The Largest Suslin Axiom [1] [2]

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To our mothers

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Chapter 1

Introduction

The main goal of this manuscript is to advance descriptive inner model theoretic methods to the level of the *Largest Suslin Axiom* (LSA), which is a descriptive set theoretic axiom asserting that there is a largest Suslin cardinal and that the largest Suslin cardinal is a member of the Solovay sequence. The underlying theory is Woodin's AD^+. For all illustrative purposes, we can ignore the "+" and assume AD. The effect of the "+" is that if we also additionally assume that $V = L(\wp(\mathbb{R}))$ then the fragment of V coded by the Suslin, co-Suslin sets of reals is Σ_1 elementary in V. The Solovay sequence is a closed-in-Θ sequence $(\theta_\alpha : \alpha \leq \Omega)$ such that

1. $\theta_0 = \sup\{\beta : \exists f : \wp(\omega) \to \beta(f \text{ is an } OD \text{ surjection})\}$,
2. if $\theta_\alpha < \Theta$ then $\theta_{\alpha+1} = \sup\{\beta : \exists f : \wp(\theta_\alpha) \to \beta(f \text{ is an } OD \text{ surjection})\}$,
3. for limit $\lambda \leq \Omega$, $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

We can now state LSA more precisely. LSA is the conjunction of the following axioms:

1. AD^+.
2. For some ordinal α, $\Theta = \theta_{\alpha+1}$ and θ_α is the largest Suslin cardinal $< \Theta$.

By a result of Woodin, LSA implies that $\mathsf{AD}_{\mathbb{R}}$ fails. The aforementioned result of Woodin says that under AD^+, $\mathsf{AD}_{\mathbb{R}}$ implies that $\Theta = \theta_\alpha$ for some limit ordinal α.

Suppose there is a transitive model of LSA containing the reals and ordinals, call it M. Because the Wadge order is well-founded, we can find a $\Gamma \subseteq \wp(\mathbb{R})^M$ such that

1. $\wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma$,
2. $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$ and

3. for any $\Gamma^* \subseteq \Gamma$, if Γ^* has the above two properties then $\Gamma^* = \Gamma$.

We then say that $L(\Gamma, \mathbb{R})$ is *the minimal model* of LSA. The terminology makes sense: for instance, if $\Gamma^\#$ exists then it is easy to see that $L(\Gamma, \mathbb{R})$ is the hull of $\mathbb{R}$ and a class club of indiscernibles. What is controversial is our use of "the". There could be two models of AD^+ whose sets of reals are not Wadge compatible, making the coexistence of two incompatible "minimal models" of LSA possible. However, we will show (see the proof of Theorem 10.3.1) that $L(\Gamma, \mathbb{R})$ is contained in both of these models, provided they exist.

In this manuscript, we establish three kinds of results that can be stated without mentioning the technical technology developed to prove them. The first set of results deals with the minimal model of LSA. Assume V is the minimal model of LSA. Then the following holds.

(A) (Theorem 7.2.2) HOD $\vDash \mathsf{GCH}$.
(B) (Theorem 10.2.1) The Mouse Set Conjecture holds.

The second set of results contains a single result which shows the consistency of LSA relative to large cardinals. We will show the following.

(C) (Corollary 10.3.1) Suppose the theory $\mathsf{ZFC}+$ "there is a Woodin cardinal that is a limit of Woodin cardinals" is consistent. Then so is LSA.

The third type of results establishes the existence of the minimal model of LSA assuming combinatorial principles or forcing axioms. The following set of results belong to this group.

(D) (Corollary 12.0.23) Assume PFA. Then there is a transitive M such that $\mathbb{R}, Ord \subseteq M$ and $M \vDash \mathsf{LSA}$.

The precursors of these results already exist in print. The first author demonstrated versions of (A), (B), and (C) for the theory $\mathsf{AD}_{\mathbb{R}}+$ "Θ is a regular cardinal". The second author proved the version of (D) for the same theory. The interested reader should consult [10], [11] and [31]. The reason to prove such results is to demonstrate that the underlying technical theory is robust and can be used in a wide range of situations.

A few words on the goal of inner model theory and its descriptive set theoretic counterpart, descriptive inner model theory, are perhaps in order. However, what follows is not a historical exposition. A more accessible introduction can be found in [13].

As is well known, the main goal of the two aforementioned subjects is the construction of canonical inner models for large cardinals. The meaning of "canonical" must be clarified. While there may be other approaches, the current interpretation of "canonical model" is a model that is a *mouse*, i.e., a model constructed from a sequence of extenders $\vec{E}$. Thus, mice have the form $L_\alpha[\vec{E}]$. To avoid coding extra information into mice, $\vec{E}$ must satisfy several conditions. Iterability of mice guarantees that mice are canonical. For instance, given two mice $\mathcal{M}$ and $\mathcal{N}$, either $\mathbb{R}^{\mathcal{M}} \subseteq \mathbb{R}^{\mathcal{N}}$ or $\mathbb{R}^{\mathcal{N}} \subseteq \mathbb{R}^{\mathcal{M}}$ and the constructibility order of $\mathbb{R}^{\mathcal{M}}$ and $\mathbb{R}^{\mathcal{N}}$ are compatible.

Among the reasons that one might like to build canonical models for set theory, one that stands out is the following. Inner model theory and its more modern sister, descriptive inner model theory, have been used to establish lower bounds for various set theoretic statements. It has been the most successful tool for attacking *the* PFA *Conjecture*.

Conjecture 1.0.1 (The PFA Conjecture) *The following theories are equiconsistent.*

1. PFA.

2. ZFC *+"There is a supercompact cardinal".*

It is a well known theorem of Baumgartner that the consistency of clause 2 implies the consistency of clause 1. As for the converse, inner model theoretic methods have been used since late 60s to establish partial results. The current best known result is (D) stated above. While there can be other methods free of inner model theory that settle the PFA Conjecture (see for instance [32]), it is hard to conceive another method that will solve the descriptive counterpart of the PFA Conjecture. Below uB stands for the set of universally Baire sets. Recall that these are exactly those sets of reals whose continuous preimages in compact Hausdorff spaces have the Baire property.

Conjecture 1.0.2 *Assume* PFA. *There is $\Gamma \subseteq uB$ such that $L(\Gamma, \mathbb{R}) \vDash$ LSA and if $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ and $\Theta = \Theta^{L(\Gamma,\mathbb{R})}$, then $V_\Theta^{\mathcal{H}} \vDash$ "there is a superstrong cardinal".*

The advantage of Conjecture 1.0.2 is that instead of postulating the existence of a model with a large cardinal it specifies the model that should satisfy the large cardinal axioms. To prove Conjecture 1.0.2, we need to analyze the model $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$, and the technical aspect of this manuscript does exactly that but for the minimal model of LSA. While the analysis of the model $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ without extra minimality assumptions on $L(\Gamma, \mathbb{R})$ alone will not solve Conjecture 1.0.2, it is an essential step towards its resolution. The additional step is to develop the theory behind the core model induction without any minimality assumptions. The core model induction is the technique that allows us to prove results like (D) and its variations. Completing these two steps without minimality assumptions is the main objective of descriptive inner model theory.

What the analysis of $\mathcal{H}$ yields is that it is a *hod mouse*. These are models constructed from extender sequences and also from iteration strategies, so they are of the form $L_\alpha[\vec{E}, \vec{\Sigma}]$. The iteration strategies coded in $\vec{\Sigma}$ are iteration strategies for (initial segments of) the model itself. Hod mice just like mice satisfy GCH. Thus, statement (A) is just a direct corollary of the analysis of $\mathcal{H}$.

The first author developed theory of hod mice assuming that the minimal model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" doesn't exist (see [10]). The next nice closure point is LSA, and developing the theory of hod mice assuming that the minimal model of LSA doesn't exist is the technical part of this manuscript. The main new problem that we need to deal with here is the notion of "short tree strategy mice".

Let us explain what this is. The analysis of the model $\mathcal{H}$ goes by inductive characterization of sets of reals of various Wadge ranks. Suppose $(\theta_\alpha : \alpha \leq \Omega)$ is the Solovay sequence. What one shows is that for each $\alpha < \Omega$ a set of reals of Wadge rank θ_α codes an iteration strategy Σ for some countable hod mouse $\mathcal{P}$. It then follows (non-trivially) that the direct limit of all Σ-iterates of $\mathcal{P}$ is $\mathcal{H}|\theta_\alpha$.

What we said is true except in one case. When θ_α is the largest Suslin cardinal below $\theta_{\alpha+1}$, any set of Wadge rank θ_α cannot code an iteration strategy for a hod mouse. Fix an α such that θ_α is the largest Suslin cardinal below $\theta_{\alpha+1}$. Let Σ be the strategy for some hod mouse $\mathcal{P}$ such that the Wadge rank of Σ is $\theta_{\alpha+1}$. What can be shown is that $\mathcal{P}$ satisfies the following two conditions.

(i) $\mathcal{P}$ has a largest Woodin cardinal denoted by $\delta^{\mathcal{P}}$.
(ii) Working in $\mathcal{P}$, let $\kappa^{\mathcal{P}}$ be the least $< \delta^{\mathcal{P}}$-strong cardinal. Then in $\mathcal{P}$, $\kappa^{\mathcal{P}}$ is not $\delta^{\mathcal{P}}$-strong and $\kappa^{\mathcal{P}}$ is a limit of Woodin cardinals.

Define the short tree component, Σ^{stc}, of Σ as follows.

1. $dom(\Sigma^{stc}) = dom(\Sigma)$
2. Suppose $\vec{\mathcal{T}} \in dom(\Sigma)$ and $b = \Sigma(\vec{\mathcal{T}})$. Then $\Sigma^{stc}(\vec{\mathcal{T}}) = b$ provided $\pi^{\vec{\mathcal{T}}}(\delta^{\mathcal{P}}) > \delta(\vec{\mathcal{T}})$. Otherwise, $\Sigma^{stc}(\vec{\mathcal{T}}) = \mathcal{M}_b^{\vec{\mathcal{T}}}$.

The short tree component of a strategy is not an unfamiliar object at least to those familiar with the core model induction. For instance, let $\mathcal{P}$ be $\mathcal{M}_\omega|(\delta^{+\omega})^{\mathcal{M}_\omega}$

where $\mathcal{M}_\omega$ is the minimal proper class mouse with infinitely many Woodin cardinals and δ is its least Woodin cardinal. Let Λ be the canonical strategy of $\mathcal{P}$, the one induced by the strategy of $\mathcal{M}_\omega$. Then $\Lambda^{stc} \in L(\mathbb{R})$, and (i) if $\mathcal{T}$ is a short tree on $\mathcal{P}$ then $\Lambda^{stc}(\mathcal{T})$ is the unique branch b of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ exists and (ii) if $\mathcal{T}$ is a maximal tree on $\mathcal{P}$ then $\Lambda^{stc}(\mathcal{T}) = (Lp_\omega(\mathcal{M}(\mathcal{T})))^{L(\mathbb{R})}$.

Getting back to our discussion, the set of reals of Wadge rank θ_α, at least in the minimal model of LSA, is Σ^{stc}. This fact forces us to consider mice relative to Σ^{stc}, and this is a rather complicated matter. The basic issue is that we cannot close mice under Σ^{stc} using the usual procedure for feeding in a strategy. For instance, suppose we are performing a construction producing mice relative to Σ^{stc}. Suppose our method of feeding Σ^{stc} is the most naive one. At stage β we consider the least tree $\mathcal{T}$ such that $\Sigma^{stc}(\mathcal{T})$ has not been told to the model. Suppose $\mathcal{T}$ is maximal, so we must not tell the model any branch of $\mathcal{T}$. It could be the case that later on in the construction while taking fine structural cores, $\mathcal{T}$ collapses to a short tree. Thus we have $\pi : \mathcal{N} \to \mathcal{M}$, $\mathcal{T} \in \mathcal{M} \cap rng(\pi)$, $\mathcal{N}$ is a core of $\mathcal{M}$ and π is the core embedding. By elementarity, $\pi^{-1}(\mathcal{T})$ doesn't have a branch indexed in $\mathcal{N}$. However, $\pi^{-1}(\mathcal{T})$ is short and hence it must have a branch indexed in $\mathcal{N}$. A large portion of this manuscript deals with this issue. We present a solution to this problem in Section 3.8.

