathcal{T}}$ it is enough to choose the unique branch that moves all the terms for A^* correctly. That is to say

Lemma 38. *Suppose $\mathcal{T}$ is a correct maximal tree on M_∞^- as in the sense of $L(\mathbb{R}^*, \mathcal{F})$. Then $\Sigma(\mathcal{T}) = b$ if and only if $\mathcal{T} \frown b$ is an A^*-iteration for all A in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$.*

Proof. This is claim 4 of Chapter 6 in [9]. Here we use Lemma 34 instead of Lemma 5.8 of [9], everything else follows word by word. □

From this and the homogeneity of the collapse it follows that $L[M_\infty, \Sigma \restriction X] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$. Also, note that if M^*_∞ is the direct limit defined in $L(\mathbb{R}^*, \mathcal{F})$, then there is an embedding $\sigma : M^- \to M^*_\infty$, where $\sigma = \bigcup_{A \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}} \sigma_{(M^-,A^*),\infty}$.

We have then our HOD Analysis' result.

Theorem 39. *Suppose $\mathcal{M}^\sharp_{\omega^2}$ exists and its iteration strategy is $(2^{\mathfrak{c}})^+$-universally Baire. Then the following are the same model.*

(1) $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$
(2) $L[M_\infty, \Sigma \restriction X]$
(3) $L[M_\infty, \sigma]$

Proof. This proof follows exactly as Claims 6 and 7 in chapter 6 of [9]. Here instead of Lemma 6.2 of [9] we use the corresponding fact about structures $L(\mathbb{R}, \mathcal{F})$ given by Σ-iterates of $\mathcal{M}^\sharp_{\omega^2}$. More precisely, note that if N_1 and N_2 are countable Σ-iterates of $\mathcal{M}^\sharp_{\omega^2}$, G_i is N_i-generic for $\mathrm{col}(\omega, \lambda^{N_i}_{\omega^2})$ and $L(\mathbb{R}_i, \mathcal{F}_i)$ are the associated models (for $i = 1, 2$), then given $x \in \mathbb{R}_1 \cap \mathbb{R}_2$ we have

$$\langle L(\mathbb{R}_1, \mathcal{F}_1), x, T^1_n\rangle \equiv \langle L(\mathbb{R}_2, \mathcal{F}_2), x, T^2_n\rangle,$$

where T^i_n is the theory of n indiscernibles for $L(\mathbb{R}_i, \mathcal{F}_i)$. Another key fact is that $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})} = L[B]$ for some $B \subset \Theta^{L(\mathbb{R},\mathcal{C})}$ but this is just Theorem 3.1 in [15]. □

Also for an arbitrary $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact we can define the corresponding $\mathcal{D}^-_\mu$. Let $M^-_{\infty,\mu}$ be its direct limit (see for example Theorem 3.13. and subsequent discussion in [15]). Furthermore by [15] we have the following result.

Theorem 40. *Suppose $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then*

$$\mathrm{HOD}^{L(\mathbb{R},\mu)} = L[M^-_{\infty,\mu}, \Sigma_\mu]$$

Where Σ_μ is defined in $L(\mathbb{R}, \mu)$ using the corresponding definition given in Theorem 38.

Let us fix a μ as in the discussion above from now on. So, the construction recovering HOD can be relativized to any particular real y as follows. Note that the existence of $\mathcal{M}^\sharp_{\omega^2}$ implies the existence of $\mathcal{M}^\sharp_{\omega^2}(y)$ and so one has $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}_y = L[M^-_{\infty,\mu}(y), \Sigma_\mu(y)]$, where $M^-_{\infty,\mu}(y)$ is the direct limit of $D^-_\mu(y)$, where

$$D^-_\mu(y) = \{H(P, A) \mid P \text{ is a strongly } A\text{-iterable } y\text{-mouse and } A \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}_y\}.$$

And $\Sigma_\mu(y)$ is the strategy whose domain are the finite full stacks on $M^-_{\infty,\mu}(y)$ that are in $M^-_{\infty,\mu}(y)|(\lambda^{M^-_{\infty,\mu}(y)}_{\omega^2})$ and it picks the branches b such that they respect every A^* [3] for $A \in \mathrm{OD}_y^{L(\mathbb{R},\mathcal{C})}$. The crux of the main theorem of this section is the following observation.