Why do we need minimality assumptions? The reason is that the theory of hod mice has been developed using examples. In many models of AD^+, we have been able to identify patterns that led to a successful theory. Without a minimality assumption, it is hard to understand every pattern that could exist. Of course, one hopes that after understanding enough patterns and special cases, we can lay down a complete theory without minimality assumptions. There has been a recent success in this direction. In an unpublished work, John Steel has proven a general comparison theorem for hod mice without any minimality assumptions. However, Steel's comparison argument, to the authors' best knowledge, does not shed light on how to construct hod mice whose strategies have the desired Wadge rank, at least for now.

Nevertheless, there is a method for constructing hod mice whose strategies have a desired Wadge rank, a method that doesn't assume any minimality assumptions. The following conjecture is at the heart of it.

Conjecture 1.0.3 *Assume there is no mouse with a superstrong cardinal. Suppose δ is a Woodin cardinal and $\lambda > \delta$ is an inaccessible cardinal. Suppose further that V_λ has an iteration strategy Σ that acts on trees that are based on V_δ, and that Σ has a unique extension in $V^{Coll(\omega,\delta)}$. Suppose also that $A \subseteq \mathbb{R}$ is a universally Baire set. Let $\mathcal{N}$ be the mouse constructed by the fully backgrounded construction of V_δ and let*

$g \subseteq Coll(\omega, \delta)$ be generic. Then in $V[g]$, the strategy of $\mathcal{N}$ induced by the extension of Σ has Wadge rank at least the Wadge rank of A_g where A_g is the extension of A to $V[g]$.

In fact, in a sense even at stages where we do have a successful theory of hod mice, proving Conjecture 1.0.3 plays a fundamental role. Unfortunately, all known proofs of Conjecture 1.0.3 use minimality hypotheses.

Chapters 2-8 develop the basic theory of hod mice for AD^+ models up to the minimal model of LSA; a consequence of this analysis is (A). The last four chapters focus on applications. Chapter 11 proves that $\square_{\kappa,2}$ holds in HOD of AD^+ models up to the minimal model of LSA for all HOD-cardinals κ. Our main use of this chapter is Chapter 12, where a proof of (D) is given. Chapter 9 develops the basic theory of *condensing sets*, which is needed in constructions of hod mice in various situations. Chapter 10 uses the material in developed in the previous chapters to prove (B) and (C). The last chapter (Chapter 12) proves (D) by constructing a hybrid version of K^c. This chapter uses methods developed in the previous chapters and [31].

The history of the manuscript is as follows. The first author started the technical portion of this work sometime in 2007-2008. Later, sometime in 2008-2009, John Steel joined the project. Some of the material presented in this manuscript goes back to this time. However, to the first author's best knowledge, there were several gaps in the proofs from this period. In particular, the notion of short tree strategy mouse was not defined correctly. The definition of short tree strategy mouse given in this manuscript is due to the first author (see Definition 3.8.5). This notion was introduced during Spring of 2012. Many of the ideas that appeared in Chapter 4-7 go back to 2007-2012 period. Several important ideas came after Spring 2012. It is truly difficult to say what idea came when, and it is best to leave such matters alone. The material in Chapter 8 (due to first author) was proved in the Fall 2015. The material in Chapter 11 (due to the second author) was proved in 2014-2015; as mentioned above, it is used in arguments in Chapter 12 and has potential applications elsewhere. The material in Chapter 9 has precursor in the first author's work [11] that proves a version of (D) under an additional large cardinal hypothesis for $\mathsf{AD}_{\mathbb{R}}$ + "Θ is a regular cardinal"; it then was adapted by the second author in [31] to prove the version of (D) for $\mathsf{AD}_{\mathbb{R}}$ + "Θ is a regular cardinal". Though the terminology has been changed somewhat, the material in Chapter 9 is a straightforward adaptation of the aforementioned papers. Chapter 10 is due to the first author and was done mostly in the Fall of 2015. Chapter 12 is joint work of the two authors and was done in the Fall of 2015 when the authors visited the Isaac Newton Institute for Mathematical Sciences.

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Chapter 2

Hybrid $\mathcal{J}$-structures

The main goal of this chapter is to prepare some terminology to be used for the rest of this manuscript. One important notion introduced in this chapter is that of the *un-dropping game* (see Definition 2.7.3). We will use it in the next chapter to prove a comparison theorem for hod mice (see Corollary 4.6.10).

2.1 Layered hybrid $\mathcal{J}$-structures

In what follows, given a transitive set (or structure) M we will use $o(M)$ to denote the ordinal height of M. Also, given a set X, we let $trc(X)$ be the transitive closure of X. We also let $trc^X = (trc(X \cup \{X\}), X, \in)$.

Definition 2.1.1 (Definition 1.1 of [10]) *Given a function f, we say f is amenable if the domain of f consists of transitive structures and for some formula ϕ and for all $a = (M, A, \in) \in \mathrm{dom}(f)$*

1. *$f(a) \subseteq o(a)$ and $0 \in f(a)$,*
2. *letting $\beta = \sup f(a)$, $\beta < o(a)$ is the unique ordinal γ such that $a \vDash \phi[\gamma]$,*[1]
3. *whenever $\eta < \sup f(a)$, $f(a) \cap \eta \in M$.*

We let ϕ_f be the formula ϕ above.

[1] We seem to need this condition in order to develop fine structure of models of the form $\mathcal{J}^{\vec{E},f}$ where f is a shifted amenable function. These are introduced below.

We say f is a *shift of an amenable function* or a *shifted amenable function* if for all $a = (M, A, \in) \in \text{dom}(f)$, $f(a) \subseteq Ord$, $f(a) \subseteq [min(f(a)), min(f(a)) + o(a))$, and there is an amenable function g such that (i) $\text{dom}(f) = \text{dom}(g)$ and (ii) for all $a \in \text{dom}(f)$ and $\gamma < o(a)$, $f(a) = \{min(f(a)) + \gamma : \gamma \in g(a)\}$. Notice that if f is a shift of an amenable function then it uniquely determines g. We say that g is the *amenable component* of f.

Jumping ahead, we remark that iteration strategies and mouse operators provide an ample source of amenable functions. For instance, let $\mathcal{M} = \mathcal{M}_1^{\#}$ and let Σ be its canonical iteration strategy. We define f as follows. Let first $\text{dom}(f)$ be the set of structures of the form $\mathcal{J}_\omega(\mathcal{T})$ where $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ of limit length and is according to Σ. Next, define $f(\mathcal{J}_\omega(\mathcal{T})) = b$ where $b = \Sigma(\mathcal{T})$. Then f is amenable. We will refer to such an f as *an amenable function given by an iteration strategy*. The reason we define the domain of f to be the set of $\mathcal{J}_\omega(\mathcal{T})$ instead of just the set of $trc^{\mathcal{T}}$ is that the later may not satisfy clause 2 of Definition 2.1.1.

Recall that a transitive structure $\mathcal{M} = (M, A)$ is called *amenable* if for every $X \in M$, $A \cap X \in M$. Following [35], we say $\mathcal{M}$ is a $\mathcal{J}$-structure over X if $\mathcal{M} = (\mathcal{J}_\alpha^A(X), B) = (\left|\mathcal{J}_\alpha^A(X)\right|, A, B)$ is an amenable structure. Keeping the notation, we also say $\mathcal{M}$ is an *acceptable $\mathcal{J}$-structure* if for all $\beta < \alpha$ and for all $\tau < \omega\beta$, if $\wp(\tau) \cap \mathcal{J}_{\beta+1}^A \not\subseteq \mathcal{J}_\beta^A$ then there is a surjection $f : \tau \to \omega\beta$ in $\mathcal{J}_{\beta+1}^A$. Finally, we say X is *self-wellordered* if there is a wellordering of X in $\mathcal{J}_1(X)$. We are now in a position to introduce the *hybrid $\mathcal{J}$-structures*.

Definition 2.1.2 (Hybrid $\mathcal{J}$-structures) *We say $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a hybrid $\mathcal{J}$-structure over a self-well-ordered set X with indexing scheme $\phi(x)$ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure such that in $\mathcal{M}$, f is a shift of an amenable function with amenable component g such that*

1. *for all $a \in \mathcal{M}$, $a \in \text{dom}(f)$ if and only if in $\mathcal{M}$, there is β such that a is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}_\beta^{A,f}(X)$ such that*

$$\mathcal{J}_\beta^{A,f} \vDash \text{“}ZFC + \phi[b]\text{”}$$

   and if γ is such that $b \vDash \phi_g[\gamma]$ then $\beta + \gamma \leq \alpha$ and $\mathcal{M} \vDash$ “$\text{cf}(\gamma)$ is not a measurable cardinal”, and

2. *for all $a \in \mathcal{M}$, if $a \in \text{dom}(f)$ then $min(f(a))$ is the least ordinal β satisfying clause 1 above.*

Suppose $\mathcal{M}$ is a hybrid $\mathcal{J}$-structure with an indexing scheme ϕ. We will often say that “$\mathcal{M}$ is indexed according to ϕ” or that “$\mathcal{M}$ is ϕ-indexed”. The following is an easy but important lemma. We leave its straightforward proof to the reader.

Lemma 2.1.3 *If a is as in clause 1 of Definition 2.1.2 then $f(a)$ is indexed at $\beta+\gamma$ where $\beta = min(f(a))$ and γ is such that $a \vDash \phi_g[\gamma]$.*

Remark 2.1.4 *Notice that it follows from clause 1 of Definition 2.1.2 that the function $a \to min(f(a))$ is injective on $\mathrm{dom}(f)$.*

Hod mice are a special blend of *layered hybrid $\mathcal{J}$-structures* introduced below. Before introducing them we establish some notation. Suppose that $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a hybrid $\mathcal{J}$-structure over X and $\xi \leq \alpha$. Then we let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ, i.e., we keep the predicate indexed at ξ. We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. Also, recall that if $\beta < \alpha$ then we write $\mathcal{J}_\beta^{\mathcal{M}}$ instead of $\mathcal{J}_\beta^{A,f}$ and, we say $\mathcal{N}$ is an (a proper) initial segment of $\mathcal{M}$ and write $\mathcal{N} \trianglelefteq \mathcal{M}$ ($\mathcal{N} \triangleleft \mathcal{M}$) if there is $\beta \leq \alpha$ ($\beta < \alpha$) such that $\mathcal{N} = \mathcal{J}_\beta^{\mathcal{M}}$.

Definition 2.1.5 (Layered hybrid $\mathcal{J}$-structure) *We say $\mathcal{M} = (\mathcal{J}_\alpha^{A,f}(X), B)$ is a layered hybrid $\mathcal{J}$-structure over self-well-ordered set X with indexing scheme $\phi(x,y)$ if $\mathcal{M}$ is an acceptable $\mathcal{J}$-structure over X such that in $\mathcal{M}$, f is a function with domain $Y^{\mathcal{M}} \subseteq \{\mathcal{Q} : \mathcal{Q} \triangleleft \mathcal{M}\}$ such that for all $\mathcal{Q} \in Y^{\mathcal{M}}$, $f(\mathcal{Q})$ is a shift of an amenable function with amenable component $g_{\mathcal{Q}}$ such that*

1. *for all $a \in \mathcal{M}$, $a \in \mathrm{dom}(f(\mathcal{Q}))$ if and only if in $\mathcal{M}$, there is β such that a is the unique transitive structure $b = (M, A, \in) \in \mathcal{J}_\beta^{A,f}(X)$ such that*

$$\mathcal{J}_\beta^{A,f} \vDash \text{“}ZFC + \phi[\mathcal{Q}, b]\text{”}$$

   and if ξ is such that $b \vDash \phi_{g_{\mathcal{Q}}}[\xi]$ then $\beta + \xi \leq \alpha$, and

2. *for all $a \in \mathcal{M}$, if $a \in \mathrm{dom}(f(\mathcal{Q}))$ then $min(f(\mathcal{Q})(a))$ is the least ordinal β satisfying clause 1 above.*

Suppose $\mathcal{M}$ is a layered hybrid $\mathcal{J}$-structure with an indexing scheme ϕ. We will often say that “$\mathcal{M}$ is indexed according to ϕ” or that “$\mathcal{M}$ is ϕ-indexed”.

We will often omit ϕ when discussing a particular layered hybrid $\mathcal{J}$-structure. If $\mathcal{M}$ is a layered hybrid $\mathcal{J}$-structure then we let $f^{\mathcal{M}}$ and $Y^{\mathcal{M}}$ be as in Definition 2.1.5. We again have that for each $\mathcal{Q} \in Y^{\mathcal{M}}$, the function $a \to min(f^{\mathcal{M}}(\mathcal{Q})(a))$ is injective on $\mathrm{dom}(f(\mathcal{Q}))$.

Notice that hybrid $\mathcal{J}$-structures can be viewed as a special case of layered hybrid $\mathcal{J}$-structures. Because of this, in the sequel we will only establish terminology for layered hybrid $\mathcal{J}$-structures though we might use the same terminology for hybrid $\mathcal{J}$-structures.

Typically, when discussing hybrid $\mathcal{J}$-structures, X will be an *iterable* structure and f will be the predicate coding its strategy.[2] As mentioned above, hod mice are a special type of layered hybrid $\mathcal{J}$-structures: the f predicate of a hod mouse codes a strategy for its layers. When the A predicate of a layered hybrid $\mathcal{J}$-structure is a coherent sequence of extenders then the resulting model is called a *hybrid layered premouse*.

Definition 2.1.6 (Layered hybrid premouse) *Suppose $\mathcal{M} = \mathcal{J}^{\vec{E},f}(X)$ is a ϕ-indexed layered hybrid $\mathcal{J}$-structure over self-well-ordered set X. $\mathcal{M}$ is called a ϕ-indexed layered hybrid premouse (lhp) if $\vec{E}$ is a fine extender sequence as in Definition 2.4 of [28] with one exception described below. We write $\vec{E}^{\mathcal{M}}$ for $\vec{E}$ etc.*

Suppose κ is a limit of cutpoint cardinals of $\mathcal{M}$ such that there is an extender $E \in \vec{E}^{\mathcal{M}}$ with $\operatorname{crit}(E) = \kappa$. Then whenever $E \in \vec{E}$ is an extender with critical point κ, the index of E is the cardinal successor of the least cutpoint of $Ult(\mathcal{M}, E)$ greater than κ.

Here κ is a cutpoint of a layered hybrid premouse $\mathcal{N}$ if there is no extender $F \in \vec{E}^{\mathcal{N}}$ such that $\operatorname{crit}(F) < \kappa \leq lh(F)$. κ is a strong cutpoint of a layered hybrid premouse $\mathcal{N}$ if there is no extender $F \in \vec{E}^{\mathcal{N}}$ such that $\operatorname{crit}(F) \leq \kappa \leq lh(F)$.

The significance of the last clause of Definition 2.1.6 will be apparent later. It was independently noticed by the first author and John Steel. Essentially it comes up as follows. Suppose κ is as in Definition 2.1.6 and suppose we have an embedding $j : \mathcal{M}|(\kappa^+)^{\mathcal{M}} \to \mathcal{N}$. Often times we will use such embeddings to guess or reconstruct extenders on the sequence of $\mathcal{M}$ that have critical point κ (see for instance Lemma 4.9.6 and Lemma 4.9.7). In the old indexing scheme (i.e., Mitchell-Steel indexing scheme) to describe extender E we need to first construct $Ult(\mathcal{M}, E)$, which in many cases has longer height than the index of E. This mismatch of heights creates many unwanted complications. Similar complications arise when we index extenders using Jensen indexing (recall that this means that extenders are indexed at the successor of the image of the critical point). In this case, while the two ordinals match, we need to guess what $Ult(\mathcal{M}, E)$ is up to the image of κ.

We finish this section by introducing hp that are closed under sharps. We will use such a closure to introduce short tree strategy premice (see Definition 2.2.3 and Definition 3.8.4).

[2] In this case, the γ defined in Definition 2.1.2 is the length of a tree $\mathcal{T}$ according to f. The condition "$\mathcal{M} \vDash \operatorname{cof}(\gamma)$ is not measurable" in Definition 2.1.2 ensures the structure we build has sufficient condensation.

Definition 2.1.7 (Closed hp) *Suppose $\mathcal{M}$ is an hp and $\alpha \leq o(\mathcal{M})$. Then we say $\mathcal{M}$ is closed under sharps below α if for all $\beta < \alpha$ there is $\gamma \in \mathrm{dom}(\vec{E}^{\mathcal{M}})$ such that $\mathrm{crit}(E^{\mathcal{M}}_{\gamma}) > \beta$. We say $\mathcal{M}$ is closed under sharps if $\mathcal{M}$ is closed under sharps below $o(\mathcal{M})$.*

2.2 Layered strategy premice

In this paper, we are concerned with lhp whose f predicate codes a strategy. The goal of this section is to introduce the language used to describe such structures.

Suppose that $\mathcal{M}$ is an lhp. We then say that a shifted amenable function f *codes a partial strategy function* for $\mathcal{M}$ if

1. $\mathrm{dom}(f) \subseteq \{\mathcal{J}_\omega(\vec{\mathcal{T}}) : \vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ without a last model$\}$,
2. whenever $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}$ such that $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in \mathrm{dom}(f)$ and whenever $\vec{\mathcal{U}}$ is an initial segment of $\vec{\mathcal{T}}$ without a last model, $\mathcal{J}_\omega(\vec{\mathcal{U}}) \in \mathrm{dom}(f)$,
3. if g is the amenable component of f then for all $\mathcal{J}_\omega(\vec{\mathcal{T}}) \in \mathrm{dom}(f)$, $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is a cofinal branch of $\vec{\mathcal{T}}$, and
4. ϕ_f is the formula defining $sup(g(\mathcal{J}_\omega(\vec{\mathcal{T}})))$ over $\mathcal{J}_\omega(\vec{\mathcal{T}})$.

Notice that we do not require that $g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is a well-founded branch of $\vec{\mathcal{T}}$, which is why we call the resulting function just a strategy function.

When defining short tree strategy mice, we will encounter hybrid structures whose f predicate doesn't necessarily code a strategy but a partial strategy. We make this notion more precise. First we make a useful notation.

Notation 2.2.1 *Suppose M is a transitive model of a fragment of set theory and $\mathcal{T}$ is an iteration tree on M of limit length. Then we let*

$$\mathcal{M}^+(\mathcal{T}) = (\mathcal{M}(\mathcal{T}))^\#.$$

In general, given a transitive self-well-ordered set X, we let $\mathcal{M}^+(X)$ be the minimal active X-mouse.

Remark 2.2.2 *Suppose $\mathcal{M}$ is an lhp. We then say that Σ is a semi-strategy for $\mathcal{M}$ if the domain of Σ consists of quadruples $(\mathcal{M}_0, \mathcal{T}_0, \mathcal{M}_1, \vec{\mathcal{U}})$ such that $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{T}_0$ is a normal tree on $\mathcal{M}_0$, $\mathcal{M}_1$ is either the last model of $\mathcal{T}_0$ or $\mathcal{T}_0$ doesn't have a last model and $\mathcal{M}_1 = \mathcal{M}^+(\mathcal{T}_0)$, and $\vec{\mathcal{U}}$ is a stack on $\mathcal{M}_1$. We can then consider amenable functions that code partial semi-iteration strategies. We will abuse our terminology and will treat semi-iteration strategies as if they were just strategies.*

Suppose then a shifted amenable function f codes a partial strategy function for $\mathcal{M}$. We then let Σ^f be the partial strategy function coded by f. More precisely, letting g be the amenable component of f,

1. $\mathrm{dom}(\Sigma^f) = \mathrm{dom}(f)$ and
2. for all $\vec{\mathcal{T}} \in \mathrm{dom}(\Sigma^f)$, $\Sigma^f(\vec{\mathcal{T}}) = g(\mathcal{J}_\omega(\vec{\mathcal{T}}))$.

We say f *codes a partial strategy* if Σ^f chooses cofinal and well-founded branches. We say f *codes a total strategy* if Σ^f is a total strategy.

Recall that if $\mathcal{M}$ is an lhp, $\mathcal{N} \trianglelefteq \mathcal{M}$ and Σ is an iteration strategy for $\mathcal{M}$ then $\Sigma_{\mathcal{N}}$ is the strategy of $\mathcal{N}$ we get by the copy construction. More precisely, $\Sigma_{\mathcal{N}}$ is the id-pullback of Σ.

Definition 2.2.3 (Strategy premouse, sp) *Suppose $\mathcal{P}$ is a transitive model of some fragment of* ZFC*, X is a self-well-ordered set such that $\mathcal{P} \in X$ and $\mathcal{M}$ is a ϕ-indexed hp. We say $\mathcal{M}$ is a ϕ-indexed strategy premouse (sp) over X based on $\mathcal{P}$ if $f^{\mathcal{M}}$ codes a partial iteration strategy for $\mathcal{P}$ and for any $a \in dom(f^{\mathcal{M}})$ if $\beta = min(f^{\mathcal{M}}(a))$ then $\mathcal{M}|\beta$ is closed under sharps (see Definition 2.1.7).*

Definition 2.2.4 (Layered strategy premouse, lsp) *Suppose $\mathcal{M}$ is a ϕ-indexed lhp. We say $\mathcal{M}$ is a ϕ-indexed layered strategy premouse (lsp) if for all $\mathcal{Q} \in Y^{\mathcal{M}}$, in $\mathcal{M}$,*

1. *$f^{\mathcal{M}}(\mathcal{Q})$ codes a partial strategy function for $\mathcal{Q}$ such that for every $a \in dom(f^{\mathcal{M}}(\mathcal{Q}))$, if $\beta = min(f^{\mathcal{M}}(\mathcal{Q})(a))$ then $\mathcal{M}|\beta$ is closed under sharps, and*
2. *if $\mathcal{Q}_0 \trianglelefteq \mathcal{Q}_1 \in Y^{\mathcal{M}} - \{\mathcal{J}_0(\mathcal{M})\}$ then letting, for $i \in 2$, Σ_i be the partial strategy function coded by $f^{\mathcal{M}}(\mathcal{Q}_i)$, then $(\Sigma_1)_{\mathcal{Q}_0}$ is id-pullback of Σ_0.*

We can also introduce lsp that are over some self-well-ordered set X and are based on some $\mathcal{P} \in X$. We leave this to the reader.

Notice that the fact that $\mathcal{M}$ is a layered strategy premouse depends on what ϕ says. Thus, the clauses above should be viewed as part of ϕ. The strategy premice are a special case of layered strategy premice, and we leave the exact definition to the reader. We let $\Sigma^{\mathcal{M}}$ be the partial strategy function coded by $f^{\mathcal{M}}$. If $\mathcal{Q} \in Y^{\mathcal{M}}$ then we let $\Sigma^{\mathcal{M}}_{\mathcal{Q}}$ be the partial strategy function coded by $f(\mathcal{Q})$.

In most applications, lsps have a very canonical indexing scheme which is originally due to Woodin. At each stage the stack whose branch is being indexed by f is the least stack whose branch hasn't yet been indexed. Here and in future definitions, for any lsp (sp) $\mathcal{M}$ (over a self-well-ordered set), we say "$\mathcal{M}$-least" to mean "$<_{\mathcal{M}}$-least", where $<_{\mathcal{M}}$ is the canonical (constructible) well order on $\mathcal{M}$. We call this the *standard indexing scheme.*

Definition 2.2.5 (Standard indexing scheme) *We say $\phi(x, y)$ is the standard indexing scheme if whenever $\mathcal{M}$ is an lsp and $\mathcal{Q} \in Y^{\mathcal{M}}$ then $\mathcal{M} \vDash \phi[\mathcal{Q}, a]$ if and only if*

1. *a is the $\mathcal{M}$-least set of the form $\mathcal{J}_\omega(\vec{\mathcal{T}})$ where $\vec{\mathcal{T}}$ is a stack on $\mathcal{Q}$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{M}}_{\mathcal{Q}}$, $\vec{\mathcal{T}}$ doesn't have a last model, if $\vec{\mathcal{T}}$ has the last normal component $\mathcal{T}$, then cof(lh($\mathcal{T}$)) is not measurable,*[3] *and $f^{\mathcal{M}}(\mathcal{Q})(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined, and*

2. *for every $\mathcal{R} \lhd \mathcal{M}$ such that $\mathcal{R} \vDash ZFC$, in $\mathcal{R}$, $(\mathcal{Q}, a)$ isn't the lexicographically $\mathcal{R}$-least set of the form $(\mathcal{Q}^*, \mathcal{J}_\omega(\vec{\mathcal{T}}))$ where $\mathcal{Q}^* \in Y^{\mathcal{R}}$ and $\vec{\mathcal{T}}$ is a stack on $\mathcal{Q}^*$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{R}}_{\mathcal{Q}^*}$, $\vec{\mathcal{T}}$ doesn't have a last model and $f^{\mathcal{R}}(\mathcal{Q}^*)(\mathcal{J}_\omega(\vec{\mathcal{T}}))$ is undefined.*

We write ϕ_{std} for ϕ.