Note that by Lemma 20 we have that $\mathcal{P}_{\mathcal{C}}(\mathbb{R}) = \mathcal{P}_\mu(\mathbb{R})$, also this implies that the notion of suitability is the same in $L(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\mu)$; the notion of ordinal definability might however be different. Define $\mathcal{T}_n^\mu$ to be the theory of n many indiscernibles of $L(\mathbb{R},\mu)$. So, we have that for any n there is k such that $\mathcal{T}_n^{\mathcal{C}} \leq_W \mathcal{T}_k^\mu$, and vice versa, for any n there is a k such that $\mathcal{T}_n^\mu \leq_W \mathcal{T}_k^{\mathcal{C}}$. From now on lets fix a real number x that codes each of these reductions in a natural way[4]. If P is a suitable initial segment of a Σ_x-iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$ then by Lemmas 26 and 32 we have that P is strongly $\mathcal{T}_n^{\mathcal{C}}$-iterable. Furthermore as $x \in P$ by Lemma 5.9 of [9] we have that P captures $\mathcal{T}_n^\mu$ for every natural n. Moreover we have that if x codes a reduction $\mathcal{T}_n^\mu \leq_W \mathcal{T}_k^{\mathcal{C}}$ then for any $i < o(P)$, $\tau^P_{\mathcal{T}_n^\mu, i} \in H(P, \mathcal{T}_k^{\mathcal{C}})$ and moreover every $\mathcal{T}_k^{\mathcal{C}}$-iteration of P is also a $\mathcal{T}_n^\mu$ one. The following lemma will show that as in the case of $L(\mathbb{R},\mathcal{C})$ the pairs of the form $(P, \mathcal{T}_n^{\mathcal{C}})$ are dense in $\mathcal{D}_\mu^-$ in the sense of Lemma 34. In other words.

Lemma 41. *Suppose $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Let A be $\mathrm{OD}_x^{L(\mathbb{R},\mu)}$ and P is an x-mouse that is A-iterable. Then there is a natural number n and a suitable initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$, say Q, that is $A \oplus \mathcal{T}_n^{\mathcal{C}}$-iterable, $\tau_A^Q \in H(Q, \mathcal{T}_n^{\mathcal{C}})$ and is an A-iterate of P.*

Proof. Here just note that $\{\mathcal{T}_n^{\mathcal{C}} \mid n \in \omega\}$ is Wadge cofinal in the Wadge hierarchy of $L(\mathbb{R},\mu)$. Also for every n we have that $\mathcal{T}_n^{\mathcal{C}}$ is $\mathrm{OD}_x^{L(\mathbb{R},\mu)}$. We can then apply Lemma 35 and comparison to get the desired Q. □

Theorem 42. *Suppose that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then for a Turing cone of $y \in \mathbb{R}$ we have that $\mathrm{HOD}_y^{L(\mathbb{R},\mu)} = \mathrm{HOD}_y^{L(\mathbb{R},\mathcal{C})}$.*

Proof. Using the previous Lemma, the proof of Theorem 36 follows in the same way giving that $M^-_{\infty,\mu}(x) = M^+_\infty(x) | \lambda_{\omega^2}^{M^+_\infty(x)}$. But then Theorem 36 gives that $M^-_\infty(x) = M^-_{\infty,\mu}(x)$. Hence the covering limits agree. Now, we turn to see that the strategies agree as well.

Claim: $\Sigma_{\mu,x} = \Sigma_x$ when restricted to the relevant trees [5].

Proof of the Claim. We will prove inductively that if $\vec{\mathcal{T}}$ is a stack of n trees, and is according to both Σ_x and $\Sigma_{\mu,x}$ then these strategies pick the next branch the same way. Note that by the definitions of Σ_x and $\Sigma_{\mu,x}$ we

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[3] As defined in $L(\mathbb{R}^*, \mathcal{F})$, the model given by a generic filter over $M_{\infty,\mu}$ for the collapse up to the sup of the Woodin cardinals of $M_{\infty,\mu}$.