Suppose $\mathcal{M}$ is an lsp and Σ is a (κ, θ)-iteration strategy for $\mathcal{Q}$ for some $\mathcal{Q} \in Y^{\mathcal{M}}$. Then it can be the case that $\Sigma^{\mathcal{M}}_{\mathcal{Q}} \subseteq \Sigma$. When this happens we get structures relative to Σ.

Definition 2.2.6 ((Σ, ϕ)-premouse) *Suppose X is a transitive self-well-ordered structure such as hp, lhp, sp or lsp or just a model of some fragment of* ZFC. *Suppose further that Σ is a (κ, θ)-iteration strategy for X and $\mathcal{M}$ is a ϕ-indexed sp over X. Then $\mathcal{M}$ is called a (Σ, ϕ)-premouse if $\Sigma^{\mathcal{M}} \subseteq \Sigma \restriction \mathcal{M}$.*

Definition 2.2.7 ((Σ, ϕ)-mouse) *Keeping the notation of Definition 2.2.6, we say $\mathcal{M}$ is a (Σ, ϕ)-mouse if $\mathcal{M}$ has an $\omega_1 + 1$-iteration strategy Λ such that whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$ then $\mathcal{N}$ is a (Σ, ϕ)-premouse.*

We warn the reader that we will often omit ϕ from our notation and say "$\mathcal{M}$ is a Σ-mouse" instead of "$\mathcal{M}$ is a (Σ, ϕ)-mouse" if ϕ is clear from the context.

[3] If $\mathrm{cof}(\mathrm{lh}(\mathcal{T})) = \kappa$ is measurable in $\mathcal{M}$, then $\mathcal{M}$ can figure out the (necessarily unique) cofinal branch b of $\mathcal{T}$ by taking the ultrapower of an extender with critical point κ on $\mathcal{M}$'s sequence. Furthermore, we do not want to index b for reasons discussed in [20].

2.3 Iterations of (Σ, ϕ)-mice

Suppose X is a transitive self-well-ordered structure such as hp, lhp, sp or lsp or just a model of some fragment of ZFC. Suppose further that Σ is an (ω_1, ω_1)-iteration strategy for X and ϕ is an indexing scheme. Given two (Σ, ϕ)-mice, we can compare them using the usual comparison argument.

Theorem 2.3.1 (Theorem 3.11 of [28]) *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two countable k-sound (Σ, ϕ)-mice with $(\omega_1 + 1)$-iteration strategies Λ and Γ respectively. Then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{M}$ and $\mathcal{N}$ respectively according to Λ and Γ respectively, having last models $\mathcal{M}^{\mathcal{T}}_{\alpha}$ and $\mathcal{N}^{\mathcal{N}}_{\eta}$ such that either*

1. *the iteration embedding $\pi^{\mathcal{T}}_{0,\alpha}$-exists[4], and $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_{\eta}$, or*
2. *the iteration embedding $\pi^{\mathcal{U}}_{0,\eta}$-exists, and $\mathcal{M}^{\mathcal{U}}_{\eta}$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_{\alpha}$.*

Comparison for lsp is more involved and we do not know how to do it in general. Below we recall our primary method of identifying the good branches of iteration trees. Recall that the strategy for a sound mouse projecting to ω is determined by *$\mathcal{Q}$-structures*. For $\mathcal{T}$ normal, let $\Phi(\mathcal{T})$ be the phalanx of $\mathcal{T}$ (see Definition 6.6 of [24]).

Definition 2.3.2 *Let $\mathcal{T}$ be a k-normal tree of limit length on a k-sound lsp, and let b be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $\mathcal{M}^{\mathcal{T}}_b$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ (i.e. $\rho(\mathcal{Q}) < \delta(\mathcal{T})$) or defines a function witnessing $\delta(\mathcal{T})$ is not a Woodin cardinal as witnessed by the extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Next we would like to state a general result stating that branches identified by $\mathcal{Q}$-structures are unique. Suppose that $\mathcal{M}$ is an lsp and Σ is a strategy for $\mathcal{M}$. If $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then we let $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ be the strategy of $\mathcal{N}$ given by $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

Definition 2.3.3 *Suppose $\mathcal{M}$ is a ϕ-indexed lsp (perhaps over some set X) and Σ is an iteration strategy for $\mathcal{M}$. We say $(\mathcal{M}, \Sigma)$ is a layered strategy ϕ-mouse (ϕ-lsm) pair if Σ has hull condensation (see Definition 1.30 of [10]) and whenever $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via $\vec{\mathcal{T}}$ then $\mathcal{N}$ is a ϕ-indexed lsp and $\Sigma^{\mathcal{N}} \subseteq \Sigma_{\mathcal{N},\vec{\mathcal{T}}}$.*

[4] In [28], this is stated in a somewhat stronger form, namely that $[0, \alpha]_T$ doesn't drop in model or degree.

We say an iteration tree $\mathcal{T}$ is above η if all the extenders used in $\mathcal{T}$ have critical points $> \eta$.

Theorem 2.3.4 *Suppose $(\mathcal{M}, \Sigma)$ is a ϕ-lsm pair. Suppose $\gamma < o(\mathcal{M})$ is such that $\sup(Y^{\mathcal{M}}) < \gamma$ and $\rho(\mathcal{M}) \leq \gamma$. Then $\mathcal{M}$ has at most one $(k, \omega_1 + 1)$ iteration strategy Λ that acts on iteration trees that are above γ and whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$ then $\mathcal{N}$ is a ϕ-indexed lsp and $\Sigma^{\mathcal{N}} \subseteq \Sigma \restriction \mathcal{N}$. Moreover, any such strategy Λ is determined by: $\Lambda(\mathcal{T})$ is the unique cofinal b such that the phalanx $\Phi(\mathcal{T})^\frown(\delta(\mathcal{T}), \deg^{\mathcal{T}}(b), \mathcal{Q}(b, \mathcal{T}))$ is $\omega_1 + 1$-iterable (as a (Σ, ϕ)-phalanx).*[5]

In some cases, however, it is enough to assume that $\mathcal{Q}(b, \mathcal{T})$ is countably iterable. This happens, for instance, when $\mathcal{M}$ has no local Woodin cardinals with extenders overlapping it.[6] While the lsp we will consider do have local overlapped Woodin cardinals (that is, some strict initial segment of the lsp has overlapped Woodin cardinals), the lsp themselves will not have such Woodin cardinals. This simplifies our situation somewhat, and below we describe exactly how this will be used.

Definition 2.3.5 (Definition 2.1 of [29]) *Let $(\mathcal{M}, \Sigma)$ be a ϕ-lsm pair and let $\gamma < o(\mathcal{M})$ be such that $\sup(Y^{\mathcal{M}}) < \gamma$. Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ above γ; then $\mathcal{Q}(\mathcal{T})$ is the unique $\oplus_{\nu \in Y^{\mathcal{M}}}(\Sigma_{\mathcal{M}|\nu}, \phi)$-mouse, if there is any, extending $\mathcal{M}(\mathcal{T})$ that has $\delta(\mathcal{T})$ as a strong cutpoint, is $\omega_1 + 1$-iterable above $\delta(\mathcal{T})$ and either projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ is not a Woodin cardinal as witnessed by the extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

Countable iterability is usually enough to guarantee there is at most one hp with the properties of $\mathcal{Q}(\mathcal{T})$. If it exists, $\mathcal{Q}(\mathcal{T})$ might identify the good branch of $\mathcal{T}$, the one any sufficiently powerful iteration strategy must choose. This is the content of the next lemma which can be proved by analyzing the proof of Theorem 6.12 of [28]. To state it we need to introduce fatal drops and also the following useful notation.

Definition 2.3.6 ($\mathcal{O}^{\mathcal{P}}$-stack) *Suppose $\mathcal{P}$ is an lsp, $\eta, \alpha < o(\mathcal{P})$ and $\mathcal{Q} \trianglelefteq \mathcal{P}||\eta$. We then let*

$$\mathcal{O}^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha} = \cup\{\mathcal{M} \trianglelefteq \mathcal{P} : \mathcal{P}|\eta \trianglelefteq \mathcal{M}, \rho(\mathcal{M}) \leq \eta, \bigcup(Y^{\mathcal{M}}) \triangleleft \mathcal{Q} \text{ and for all } E \in \vec{E}^{M}, \text{ if } \eta \in [\mathrm{crit}(E), lh(E)) \text{ then } \mathrm{crit}(E) \leq \alpha\}.$$

Next we define the stack $(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\mathcal{Q},\alpha} : \xi \leq \Omega^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha})$ according to the following recursion:

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[5]The meaning of this is left to the reader.

[6]An extender E overlaps κ if $\mathrm{crit}(E) \leq \kappa \leq \mathrm{lh}(E)$.

1. *$\mathcal{O}^{\mathcal{P},0}_{\eta,\mathcal{Q},\alpha} = \mathcal{O}^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha}$,*

2. *for $\xi + 1 \leq \Omega^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha}$, $\mathcal{O}^{\mathcal{P},\xi+1}_{\eta,\mathcal{Q},\alpha} = \mathcal{O}^{\mathcal{P}}_{o(\mathcal{O}^{\mathcal{P},\xi}_{\eta,\mathcal{Q},\alpha}),\mathcal{Q},\alpha}$,*

3. *for limit $\lambda \leq \Omega^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha}$, $\mathcal{O}^{\mathcal{P},\lambda}_{\eta,\mathcal{Q},\alpha} = \bigcup_{\xi<\lambda} \mathcal{O}^{\mathcal{P},\xi}_{\eta,\mathcal{Q},\alpha}$, and*

4. *$\Omega^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha}$ is the least ν such that $\mathcal{O}^{\mathcal{P},\nu+1}_{\eta,\mathcal{Q},\alpha} = \mathcal{O}^{\mathcal{P},\nu}_{\eta,\mathcal{Q},\alpha}$.*

If $\mathcal{Q} = \mathcal{P}||\kappa$, then we write $\mathcal{O}^{\mathcal{P}}_{\eta,\kappa,\alpha}$ for $\mathcal{O}^{\mathcal{P}}_{\eta,\mathcal{Q},\alpha}$. For $\xi \leq \Omega^{\mathcal{P}}_{\eta,\mathcal{P}||\eta,\alpha}$, we let $\mathcal{O}^{\mathcal{P},\xi}_{\eta} = \mathcal{O}^{\mathcal{P},\xi}_{\eta,\mathcal{P}||\eta,0}$.[7]

We can now introduce fatal drops. Suppose $\mathcal{T}$ is an iteration tree on some structure M and N is a node on $\mathcal{T}$. Then we let $\mathcal{T}_{\geq N}$ be the portion of $\mathcal{T}$ that appears after stage N.

Definition 2.3.7 (Fatal drop) *Suppose $\mathcal{M}$ is a ϕ-indexed lhp and $\mathcal{T}$ is an iteration tree on $\mathcal{M}$. We say $\mathcal{T}$ has a fatal drop if for some $\alpha < lh(\mathcal{T})$ and some $\eta < o(\mathcal{M}^{\mathcal{T}}_{\alpha})$, $\mathcal{T}_{\geq \mathcal{M}^{\mathcal{T}}_{\alpha}}$ is a normal iteration tree on $\mathcal{O}^{\mathcal{M}^{\mathcal{T}}_{\alpha}}_{\eta}$ that is above η. We then say $\mathcal{T}$ has a fatal drop at (α, η) if the pair is the lexicographically least satisfying the above condition.*

The following is the lemma mentioned above.

Lemma 2.3.8 *Let $(\mathcal{M}, \Sigma)$ be a ϕ-lsm pair and let $\gamma < o(\mathcal{M})$ be such that $\bigcup(Y^{\mathcal{M}}) \lhd \mathcal{M}||\gamma$.*

1. *Suppose $\mathcal{T}$ is a normal iteration tree on $\mathcal{M}$ above γ of limit length and suppose $\mathcal{Q}(\mathcal{T})$ exists. Then there is at most one cofinal branch b of $\mathcal{T}$ such that either $\mathcal{Q}(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b$ or $\mathcal{Q}(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b|\xi$ for some ξ in the wellfounded part of $\mathcal{M}^{\mathcal{T}}_b$.*

2. *Suppose further no measurable cardinal of $\mathcal{M}$ which is $\geq \gamma$ is a limit of Woodin cardinals. If then $\mathcal{T}$ is an iteration tree according to Σ above γ which doesn't have a fatal drop and $b = \Sigma(\mathcal{T})$ is such that $\mathcal{Q}(b, \mathcal{T})$-exists then $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.*

$\mathcal{Q}(\mathcal{T})$ identifies b because it determines a canonical cofinal subset of $rng(\pi^{\mathcal{T}}_{\alpha,b} \cap \delta(\mathcal{T}))$, for some $\alpha \in b$, to which we can apply Lemma 1.13 of [10] (which is an immediate consequence of the zipper argument from [7]).

[7] In particular, η is a strong cutpoint of $\mathcal{O}^{\mathcal{P},\xi}_{\eta}$.

Remark 2.3.9 *Suppose $(\mathcal{M}, \Sigma)$ is a ϕ-lsm pair and $\mathcal{Q} \in Y^{\mathcal{M}}$. Let $\mathcal{R} = \mathcal{M}$ if $\mathcal{Q}$ is the largest initial segment of $\mathcal{M}$ in $Y^{\mathcal{M}}$ and otherwise, let $\mathcal{R}$ be the least member of $Y^{\mathcal{M}}$ properly extending $\mathcal{Q}$. Suppose $\mathcal{T}$ is a tree on $\mathcal{M}$ which is above $o(\mathcal{Q})$ and is based on $\mathcal{R}$. Notice that in this case we can define $\mathcal{Q}(\mathcal{T})$ just as in Definition 2.3.5 by using $\mathcal{R}$ instead of $\mathcal{M}$.*

2.4 Hod-like layered hybrid premice

In this paper, we are concerned with lsp[8] whose f predicates code a fragment of their own strategy. The difference of the lsp considered here and those considered in [10] is that here we will have lsp whose predicate codes the *short tree strategy* of its initial segments. The hod mice we will consider in this paper are all *layered*, and we start by introducing these objects.

If $\mathcal{M}$ is an lsp and κ is an $\mathcal{M}$-cardinal then we let

$$X_\kappa^{\mathcal{M}} = \{\xi : E_\xi^{\mathcal{M}} \neq \emptyset \text{ and } \mathrm{crit}(E_\xi^{\mathcal{M}}) = \kappa\}.$$

We also let

$$o^{\mathcal{M}}(\kappa) = \max(\sup X_\kappa^{\mathcal{M}}, (\kappa^+)^{\mathcal{M}}).$$

Suppose M is a transitive structure and η is an ordinal. Then we let $(\eta^{+\alpha})^M$ be the αth-cardinal successor of η in M if it exists and otherwise, we let it be the ordinal height of M.

Definition 2.4.1 (Pre-hod-like) *Suppose $\mathcal{P}$ is an lsp. We say $\mathcal{P}$ is pre-hod-like if one of the following holds:*

1. *(Type I) For some δ such that $\mathcal{P} \vDash$ "δ is a Woodin cardinal or a limit of Woodin cardinals", $\mathcal{P} = \cup_{n<\omega}\mathcal{P}|(\delta^{+n})^{\mathcal{P}}$.*
2. *(Type II) For some $\mathcal{P}$-cardinal κ, letting $\delta = o^{\mathcal{P}}(\kappa)$, $\rho(\mathcal{P}) \leq \delta$ or $o(\mathcal{P})$ is a limit of ordinals ξ such that $\rho(\mathcal{P}|\xi) \leq \delta$.*

We let $\delta^{\mathcal{P}}$ be the δ above.

The next definition isolates the type of hod pairs that give rise to pointclasses satisfying the Largest Suslin Axiom.

[8] We write "lsp" for both layered hybrid premouse and layered hybrid premice.

Definition 2.4.2 (Lsa type) *Suppose $\mathcal{P}$ is a pre-hod-like lsp. We say $\mathcal{P}$ is of lsa type if there is $\kappa < \delta^{\mathcal{P}}$ such that $o^{\mathcal{P}}(\kappa) = \delta^{\mathcal{P}}$ and $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is a Woodin cardinal and κ is a limit of Woodin cardinals".*

In this paper we will consider hod mice that are lsa small.

Definition 2.4.3 (Lsa small) *Suppose $\mathcal{P}$ is a pre-hod-like lsp. We say $\mathcal{P}$ is lsa small if for all $\mathcal{P}$-cardinal κ such that $o^{\mathcal{P}}(\kappa) < \delta^{\mathcal{P}}$ and $\mathcal{P} \vDash$ "κ is a limit of Woodin cardinals", $\mathcal{P} \vDash$ "$o^{\mathcal{P}}(\kappa)$ is not a Woodin cardinal".*

The next definition is somewhat technical. The meaning of it is that we will wait until we see the sharp of a layer before we will activate the strategy.

Definition 2.4.4 (Proper Type II) *Suppose $\mathcal{P}$ is a pre-hod-like lsp of Type II. We say $\mathcal{P}$ is of proper Type II if there is $\xi \in dom(\vec{E}^{\mathcal{P}})$ such that $\mathrm{crit}(E_\xi^{\mathcal{P}}) > \delta^{\mathcal{P}}$, $\mathcal{P}|\xi = \mathcal{J}_\xi[\mathcal{P}|\delta^{\mathcal{P}}]$ and $\mathcal{P}||\xi$ is of lsa type.*

We can now isolate the layers of pre-hod-like lsp.

Definition 2.4.5 (Layers of lsp) *Suppose $\mathcal{P}$ is an lsa small pre-hod-like lsp. We say $\mathcal{Q} \trianglelefteq \mathcal{P}$ is a layer of $\mathcal{P}$ if one of the following conditions holds:*

1. *$\mathcal{P}$ is of proper Type II lsa type lsp and $\mathcal{Q} = \mathcal{M}^+(\mathcal{P}|\delta^{\mathcal{P}})$.*
2. *$\mathcal{Q} \triangleleft \mathcal{P}|\delta^{\mathcal{P}}$ is a pre-hod-like lsp and the following holds.*
   (a) *For some $\mathcal{P}$-cardinal κ such that $\mathcal{P} \vDash$ "κ is a limit of Woodin cardinals", $\delta^{\mathcal{Q}} = o^{\mathcal{Q}}(\kappa)$ and $\mathcal{Q}$ is of proper Type II.*
   (b) *Clause 2.a fails, $\delta^{\mathcal{Q}}$ is a $\mathcal{P}$ cardinal such that $\mathcal{P} \vDash$ "$\delta^{\mathcal{Q}}$ is either a Woodin cardinal or a limit of Woodin cardinals", $\mathcal{Q} = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{Q}},\delta^{\mathcal{Q}}}$ and if $\delta^{\mathcal{Q}}$ is a limit of Woodin cardinals of $\mathcal{P}$ then*
   $$\mathcal{P}|((\delta^{\mathcal{Q}})^+)^{\mathcal{P}} = \mathcal{Q}|((\delta^{\mathcal{Q}})^+)^{\mathcal{Q}}.$$

Next we introduce hod-like lsp. These will eventually turn into hod premice. To do this we need to impose conditions on the layers of lsp, which are just the members of $Y^{\mathcal{P}}$ where $\mathcal{P}$ is an lsp.

Definition 2.4.6 (Hod-like lsp) *Suppose $\mathcal{P}$ is a pre-hod-like lsp. We say $\mathcal{P}$ is hod-like if the following conditions hold.*

1. *If $\mathcal{P}$ is of Type II then $\mathcal{P}$ is of proper Type II.*

2. $Y^{\mathcal{P}} = \{\mathcal{Q} : \mathcal{Q}$ *is a layer of* $\mathcal{P}\}$.

The next definition isolates four types of proper pre-hod-like lsp that we will encounter in this paper. The types are not necessarily disjoint.

Notation 2.4.7 *Suppose $\mathcal{P}$ is a hod-like lsp. Let*

$$L^{\mathcal{P}} = \{\delta : \exists \mathcal{Q} \in Y^{\mathcal{P}}, \delta^{\mathcal{Q}} = \delta\} \cup \{\delta^{\mathcal{P}}\}.$$

Let $\lambda^{\mathcal{P}}$ be the order type of $L^{\mathcal{P}}$. We let $(\delta^{\mathcal{P}}_\alpha : \alpha \leq \lambda^{\mathcal{P}})$ be the increasing enumeration of $L^{\mathcal{P}}$. Often we will refer to the intervals $(\delta^{\mathcal{P}}_\alpha, \delta^{\mathcal{P}}_{\alpha+1})$ as the windows of $\mathcal{P}$. If $\lambda^{\mathcal{P}}$ is a successor then we often say that $(\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}})$ is the top window of $\mathcal{P}$.

Terminology 2.4.8 *Suppose $\mathcal{P}$ is a hod-like lsp.*

1. ***(Successor type)*** *We say $\mathcal{P}$ has a successor type if $\lambda^{\mathcal{P}}$ is a successor ordinal and $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is not a measurable cardinal.*

2. ***(Limit type)*** *We say $\mathcal{P}$ has a limit type if $\lambda^{\mathcal{P}}$ is a limit ordinal or $\lambda^{\mathcal{P}}$ is a successor ordinal and $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ is a measurable cardinal.*

3. ***(Lsa types)*** *Suppose $\mathcal{P}$ is of lsa type. We say $\mathcal{P}$ has lsa type I if $\mathcal{P} \vDash$ "ZFC-Powerset". Otherwise, we say $\mathcal{P}$ has lsa type II.*

4. ***(Meek)*** *We say $\mathcal{P}$ is meek if either it has a successor type or $\lambda^{\mathcal{P}}$ is a limit ordinal.*

Remark 2.4.9 *From now on we tacitly assume that all lsp considered in this paper are lsa-small. We will, from time to time, remind the reader of this.*

Definition 2.4.10 (The internal strategy) *Given $\mathcal{Q} \in Y^{\mathcal{P}}$ we let $\Sigma^{\mathcal{P}}_{\mathcal{Q}}$ be the strategy of $\mathcal{Q}$ coded by $f^{\mathcal{P}}(\mathcal{Q})$.*

Next, we isolate the bottom part of non-meek limit type hod-like lsp. This is essentially the part of $\mathcal{P}$ that is below the largest measurable limit of cutpoint Woodin cardinals.

Definition 2.4.11 (The bottom part of lsp) *Given a non-meek limit type hod-like lsp $\mathcal{P}$, we let $\mathcal{P}^b = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \mathcal{P}||\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}}$ where "b" stands for "bottom". We say that $\mathcal{P}^b$ is the bottom part of $\mathcal{P}$.*

We end this section with the definition of hod initial segments of lsp.

Definition 2.4.12 (Hod initial segment) *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod-like lsp. We then write $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ and say $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$ if $\mathcal{P} \in Y^{\mathcal{Q}}$.*

We finish this section by introducing a useful notation.

Notation 2.4.13 *Suppose $\mathcal{P}$ is a hod-like lsp and $\xi < \lambda^{\mathcal{P}}$. We define $\mathcal{P}(\xi)$ according to the following clause.*

1. *Suppose that $\delta_\xi^{\mathcal{P}}$ is a Woodin cardinal of $\mathcal{P}$ or a non-measurable limit of Woodin cardinals of $\mathcal{P}$. Then we let $\mathcal{P}(\xi) = \mathcal{O}^{\mathcal{P},\omega}_{\delta_\xi^{\mathcal{P}}, \mathcal{P}|\delta_\xi^{\mathcal{P}}}$.*

2. *Suppose $\delta_\xi^{\mathcal{P}}$ is a measurable limit of Woodin cardinals. Let $E \in \vec{E}^{\mathcal{P}}$ be the Mitchell order 0 extender with critical point $\delta_\xi^{\mathcal{P}}$. Then let $\mathcal{P}(\xi) = Ult(\mathcal{P}, E)(\xi)$.*

3. *Suppose $\xi = \gamma + 1$ and $\delta_\xi^{\mathcal{P}} = o^{\mathcal{P}}(\delta_\gamma^{\mathcal{P}})$. If $\delta_\xi^{\mathcal{P}} < \delta^{\mathcal{P}}$ then let $\mathcal{P}(\xi) = \mathcal{P}|((\delta_\xi^{\mathcal{P}})^+)^{\mathcal{P}}$. If $\delta_\xi^{\mathcal{P}} = \delta^{\mathcal{P}}$ then let $\mathcal{P}(\xi) = \mathcal{P}$.*

2.5 Analysis of stacks

Here we review the analysis of stacks of iteration trees from Section 6.2 of [10]. Suppose M is a transitive structure and $\vec{\mathcal{T}}$ is a stack of iteration trees on M[9]. Let $\mathcal{S}$ and $\mathcal{R}$ be nodes in $\vec{\mathcal{T}}$. Then we write $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ for the component of $\vec{\mathcal{T}}$ that comes after stage $\mathcal{S}$ and $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ for the component of $\vec{\mathcal{T}}$ up to stage $\mathcal{S}$. In the case $\mathcal{R}$ appears in $\vec{\mathcal{T}}$ later than $\mathcal{S}$, we also write $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ for the part of $\vec{\mathcal{T}}$ that is between $\mathcal{S}$ and $\mathcal{R}$. Notice that neither $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ nor $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ might be stacks on $\mathcal{S}$.