[4] Fix $z \mapsto \langle (z)_i \rangle_{i\in\omega}$ a recursive bijection between $\mathbb{R}$ and $\mathbb{R}^\omega$ and fix x such that given $n \in \omega$ there exists i and j naturals such that $(x)_i$ codes a continuous reduction witnessing $\mathcal{T}_n^\mu \leq_W \mathcal{T}_k^{\mathcal{C}}$ (for some k) and the similarly $(x)_j$ codes a reduction $\mathcal{T}_n^{\mathcal{C}} \leq_W \mathcal{T}_k^\mu$ (fore some other k).

[5] We refer as Σ_x the strategy given by Lemma 38 and $\Sigma_{\mu,x}$ the one given in $L(\mathbb{R},\mu)$.

have that $\Sigma_x(\vec{\mathcal{T}}) = b$ if and only if $\vec{\mathcal{T}} \frown b$ is an $\mathcal{T}_n^{\mathcal{C}*}$-iteration on M_x^- for all $n \in \omega$ (here again the key fact is that the $\xi_{\mathcal{T}_n^{\mathcal{C}}}^{M_x^-}$ and the $\xi_{\mathcal{T}_n^{\mu}}^{M_x^-}$ are cofinal in $\delta_o^{M_x^-}$). As we noted above this means that $\vec{\mathcal{T}} \frown b$ is a $\mathcal{T}_n^{\mu*}$-iteration for all $n \in \omega$, in other words $\Sigma_{\mu,x}(\vec{\mathcal{T}}) = b$, which finishes the proof of the claim. □

But then this implies $\mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})} = \mathrm{HOD}_x^{L(\mathbb{R},\mu)}$ by Theorem 42. Note also that if $y \geq_T x$ then the analogous results relative to y is still valid. This completes the proof.

□

Proof of Theorem 5. First lets suppose that Σ is $(2^{\mathfrak{c}})^+$-universally Baire, so all the previous results of this section hold. By Theorem 42 we can fix a real x such that $\mathrm{HOD}_x^{L(\mathbb{R},\mu)} = \mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})}$. Then, by results of [14] we have that

$$L(\mathbb{R},\mu) = \mathrm{HOD}_x^{L(\mathbb{R},\mu)}(\mathbb{R}) \text{ and } \mathrm{HOD}_x^{L(\mathbb{R},\mathcal{C})}(\mathbb{R}) = L(\mathbb{R},\mathcal{C}),$$

which clearly implies $L(\mathbb{R},\mathcal{C}) = L(\mathbb{R},\mu)$.

Now, if Σ is just an ω_1+1-iteration strategy, but not necessarily universally Baire. Pick γ such that V_γ reflects enough set theory, and let $N \prec V_\gamma$ be countable and H its transitive collapse. Then we are in the same situation as when proving Proposition 7. Hence the result follows word by word from the proof of Proposition 7. □

4. The AD Case

We give in this section a proof of Theorem 4. We will first assume AD^+ and for contradiction suppose that the theorem does not hold and then we reflect this statement to a Suslin co-Suslin set. Then we can use [12] and [5] to construct models with Woodin cardinals and run a version of last section's arguments. We start by noting some preliminary facts. Lastly we show how to reduce the hypotheses to $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$

Lemma 43. *Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$ and let μ be a filter such that $L(\mathbb{R},\mu)$ satisfies $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $\mathcal{P}_\mu(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.*

Proof. Otherwise we have that $V = L(\mathcal{P}(\mathbb{R}))$ believes there is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Also $V = L(\mathbb{R},\mu)$, so by Theorem 9.103. of [18] $L(\mathbb{R},\mu) \models \mathrm{AD}_{\mathbb{R}}$ but this is impossible since by [15] we have $L(\mathbb{R},\mu) \models \Theta = \theta_0$. □