Definition 2.5.1 (Cutpoint of a stack) *We say $\mathcal{S}$ is a cutpoint of $\vec{\mathcal{T}}$ if no normal component of $\vec{\mathcal{T}}_{\leq \mathcal{S}}$ has a fatal drop and $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a stack on $\mathcal{S}$.*

Suppose now that $\mathcal{T}$ is a normal tree on M.

Definition 2.5.2 (Reducible and irreducible trees) *We say $\mathcal{T}$ is reducible if it has a cutpoint. Otherwise we say $\mathcal{T}$ is irreducible.*

[9] Recall that all trees are normal.

Suppose next that $\mathcal{P}$ is a hod-like lsp. In our current context we must consider stacks with more severe dropping patterns than those considered in [10]. However, we will rule out stacks with too many bad drops. The bad drops will consist of *fatal drops* and *non-continuable drops*. The stacks that we will consider can have at most one of each such drops. We have already introduced fatal drops (see Definition 2.3.7). Below we introduce non-continuable drops.

Continuing with our $\mathcal{P}$ suppose $\mathcal{T}$ is a normal irreducible tree on $\mathcal{P}$ which has a last model but $\pi^{\mathcal{T}}$ doesn't exist.

Definition 2.5.3 (Continuable drop) *We then say $\mathcal{T}$ has a continuable drop if $\mathcal{T}$ doesn't have a fatal drop and for some limit type $\mathcal{Q} \in Y^{\mathcal{P}}$, $\mathcal{T}$ is based on $\mathcal{Q}$ and is above $o(\mathcal{Q}^b)$.*

Besides fatal drops, continuable drops rule out drops in windows $(\delta_\xi^{\mathcal{P}}, \delta_{\xi+1}^{\mathcal{P}})$ where $\delta_\xi^{\mathcal{P}}$ is not a measurable cardinal of $\mathcal{P}$. Notice that it is not required that there be no such drops, but rather that the final branch doesn't have such a drop.

Definition 2.5.4 (Continuable stack) *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is continuable if for every two successive cutpoints $\mathcal{S}$ and $\mathcal{R}$, either $\pi^{\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}}$ exists or $\vec{\mathcal{T}}_{\mathcal{S},\mathcal{R}}$ (which is a normal irreducible tree) has a continuable drop.*

Say $\mathcal{T}$ has a non-continuable drop if $\mathcal{T}$ has a drop which is not a continuable drop. The next definition blocks iterations of hod-like lsp that have more than one non-continuable drops.

Definition 2.5.5 (Stack on hod-like lsp) *We say $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ if it is produced according to the rules of the usual iteration game except that for every $\alpha < \eta$, $\vec{\mathcal{T}} \restriction \alpha$ is continuable.*

Continuing with our $\mathcal{P}$, let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$. Given a node $\mathcal{R}$ in $\vec{\mathcal{T}}$ we say $\mathcal{R}$ is a *terminal node* in $\vec{\mathcal{T}}$ if player I can legitimately continue $\vec{\mathcal{T}}_{\leq\mathcal{R}}$ by starting a new round of the iteration game. We say $\mathcal{R}$ is a *non-trivial* terminal node if it is a terminal node and the extender chosen from $\mathcal{R}$ is applied to $\mathcal{R}$. The following is an easy lemma.

Lemma 2.5.6 *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ and $\mathcal{S}$ is a cutpoint of $\vec{\mathcal{T}}$. Then $\mathcal{S}$ is a non-trivial terminal node of $\vec{\mathcal{T}}$.*

Suppose again that $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$. We then let

$$tn(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a terminal node in } \vec{\mathcal{T}}\}$$
$$ntn(\vec{\mathcal{T}}) = \{\mathcal{R} : \mathcal{R} \text{ is a non-trivial terminal node in } \vec{\mathcal{T}}\}.$$

Given two $\mathcal{Q}, \mathcal{R} \in \mathrm{tn}(\vec{\mathcal{T}})$, we write $\mathcal{Q} \preceq^{\vec{\mathcal{T}},w} \mathcal{R}$ if[10], in $\vec{\mathcal{T}}$, $\mathcal{R}$ appears later than $\mathcal{Q}$. We write $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ if, in $\vec{\mathcal{T}}$, $\mathcal{Q}$-to-$\mathcal{R}$ iteration embedding exists. If $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ then we let $\pi^{\vec{\mathcal{T}}}_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the iteration embedding given by $\vec{\mathcal{T}}$. If $\mathcal{Q} = \mathcal{P}$ then we just write $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}}$. We write $\mathcal{Q} \preceq^{\vec{\mathcal{T}},s} \mathcal{R}$ if[11] $\mathcal{Q} \preceq^{\vec{\mathcal{T}}} \mathcal{R}$ and $\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{R}}$ is a stack on $\mathcal{Q}$.

Continuing with $\mathcal{P}$ and $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$, suppose $C \subseteq tn(\vec{\mathcal{T}})$. We say C is *linear* if it is linearly ordered by $\preceq^{\vec{\mathcal{T}},s}$.

Suppose now that C is linear and $(\mathcal{R}_\alpha : \alpha < \eta)$ is a $\preceq^{\vec{\mathcal{T}},s}$-increasing enumeration of C. We let $lh(C) = \eta$. Suppose further that η is a limit ordinal. Then we let $\mathcal{R}^{\vec{\mathcal{T}}}_C$ be the direct limit of the $\mathcal{R}_\alpha$ under the iteration embeddings $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}_\alpha,\mathcal{R}_\beta}$. We then say $C \subseteq tn(\vec{\mathcal{T}})$ is *closed* if it is linear and for every limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha} \in C$. Notice that linearity implies that for each limit $\alpha < lh(C)$, $\mathcal{R}^{\vec{\mathcal{T}}}_{C\restriction\alpha}$ is a node in $\vec{\mathcal{T}}$.

Next, we say C is *cofinal* if for every node $\mathcal{S}$ of $\vec{\mathcal{T}}$ either $\mathcal{S} \in C$ or there are $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q} \in C$ such that $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$. The following is another easy lemma.

Lemma 2.5.7 *If C is cofinal then every node in C is a cutpoint.*

We say C is a *club* if it is closed and cofinal. Notice that if C is closed and cofinal and $\mathcal{S} \notin C$ then there is a $\preceq^{\vec{\mathcal{T}},s}$-largest $\mathcal{R} \in C$ such that for any $\mathcal{Q} \in C$ such that $\mathcal{R} \preceq^{\vec{\mathcal{T}},s} \mathcal{Q}$, $\mathcal{S}$ is a node in $\vec{\mathcal{T}}_{\mathcal{R},\mathcal{Q}}$.

Continuing with our fixed $\mathcal{P}$, suppose $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ is a stack on $\mathcal{P}$. If $\mathcal{R}$ is a non-trivial terminal node of $\vec{\mathcal{T}}$ then we let $\xi^{\vec{\mathcal{T}},\mathcal{R}}$ be the least such that $E^{\mathcal{T}_\alpha}_\beta \in \mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}} + 1)$, where $\mathcal{R} = \mathcal{M}^{\mathcal{T}_\alpha}_\beta$. We also let $\vec{\mathcal{T}}_{\mathcal{R}}$ be the largest initial segment of $\vec{\mathcal{T}}_{\geq\mathcal{R}}$ that can be regarded as a stack on $\mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}} + 1)$.

Notice that if $\vec{\mathcal{T}}$ doesn't have a last model but there is a club $C \subseteq tn(\vec{\mathcal{T}})$ then C uniquely identifies the branch of $\vec{\mathcal{T}}$. Indeed, let $D = \{\mathcal{S} \in tn(\vec{\mathcal{T}}) : \exists \mathcal{R}, \mathcal{Q} \in C(\mathcal{R} \preceq^{\vec{\mathcal{T}}} \mathcal{S} \preceq^{\vec{\mathcal{T}}} \mathcal{Q})\}$. Let $\mathcal{R} \in D$ be the $\preceq^{\vec{\mathcal{T}}}$-minimal member of D and let b be the set of indices of the nodes of $\vec{\mathcal{T}}$ between $\mathcal{P}$ and $\mathcal{R}$. Then the union of b with the indices of the nodes of D constitute a branch b_C of $\vec{\mathcal{T}}$. It is not hard to see that we have $\mathcal{M}^{\vec{\mathcal{T}}}_{b_C} = \mathcal{R}^{\vec{\mathcal{T}}}_C$.

Suppose now that $\vec{\mathcal{T}}$ doesn't have a last model and there is no club $C \subseteq tn(\vec{\mathcal{T}})$. Let then $D = \{\mathcal{S} \in tn(\mathcal{T}) : \mathcal{S} \text{ is a cutpoint}\}$. It follows from our discussion that D

[10] "w" stands for "weakly"
[11] "s" stands for "strongly".

has a $\preceq^{\vec{\mathcal{T}},s}$-largest element. Let $\mathcal{S}_{\vec{\mathcal{T}}}$ be this largest element. The following is our last easy lemma.

Lemma 2.5.8 *$\vec{\mathcal{T}}_{\geq \mathcal{S}_{\vec{\mathcal{T}}}}$ is a normal tree on $\mathcal{S}_{\vec{\mathcal{T}}}$ such that if it is reducible then it has either a fatal or a non-continuable drop.*

2.6 The iteration embedding $\pi^{\vec{\mathcal{T}},b}$

Continuing with $\mathcal{P}$ and $\vec{\mathcal{T}}$, assume that $\mathcal{P}$ is a limit type hod-like lsp which isn't meek. Again, we will not be concerned with the particular indexing scheme that $\mathcal{P}$ has. In some cases, regardless of whether $\vec{\mathcal{T}}$ has a last model or not, it is possible to extract an embedding out of the iteration embeddings given by $\vec{\mathcal{T}}$ that acts on $\mathcal{P}^b$. We describe this embedding below. First we define it by assuming that $\vec{\mathcal{T}} = \mathcal{T}$ is a normal irreducible tree. Recall that our lsp are lsa-small (see Definition 2.4.3).

Definition 2.6.1 ($\pi^{\mathcal{T},b}$ fo irreducible trees) *Let $\lambda = \lambda^{\mathcal{P}}$. Let $\mathcal{M} = \mathcal{M}(\mathcal{T})$ if $\mathcal{T}$ is of limit length and let $\mathcal{M}$ be the last model of $\mathcal{T}$ otherwise. Then letting $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$, we let $\pi^{\mathcal{T},b}$ be*

1. *undefined if $\mathcal{T}$ is below δ and $\pi^{\mathcal{T}}$ doesn't exist,*
2. *$\pi^{\mathcal{T}} \restriction \mathcal{P}^b$ if $\pi^{\mathcal{T}}$ exist,*
3. *id if $\mathcal{T}$ is above δ, $\pi^{\mathcal{T}}$ doesn't exist and $\mathcal{M}|(\delta^+)^{\mathcal{P}} = \mathcal{P}|(\delta^+)^{\mathcal{P}}$,*
4. *undefined if $\mathcal{T}$ is above δ, $\pi^{\mathcal{T}}$ doesn't exist and $\mathcal{M}|(\delta^+)^{\mathcal{P}} \neq \mathcal{P}|(\delta^+)^{\mathcal{P}}$.*

Remark 2.6.2 *Notice that in Definition 2.6.1, because $\mathcal{T}$ is irreducible and $\delta^{\mathcal{P}}_{\lambda-1}$ is a limit of cutpoints, it cannot be the case that for some $\alpha < lh(\mathcal{T})$, $\mathrm{crit}(E^{\mathcal{T}}_\alpha) = \delta^{\mathcal{P}}_{\lambda-1}$ and $\mathrm{crit}(E^{\mathcal{T}}_{\alpha+1}) \geq \pi_{E^{\mathcal{T}}_\alpha}(\delta^{\mathcal{P}}_{\lambda-1})$ (this is because otherwise $\mathcal{T}_{\geq \mathcal{M}^{\mathcal{T}}_{\alpha+1}}$ would be a normal tree on $\mathcal{M}^{\mathcal{T}}_{\alpha+1}$). This observation implies that the above clauses are all possible clauses.*

Next we define $\pi^{\mathcal{T},b}$ for trees $\mathcal{T}$.

Definition 2.6.3 ($\pi^{\mathcal{T},b}$ for trees) *Suppose $\mathcal{T}$ is a tree on $\mathcal{P}$. We define $\pi^{\mathcal{T},b}$ by induction on cutpoints of $\mathcal{T}$. If there is a cutpoint $\mathcal{R}$ of $\mathcal{T}$ such that $\pi^{\mathcal{T}_{\mathcal{P},\mathcal{R}},b}$ is undefined then let $\pi^{\mathcal{T},b}$ be undefined. Otherwise let $C = (\mathcal{R}_\alpha : \alpha < \eta)$ be the sequence of cutpoints of $\mathcal{T}$. If C is a club then letting c be the unique branch of $\mathcal{T}$, we let $\pi^{\mathcal{T},b} = \pi^{\mathcal{T}}_c \restriction \mathcal{P}^b$. Otherwise letting $\eta = \gamma + 1$[12], let $\pi^{\mathcal{T},b} = \pi^{\mathcal{T}_{\geq \mathcal{R}_\gamma},b} \circ \pi^{\mathcal{T}_{\mathcal{P},\mathcal{R}_\gamma},b}$.*

[12]Notice that η is always a successor ordinal.

Finally we define $\pi^{\vec{\mathcal{T}},b}$ for stacks.

Definition 2.6.4 ($\pi^{\mathcal{T},b}$ for stacks) *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with normal components $(\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$. If for some $\alpha < \eta$, $\pi^{\mathcal{T}_\alpha,b}$ is undefined then we let $\pi^{\vec{\mathcal{T}},b}$ be undefined. Suppose then for every $\alpha < \eta$, $\pi^{\mathcal{T}_\alpha,b}$ is defined. Then if η is a limit ordinal then, letting c be the unique branch of $\vec{\mathcal{T}}$, we let $\pi^{\vec{\mathcal{T}},b} = \pi_c^{\vec{\mathcal{T}}} \restriction \mathcal{P}^b$. If $\eta = \gamma + 1$ then let $\pi^{\vec{\mathcal{T}},b} = \pi^{\mathcal{T}_\gamma,b} \circ \pi^{\vec{\mathcal{T}}\restriction\gamma,b}$.*

Notice that in Definition 2.6.5 we are not assuming that the stack has a last model. The fragment of the eventual iteration embedding $\pi^{\vec{\mathcal{T}}}$ restricted to $\mathcal{P}^b$ can be seen without actually having the last branch. Notice also that the actual branch embedding may not agree with $\pi^{\vec{\mathcal{T}},b}$.

Definition 2.6.5 (Almost non-dropping stacks) *Suppose $\mathcal{P}$ is a non-meek hod-like lsp and $\vec{\mathcal{T}}$ is a stack of iteration trees on $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is almost non-dropping if $\pi^{\vec{\mathcal{T}},b}$ is defined on $\mathcal{P}^b$. Suppose Σ is an iteration strategy for $\mathcal{P}$[13]. We then let*

$$I(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \vec{\mathcal{T}} \text{ is according to } \Sigma,\ \mathcal{R} \text{ is the last model of } \vec{\mathcal{T}} \text{ and } \pi^{\vec{\mathcal{T}}} \text{ is defined}\}.$$

$$I^b(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \vec{\mathcal{T}} \text{ is according to } \Sigma,\ \mathcal{R} \text{ is the last model of } \vec{\mathcal{T}} \text{ and } \pi^{\vec{\mathcal{T}},b} \text{ is defined}\}.$$

Remark 2.6.6 *Notice that if $\vec{\mathcal{T}}$ is almost non-dropping then it may only have drops in some image of the top window of $\mathcal{P}$.*

The following notion will be used throughout this paper.

Definition 2.6.7 (Canonical singularizing sequences) *Suppose $\mathcal{P}$ is a non-meek hod-like lsp and $\vec{\mathcal{T}}$ is an almost non-dropping stack on $\mathcal{P}$. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}},b}(\mathcal{P}^b)$. Then $\mathcal{Q}$ is an lsp. For $\xi + 1 \leq \lambda^{\mathcal{Q}}$, we let*

$$s(\vec{\mathcal{T}}, \xi) = \{\alpha : \exists a \in (\delta_\xi^{\mathcal{Q}} + 1)^{<\omega} \exists f \in \mathcal{P}^b(\alpha = \pi^{\vec{\mathcal{T}},b}(f)(a))\} \cap \delta_{\xi+1}^{\mathcal{Q}}$$

The following is an easy lemma.

Lemma 2.6.8 *Suppose $\mathcal{P}$ is a non-meek hod-like lsp and $\vec{\mathcal{T}}$ is an almost non-dropping stack on $\mathcal{P}$. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}},b}(\mathcal{P}^b)$. Then for any $\xi + 1 \leq \lambda^{\mathcal{Q}}$, $\sup(s(\vec{\mathcal{T}}, \xi)) = \delta_{\xi+1}^{\mathcal{Q}}$.*

[13] It is worth remembering that this entails that Σ-iterates of $\mathcal{P}$ have the same indexing scheme as $\mathcal{P}$.

2.7 The un-dropping game

Before we proceed, we explain the meaning of the un-dropping game. Suppose we are comparing the strategies of two lsa type hod-like lsp $\mathcal{P}$ and $\mathcal{Q}$. Let Σ be the strategy of $\mathcal{P}$ and Λ be the strategy of $\mathcal{Q}$. Let us assume that the pointclasses generated by $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are the same. We are then searching for $\mathcal{R}$ which is an iterate of $\mathcal{P}$ and $\mathcal{Q}$ and $\Sigma_{\mathcal{R}} = \Lambda_{\mathcal{R}}$. In this comparison we might be forced to consider iteration trees $\mathcal{T}$ and $\mathcal{U}$ with last models $\mathcal{M}$ and $\mathcal{N}$ such that $\pi^{\mathcal{T}}$ and $\pi^{\mathcal{U}}$ don't exist and for some $\xi < \min(\lambda^{\mathcal{M}}, \lambda^{\mathcal{N}})$, $\mathcal{M}(\xi+1) = \mathcal{N}(\xi+1)$ but $\Sigma_{\mathcal{M}(\xi+1)} \neq \Lambda_{\mathcal{N}(\xi+1)}$. We can continue the comparison by comparing $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$ and producing $(\mathcal{S}, \Phi)$ which is a common tail of $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$. However, $(\mathcal{S}, \Phi)$ cannot be thought of as a last model of a successful comparison of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ simply because $\pi^{\mathcal{T}}$ and $\pi^{\mathcal{U}}$ do not exist. What we need to do is to compare $(\mathcal{M}, \Sigma_{\mathcal{M}})$ and $(\mathcal{N}, \Lambda_{\mathcal{N}})$ and then somehow get back to $\mathcal{P}$ and $\mathcal{Q}$. This is what the un-dropping game achieves.

To define the un-dropping game, we need to define the sequence of *main drops*. It is the sequence of stages in an iteration at which there is a drop below the top window.

Definition 2.7.1 (The main drops of a continuable stack) *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is a continuable stack. We say $md^{\vec{\mathcal{T}}} = (\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ is the sequence of main drops of $\vec{\mathcal{T}}$ if the following conditions hold:*

1. *$k < \omega$ and $\mathcal{R}_0 = \mathcal{P}$.*
2. *$(\mathcal{R}_i : i \leq k)$ is a $\preceq^{\vec{\mathcal{T}},s}$-increasing sequence of cutpoints of $\vec{\mathcal{T}}$.*
3. *For $i+1 \leq k$, $\vec{\mathcal{T}}_i = \vec{\mathcal{T}}_{\mathcal{R}_i, \mathcal{R}_{i+1}}$ and $\vec{\mathcal{T}}_k = \vec{\mathcal{T}}_{\geq \mathcal{R}_k}$.*
4. *For each $i \leq k$, $\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1 < \lambda^{\mathcal{R}_i}$, $\mathcal{R}_i(\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1)$ is a limit type hod-like lsp and $\vec{\mathcal{T}}_{\geq \mathcal{R}_i}$ is a stack on $\mathcal{R}_i(\xi^{\vec{\mathcal{T}}, \mathcal{R}_i} + 1)$[14].*
5. *For each $i < k$, $\mathcal{R}_{i+1}$ is $\preceq^{\vec{\mathcal{T}},s}$-least cutpoint $\mathcal{Q}$ of $\vec{\mathcal{T}}$ such that $\mathcal{R}_i \preceq^{\vec{\mathcal{T}},s} \mathcal{R}_{i+1}$, $\pi^{\mathcal{T}_{\mathcal{R}_i, \mathcal{Q}}, b}$ exists and for every node $\mathcal{S} \neq \mathcal{Q}$ of $\vec{\mathcal{T}}_{\geq \mathcal{Q}}$, $\pi^{\vec{\mathcal{T}}_{\mathcal{R}_i, \mathcal{S}}, b}$ doesn't exist.*
6. *For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}_{\geq \mathcal{R}_k}$, $\pi^{(\vec{\mathcal{T}}_k)_{\mathcal{R}_k, \mathcal{S}}, b}$ exists.*

[14] Recall the definition of $\xi^{\vec{\mathcal{T}}, \mathcal{R}_i}$. It was defined a few paragraphs below Definition 2.5.7.

Notice that it is possible that in the above definition $\mathcal{R}_0 = \mathcal{R}_1$. This can happen, for instance, when I starts out with a drop. Next we define the un-dropping extender of $\vec{\mathcal{T}}$. This is essentially the extender given by dovetailing the embeddings $\pi^{\vec{\mathcal{T}}_i,b}$.

Definition 2.7.2 (The un-dropping extender of a stack) *Suppose $\mathcal{P}$ is a hod-like lsp and $\vec{\mathcal{T}}$ is a continuable stack with a last model. Let $(\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ be the sequence of the main drops of $\vec{\mathcal{T}}$ and suppose $\pi^{\vec{\mathcal{T}}_k,b}$ is defined. For $i \leq k$, let $\kappa_i = \delta^{\mathcal{R}_i}_{\xi^{\vec{\mathcal{T}},\mathcal{R}_i}}$, and for $i+1 \leq k$, let*

$$\sigma^{\vec{\mathcal{T}}}_{i,i+1} : (\wp(\kappa_i))^{\mathcal{R}_i} \to (\wp(\kappa_{i+1}))^{\mathcal{R}_{i+1}}$$

be given by

$$\sigma^{\vec{\mathcal{T}}}_{i,i+1}(A) = \pi^{\vec{\mathcal{T}}_i,b}(A) \cap \kappa_{i+1}.$$

Set $\sigma^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{T}}_k,b} \circ \sigma^{\vec{\mathcal{T}}}_{k-1,k} \circ \sigma^{\vec{\mathcal{T}}}_{k-2,k-1} \cdots \circ \sigma^{\vec{\mathcal{T}}}_{0,1}$ and let $E^{\vec{\mathcal{T}}}$ be the $(\kappa_0, \pi^{\vec{\mathcal{T}}_k,b}(\kappa_k))$-extender derived from $\sigma^{\vec{\mathcal{T}}}$. More precisely,

$$E^{\vec{\mathcal{T}}} = \{(a, A) : a \text{ is a finite subset of } \pi^{\vec{\mathcal{T}}_k,b}(\kappa_k),\ A \in (\wp(\kappa_0))^{\mathcal{P}},\ \text{and } a \in \sigma^{\vec{\mathcal{T}}}(A)\}.$$

$E^{\vec{\mathcal{T}}}$ is called the un-dropping extender of $\vec{\mathcal{T}}$. Suppose $\mathcal{Q} \trianglelefteq_{hod} \pi^{\vec{\mathcal{T}}_k,b}(\mathcal{R}^b_k)$. Then we let $E^{\vec{\mathcal{T}}}_{\mathcal{Q}}$ be the $(\kappa_0, \delta^{\mathcal{Q}})$-extender derived from $\sigma^{\vec{\mathcal{T}}}$. More precisely,

$$E^{\vec{\mathcal{T}}}_{\mathcal{Q}} = \{(a, A) : a \text{ is a finite subset of } \delta^{\mathcal{Q}},\ A \in (\wp(\kappa_0))^{\mathcal{P}},\ \text{and } a \in \sigma^{\vec{\mathcal{T}}}(A)\}.$$

When comparing hod premice we need to consider iterations in which at certain stages I is allowed to use the un-dropping extender of the resulting stack. The game producing such iterations is defined below.