From now on we will also assume that $V \models \Theta = \theta_0$, as otherwise there exists a non-tame mouse and hence $\mathcal{M}_{\omega^2}^{\sharp}$ exists and it is iterable so the results of last section would hold. Since $\Theta = \theta_0$ we have that, in particular, DC holds in V. We now prove the first approximation to our main result

Theorem 44. *Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$. Then there is at most one model of the form $L(\mathbb{R},\mu)$ satisfying $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Moreover if such model exists then the unique such model is $L(\mathbb{R},\mathcal{C})$ where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Proof. Suppose that there is $\mu \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu) \models \text{AD } \omega_1$ is $\mathbb{R}$-supercompact. Let μ be chosen such that $\mathcal{P}_\mu(\mathbb{R})$ is the minimal (in that given any ν such that $L(\mathbb{R},\nu) \models \text{AD} + \omega_1$ is $\mathbb{R}$-supercompact then we have that $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$). Note that by Lemma 43 we have that there is a set of reals A such that $L(\mathbb{R},\mu)$ is definable from parameters in $L(\mathbb{R},A)$ and moreover $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},A)$. Now by Σ_1 reflection we may assume that A is Suslin and co-Suslin.

Let us work from now on in $L(\mathbb{R},A)$. By minimality of μ we get that $(\mathbb{R},\mu)^\sharp$ is Suslin and co-Suslin in $L(\mathbb{R},A)$. The presence of $(\mathbb{R},\mu)^\sharp$ implies trivially the existence of $\mathcal{N} = (M_{\infty,\mu})^\sharp$ [6]. Let Γ be $\Sigma_1^{L(\mathbb{R},\mu)}$ and $\vec{B}$ a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Let us fix ζ to be the largest Suslin cardinal in $L(\mathbb{R},A)$.

Claim 1: $\mathbf{Env}(\Gamma) \subset Lp(\mathbb{R})$

Proof of Claim1: Let $B \in Env(\Gamma)$. First, note that B is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$. So, for any $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ we have that $B \cap \sigma \in M_\sigma$ where $M_\sigma = \text{HOD}^{L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))}_{\{B,\sigma\}\cup\sigma}$. By the definition of **Env** we have that $B \cap \sigma \in \text{OD}^{L(\mathbb{R},\mu)}_{\{\sigma,A\}\cup\sigma}$, and so by mouse capturing in $L(\mathbb{R},\mu)$ we have that $B \cap \sigma \in Lp(\sigma)$. Define $\mathcal{M}_\sigma \lhd Lp(\sigma)$ to be the least initial segment of $Lp(\sigma)$ having $B \cap \sigma$ as an element. Note that $\mathcal{M}_\sigma \in M_\sigma$ and $M_\sigma \models$ "$\mathcal{M}_\sigma$ is $\omega_1 + 1$ iterable" because the unique iteration strategy for $\mathcal{M}_\sigma$ is definable.

Also the club filter $\mathcal{C}$ is an ultrafilter on $\mathcal{P}_\zeta(\mathbb{R})$. So, we can define $M = \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma/\mathcal{C}$, where the functions of this ultraproduct are $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma$ and $f \in L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$. Note that by [14] $\mathcal{C}$ is normal and countably complete. Then we have that Σ_1-Łos holds, since $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ satisfies Σ_1-replacement. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_\mathcal{C}$; we claim that M believes "$\mathcal{M}$ is countably iterable". To see this let $\bar{\mathcal{M}}$ be a countable transitive hull of $\mathcal{M}$, then we have that $\bar{\mathcal{M}} \in \sigma$ for club-many σ. Also $[\sigma \mapsto \bar{\mathcal{M}}]_\mathcal{C} = \bar{\mathcal{M}}$ (by countable completeness of $\mathcal{C}$). Now by Σ_1 -Łos we have that for club-many σ, $\bar{\mathcal{M}}$ is a countable hull of $\mathcal{M}_\sigma$ and so $M_\sigma \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable". Let Σ_σ be the unique iteration strategy of $Lp(\sigma)$, then the function $\sigma \mapsto \Sigma_\sigma$ is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ and is such that $M_\sigma \models (HC, \Sigma_\sigma)$" $\models \Sigma_\sigma$is an ω_1 strategy for $\bar{\mathcal{M}}$". By Łos, again, we get that $M \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable".

Also, $B = [\sigma \mapsto B \cap \sigma]_\mathcal{C}$ hence $B \in \mathcal{M}$. Note that in $L(\mathbb{R},A)$, $\mathcal{M}$ is actually countably iterable, so we have $\mathcal{M} \lhd Lp(\mathbb{R})$ and so $B \in Lp(\mathbb{R})$. □

Arguing as in the proof of the Claim in Lemma 20 we then get that $\mathbf{Env}(\Gamma) = \mathcal{P}_\mu(\mathbb{R})$. Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Recall

[6] Here we identify $\mathcal{N}$ with the least active mouse extending $M_{\infty,\mu}$.

that $\mathcal{N}$ captures every B in $\vec{B}$, say via τ_B. Define then

$$\mathcal{M} = Hull^{\mathcal{N}}(\{\tau_B^{\mathcal{N}} \mid B \in \vec{B}\}).$$

Here we think of $\mathcal{M}$ as the transitive collapse of this Hull. Then we have that $\mathcal{M}$ is $\omega_1 + 1$ iterable and so $\mathcal{M}^{\sharp}_{\omega^2}$ exists and is $\omega_1 + 1$-iterable.

Claim 2: $L(\mathbb{R}, \mathcal{C})$ is a model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact and the only such model.

Proof of Claim 2. Here we use the results of Section 2. The key point is that the iteration strategy for $\mathcal{M}^{\sharp}_{\omega^2}$ might not extend to big generic collapses. For this though we use instead a countable elementary substructure of $L_\alpha(\mathbb{R}, A)$, where α is such that $L_\alpha(\mathbb{R}, A)$ reflects enough set theory. Let $N \prec L_\alpha(\mathbb{R}, A)$ be countable and elementary such that $\mathcal{M}^{\sharp}_{\omega^2} \in N$ (here we use that DC holds in V). Let $\bar{H}$ be the transitive collapse of N. Then as in the proof of Proposition 7 the results of Section 2 give that $\bar{H}$ models "$L(\mathbb{R}, \mathcal{C})$ satisfies $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact", but then N does and so does V.

The same argument combined with the results of Section 3 will show that since $\mathcal{M}^{\sharp}_{\omega^2}$ exists $L(\mathbb{R}, \mathcal{C})$ is the unique model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. This concludes the proof. □

□

Let us mention that the key fact about AD^+ we used in the proof of Theorem 44 is that given μ such that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, then one can reflect the existence of such a μ to the Suslin co-Suslin part of a model of the form $L(\mathbb{R}, A)$, where A is a set of reals. This is particularly useful as then one can take ultraproducts using the club filter. In the absence of AD^+ this can be a little bit more tricky, but we show how to overcome this difficulty and get the proof of the result under $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$.

Proof of Theorem 4. First let us assume AD^+ holds, and then we will use this proof to get a proof under $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$. Suppose that there are μ and ν such that $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are models of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We may assume with no loss that $V = L(\mathbb{R}, \mu, \nu)$ and $\Theta = \theta_0$, as otherwise there is a non-tame mouse [7].

Note that the proof of Lemma 20 still holds so $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$.

Claim: $\mathcal{P}(\mathbb{R})$ is strictly larger that $\mathcal{P}_\mu(\mathbb{R})$.

Proof of the Claim. Otherwise we have that $\mathcal{P}(\mathbb{R}) = \mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$. We can fix then an $\mathrm{OD}^{L(\mathcal{P}(\mathbb{R}))}$ tree T that projects to a universal Σ^2_1. Following [15] we let $\mathbb{D} = \{\langle d_i \mid i \in \omega\rangle \mid \forall i \in \omega\ d_i$ is a Σ^2_1 degree and $d_i < d_{i+1}\}$. We

[7] Here $L(\mathbb{R}, \mu, \nu)$ is constructed by induction as follows. $L_0(\mathbb{R}, \mu, \nu) = \mathbb{R}$, for $\alpha \in \mathrm{ON}$ we let $L_{\alpha+1}(\mathbb{R}, \mu, \nu)$ be the collection definable sets over $(L_\alpha(\mathbb{R}, \mu, \nu), \in, \nu \cap L_\alpha(\mathbb{R}, \mu, \nu), \mu \cap L_\alpha(\mathbb{R}, \mu, \nu))$ and taking unions at limit stages.

recall in the following lines the definition of the auxiliary measures $\bar{\mu}$ and $\bar{\nu}$ on $\mathbb{D}$.

For $A \subseteq \mathbb{D}$, let $S \subset \mathrm{ON}$ be an ∞-Borel code for A, then

$$A \in \bar{\mu} \text{ iff } \forall^*_{\mu}\sigma L[T,S](\sigma) \models \text{“}\mathrm{AD}^+ + \sigma = \mathbb{R} \text{ and } \exists(\emptyset, U) \in \bar{\mathbb{P}}(\emptyset, U) \Vdash \dot{G} \in \mathcal{A}_S\text{”}$$

Where $\bar{\mathbb{P}}$ is the usual Prikry forcing using Σ^2_1-degrees in $L[S,T](\sigma)$ and the Martin measure (see section 6.3 of [2]), also $\dot{G}$ is the name of the corresponding Prikry sequence and $\mathcal{A}_S$ is the interpretation of the set of reals coded by S.

By results of [15] we have:

- For any $S \subset \mathrm{ON}$ we have that $\forall^*_{\mu}\sigma L[T,S](\sigma) \models$ “$\mathrm{AD}^+ + \sigma = \mathbb{R}$”.
- Whether $A \in \bar{\mu}$ does not depend on the code S.
- Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and for $d \in \mathbb{D}$ let

$$\sigma_d = \{y \mid \text{there are } i \text{ and } x \text{ such that } y \leq d(i)\}.$$

  Then we have that if $\bar{A} = \{d \in \mathbb{D} \mid \sigma_d \in A\}$

$$A \in \mu \text{ if and only if } \bar{A} \in \bar{\mu}.$$

Let us recall the construction of the Prikry Forcing done in Section 2 of [15]; we however, will alternate using μ and ν when choosing measure one sets. More precisely, given $X \subseteq \mathbb{D}^{n+1}$ we say $X \in \mathcal{U}_n$ if

$$\forall^*_{\bar{\mu}}\vec{z}(0)\forall^*_{\bar{\nu}}\vec{z}(1)\cdots\forall^*_{\bar{\mu}}\vec{z}(n-1)\forall^*_{\bar{\nu}}\vec{z}(n)\,(\langle\vec{z}(i) \mid i < n+1\rangle \in X)$$

We also define $\mathbb{P}$ as follows. Conditions will be pairs $(p, \vec{U})$, with $\vec{U}(n) \in \mathcal{U}_n$ for all $n \in \omega$ and such that $p = \langle \vec{d_i} \mid i < n\rangle$ is a sequence of elements in $\mathbb{D}$, such that $\vec{d_i}$ is in $L[x,T]$ for any (all) $x \in \vec{d}_{i+1}(0)$ and it is countable there. We say $(q, \vec{W}) \leq_{\mathbb{P}} (p, \vec{U})$ if $q = p \frown r$ and $r \frown s \in \vec{U}(n+k)$ for all k, and $s \in \vec{W}(k)$. As in Section 6 of [2] we will have that $\mathbb{P}$ has the Prikry property, which is to say that given a forcing statement Φ and a condition $(p, \vec{U}) \in \mathbb{P}$, there is $\vec{W}$ such that $(p, \vec{W})$ decides Φ. We summarize the facts of this forcing that we will use (see [15]).

- For a given set a that admits a well order rudimentary in a, there is a cone of reals x such that $\mathrm{HOD}^{L[T,x]}_{T,a} \models \omega_2^{L[T,x]}$ is Woodin. For a real x we let $\delta(x) = \omega_2^{L[T,x]}$. And for a Σ^2_1-degree d, we let $\delta(d) = \delta(x)$ for any (all) $x \in d$.
- Given $\langle \vec{d_i} \mid i < n\rangle \in \mathbb{D}^n$, we let

$$Q_0(\vec{d}) = \mathrm{HOD}^{L[\vec{d}_0,T]}_{\vec{d}_0,T} \mid \sup\{\delta(d_0(n)) \mid n \in \omega\},$$

  and

$$Q_{i+1}(\vec{d}) = \mathrm{HOD}^{L[T,\vec{d}_{i+1}]}_{Q_i,\vec{d}_{i+1},T} \mid (\sup\{\delta(d_{i+1}(n)) \mid n \in \omega\}).$$

- Given G generic for $\mathbb{P}$ define $g = \bigcup\{p \mid (p, \vec{U}) \in G$ for some $\vec{U}\}$. Let $Q_i(g)$ be $Q_i(g \restriction i)$. Then $L[\cup_{i\in\omega} Q_i(g), T]$ has ω^2 many Woodin cardinals.
- If $\sigma_i = \{x \mid \exists i, n(x \in \vec{d_i}(n))\}$ then the tail filter $\mathcal{F}$ generated by σ_i is such that $L(\mathbb{R},\mathcal{F}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Let us fix G a V-generic filter for $\mathbb{P}$ and let $\mathcal{F}$ be its associated tail filter. We claim that $L(\mathbb{R},\mu) = L(\mathbb{R},\mathcal{F}) = L(\mathbb{R},\nu)$. For this, suppose that $A \in \mathcal{F} \cap V$, we will show $A \in \mu$. Otherwise we have $A \notin \mu$, let $(p, \vec{U}) \Vdash A \in \mathcal{F}$. Let $\vec{W}$ be defined as $\vec{W}(2n) = \vec{U}(2n) \cap \mathbb{D} \setminus \bar{A}$, and $\vec{W}(2n+1) = \vec{U}(2n+1)$ for $n \in \omega$ (here $\bar{A}$ is the translation to of A to $\mathbb{D}$ as defined before). But then it is clear that $(p, \vec{W}) \Vdash A \notin \mathcal{F}$, a contradiction. Hence $L(\mathbb{R},\mu) = L(\mathbb{R},\nu)$ and so $V = L(\mathbb{R},\mu)$ which is impossible. □

Hence $\mathcal{P}(\mathbb{R})$ is strictly larger than $\mathcal{P}_\mu(\mathbb{R})$, and we can choose $A \subseteq \mathbb{R}$ such that $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are definable (from parameters) in $L(\mathbb{R},A)$ and hence the result follows from Theorem 44.

Now, assume AD^+ does not hold, then we have that $\mathcal{P}_\mu(\mathbb{R})$ is strictly smaller than $\mathcal{P}(\mathbb{R})$ (because AD^+ holds in $L(\mathbb{R},\mu)$). Let Γ be $\{A \subset \mathbb{R} \mid L(\mathbb{R},A) \models \mathrm{AD}^+\}$ by Theorem 9.14. of [18] we have that $L(\mathbb{R},\Gamma) \models \mathrm{AD}^+$. We have two cases. If Γ strictly contains $\mathcal{P}_\mu(\mathbb{R})$, then we have that $L(\mathbb{R},\mu)$ is definable from parameters in $L(\mathbb{R},\Gamma)$ and hence one can work in $L(\mathbb{R},\Gamma)$ and the theorem follows from Theorem 44.

If $\Gamma = \mathcal{P}_\mu(\mathbb{R})$, then $\Gamma \neq \mathcal{P}(\mathbb{R})$ and, by Theorem 9.14 of [18] again, we get $L(\mathbb{R},\Gamma) \models \mathrm{AD}_\mathbb{R}$, and so $L(\mathbb{R},\mu) \models \mathrm{AD}_\mathbb{R}$, which is a contradiction.

□

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