Definition 2.7.3 (The un-dropping iteration game) *Suppose $\mathcal{P}$ is a hod-like lsp with an indexing scheme ϕ. The un-dropping iteration game on $\mathcal{P}$, $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$, is an iteration game satisfying the following conditions:*

1. *If any of the models produced during a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ is ill-founded or doesn't have indexing scheme ϕ then player II loses that run.*

2. *$\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ has at most κ main rounds. Player I starts the main rounds.*

3. *If p is a run of $\mathcal{G}^u_k(\mathcal{P}, \kappa, \lambda, \alpha)$ and $\mathcal{M}_\zeta$ is the model at the beginning of the ζth main round of p then the ζth main round of p is a run of $\mathcal{G}_k(\mathcal{M}_\zeta, \lambda, \alpha)$.*

4. *Suppose p is a run of $\mathcal{G}^u_k(\mathcal{P},\kappa,\lambda,\alpha)$ and $(\mathcal{M}_\xi : \xi < \zeta)$ are models at the beginning of the main rounds of p. Suppose $\xi < \gamma$ and $\gamma + 1 < \zeta$. Then the iteration embedding $\pi : \mathcal{M}_\xi \to \mathcal{M}_\gamma$ exists.*

5. *Suppose p is a run of $\mathcal{G}^u_k(\mathcal{P},\kappa,\lambda,\alpha)$. Then player I can start a main round in two different ways.*

   (a) *Suppose first p has $\zeta < \kappa$ main rounds where ζ is a limit ordinal. Let $(\mathcal{M}_\alpha : \alpha < \zeta)$ be the sequence of the models at the beginning of the main rounds. Let then $\mathcal{M}_\zeta$ be the direct limit of $\mathcal{M}_\alpha$ under the iteration embeddings. Then the ζth main round is played on $\mathcal{M}_\zeta$.*

   (b) *Next suppose $\zeta = \gamma + 1$. Then I can start a new main round only if the stack played in the γth main round is continuable. Let then $\vec{\mathcal{T}}_\gamma$ be the stack played in the γth main round and suppose $\vec{\mathcal{T}}_\gamma$ is continuable with last model $\mathcal{R}$. Then Player I chooses $\mathcal{Q} \trianglelefteq_{hod} \mathcal{R}^b$ and $\xi \leq \gamma$. Let $\pi : \mathcal{M}_\xi \to \mathcal{M}_\gamma$ be the iteration embedding in $\vec{\mathcal{T}}$ and let $F^{\vec{\mathcal{T}}}_{\xi,\gamma}$ be the $(\delta^{\mathcal{M}^b_\xi}, \delta^{\mathcal{M}^b_\gamma})$-extender derived from π. Set*

   $$\mathcal{M}_\zeta = Ult(Ult(\mathcal{M}_\xi, F^{\vec{\mathcal{T}}}_{\xi,\gamma}), E^{\vec{\mathcal{T}}_\gamma}_{\mathcal{Q}}).$$

   Then I can start a new main round, if she wishes so, on $\mathcal{M}_\zeta$.

If $\vec{\mathcal{T}}$ is a run of $\mathcal{G}^u_k(\mathcal{P},\kappa,\lambda,\alpha)$, then we let $(\mathcal{M}_\varsigma, \vec{\mathcal{T}}_\varsigma, \mathcal{Q}_\varsigma, \xi_\varsigma, F_\varsigma, E_\varsigma : \varsigma < \eta)$ be such that

1. *$\mathcal{M}_\varsigma$ is the lsp at the beginning of the ςth main round,*

2. *$\vec{\mathcal{T}}_\varsigma$ is the stack played in the ςth main round,*

3. *if $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}_\varsigma$ and $\varsigma + 1 < \eta$ then $\mathcal{Q}_\varsigma \trianglelefteq_{hod} \mathcal{R}^b$,*

4. *$\xi_\varsigma \leq \varsigma$,*

5. *F_ς is the $(\delta^{\mathcal{M}^b_\xi}, \delta^{\mathcal{M}^b_\gamma})$-extender derived from $\pi^{\vec{\mathcal{T}}_{\mathcal{M}_{\xi_\varsigma}, \mathcal{M}_\varsigma}}$,*

6. *$E_\varsigma = E^{\vec{\mathcal{T}}_\varsigma}_{\mathcal{Q}_\varsigma}$ and*

7. *both F_ς and E_ς are defined iff $\varsigma + 1 < \eta$.*

We will often omit ξ_ς and F_ς as those are not essential. If Σ is a winning strategy for II in $\mathcal{G}^u_k(\mathcal{P},\kappa,\lambda,\alpha)$ then we say Σ is a (κ,λ,α)-strategy. We say $\vec{\mathcal{T}}$ is a generalized stack if it is produced via a run of the un-dropping game.

It is important to remark that clauses 4 and 5b are in conflict. Clause 4 blocks the possibility of un-dropping to an earlier model than the model at the end of the previous main round while 5b allows one to go back. The issue is resolved by noticing that Player I can un-drop to an earlier model than the model at the end of the previous main round only once. We require that our iteration strategies be $(\omega_1, \omega_1, \omega_1)$-strategies.

Definition 2.7.4 (Hod-like lsp pair) *We say $(\mathcal{P}, \Sigma)$ is a hod-like lsp pair (with an indexing scheme ϕ) if $\mathcal{P}$ is a hod-like lsp (with an indexing scheme ϕ) and Σ is a winning strategy in $\mathcal{G}_k^u(\mathcal{P}, \omega_1, \omega_1, \omega_1)$.*

Chapter 3

Short tree strategy mice

3.1 The short tree component of a strategy

Suppose $(\mathcal{P}, \Sigma)$ is a hod-like lsp pair such that $\mathcal{P}$ is of lsa type. We suppress the indexing scheme that the pair $(\mathcal{P}, \Sigma)$ has from our notations below; the particular indexing scheme will not matter for what follows. The next definition isolates the *short tree component* of Σ denoted by Σ^{stc}. Let $\kappa = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ and $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$.

Definition 3.1.1 (The normal short tree component of a strategy) *We first define Σ^{nstc}, the portion of the short tree component that acts on normal trees. Suppose $\mathcal{T}$ is a normal tree on $\mathcal{P}$ of limit length. Let $b = \Sigma(\vec{\mathcal{T}})$. We then let*

$$\Sigma^{nstc}(\mathcal{T}) = \begin{cases} b & : \ \pi^{\mathcal{T}}_b \textit{ doesn't exist or } \pi^{\mathcal{T}}_b(\delta) > \delta(\mathcal{T}), \\ \mathcal{M}^{\mathcal{T}}_b & : \ \textit{otherwise.} \end{cases}$$

Suppose $\mathcal{Q}$ is an iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}}$ exists. We define the short tree component of Σ by concatenating all $\Sigma^{nstc}_{\mathcal{Q},\vec{\mathcal{T}}}$.

To make the next definition more intuitive, we say $\mathcal{T}$ is a Σ-*maximal* tree on $\mathcal{P}$ if $\mathcal{T}$ has a limit length, is according to Σ and the second case of Definition 3.1.1 above holds for $\mathcal{T}$. Notice that maximality of $\mathcal{T}$ depends on Σ. Also, if $\mathcal{T}$ is a tree then we let $\mathcal{T}^-$ be $\mathcal{T}$ without its last model if it exists and $\mathcal{T}$ otherwise.

The next definitions describe when a stack is according to the short tree strategy component of Σ.

Definition 3.1.2 *We let*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \mathcal{U}_\alpha : \alpha < \eta) \in \mathrm{dom}(\Sigma^{stc})$$

if there is $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta) \in \mathrm{dom}(\Sigma)$ *such that* $\vec{\mathcal{U}}$ *is the same as* $\vec{\mathcal{T}}$ *except it doesn't have the maximal branches of* $\vec{\mathcal{T}}$*; more precisely,*

1. *For every* $\alpha < \eta$, $\mathcal{N}_\alpha = \mathcal{M}_\alpha$.

2. *For every* $\alpha < \eta$ *such that* $\pi^{\vec{\mathcal{T}}\restriction\alpha}$*-exists,*

$$\mathcal{U}_\alpha = \begin{cases} \mathcal{T}_\alpha^- & : \mathcal{T}_\alpha \text{ is } \Sigma_{\mathcal{N}_\alpha, \vec{\mathcal{T}}\restriction\alpha}\text{-maximal,} \\ \mathcal{T}_\alpha & : \text{otherwise.} \end{cases}$$

3. *Letting* α *be the least, if it exists, such that* $\pi^{\vec{\mathcal{T}}\restriction\alpha}$*-doesn't exist, for all* $\beta \geq \alpha - 1$, $\mathcal{U}_\beta = \mathcal{T}_\beta$[1].

4. *There are finitely many* α *such that* $\mathcal{U}_\alpha \neq \mathcal{T}_\alpha$.

5. *Either* η *is a limit ordinal or* $\mathcal{T}_{\eta-1}$ *has a limit length.*

If $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *are as above then we write* $\vec{\mathcal{U}} = \vec{\mathcal{T}}^{sc}$ *and say that* $\vec{\mathcal{U}}$ *is the short component of* $\vec{\mathcal{T}}$.

Finally, we define the domain of the short tree component of Σ on generalized stacks.

Definition 3.1.3 (The short tree component of a strategy: the domain) *We let the generalized stack*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \vec{\mathcal{U}}_\alpha, \mathcal{Q}_\alpha, E_\alpha : \alpha < \eta) \in \mathrm{dom}(\Sigma^{stc})$$

if there is a generalized stack $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, \mathcal{R}_\alpha, F_\alpha : \alpha < \eta) \in \mathrm{dom}(\Sigma)$ *such that* $\vec{\mathcal{U}}$ *is the same as* $\vec{\mathcal{T}}$ *except it doesn't have the maximal branches of* $\vec{\mathcal{T}}$*; more precisely,*

1. *for every* $\alpha < \eta$, $\mathcal{N}_\alpha = \mathcal{M}_\alpha$, $\mathcal{Q}_\alpha = \mathcal{R}_\alpha$ *and* $E_\alpha = F_\alpha$,

2. *for every* $\alpha < \eta$ *such that* $\vec{\mathcal{U}}_\alpha = \vec{\mathcal{T}}_\alpha^{sc}$,

3. *there are finitely many* α *such that* $\vec{\mathcal{U}}_\alpha \neq \vec{\mathcal{T}}_\alpha$, *and*

4. *either* η *is a limit ordinal or the last normal component of* $\vec{\mathcal{T}}_{\eta-1}$ *has a limit length.*

[1] Notice that α is necessarily a successor ordinal.

If $\vec{\mathcal{T}}$ *and* $\vec{\mathcal{U}}$ *are as above then we write* $\vec{\mathcal{U}} = \vec{\mathcal{T}}^{sc}$ *and say that* $\vec{\mathcal{U}}$ *is the short component of* $\vec{\mathcal{T}}$*.*

Conditions (4) in 3.1.3 and (3) in 3.1.3 ensure that if the relevant stacks are of limit length, we can take the direct limit. We will not be concerned with quasi-limits (cf. [19]) here.

The next definition defines the short tree component of Σ. Recall that if $\vec{\mathcal{T}}$ is a stack of iteration trees then $\delta(\vec{\mathcal{T}})$ is the sup of the generators of $\vec{\mathcal{T}}$. It can be defined inductively on the number of normal components of $\vec{\mathcal{T}}$ (see Definition 1.15 of [10]).

Definition 3.1.4 (The short tree component of a strategy) *Given a generalized stack*

$$\vec{\mathcal{U}} = (\mathcal{N}_\alpha, \mathcal{U}_\alpha, \mathcal{Q}_\alpha, E_\alpha : \alpha < \eta) \in \mathrm{dom}(\Sigma^{stc}),$$

letting $\vec{\mathcal{T}}$ *be such that* $\vec{\mathcal{T}}^{sc} = \vec{\mathcal{U}}$ *and* $b = \Sigma(\vec{\mathcal{T}})$*, we let*

$$\Sigma^{stc}(\vec{\mathcal{U}}) = \begin{cases} b & : \ \pi_b^{\vec{\mathcal{T}}} \text{ doesn't exists or } \pi_b^{\mathcal{T}}(\delta) > \delta(\vec{\mathcal{T}}), \\ \mathcal{M}_b^{\vec{\mathcal{T}}} & : \ \text{otherwise.} \end{cases}$$

Thus, $\Sigma^{stc}(\vec{\mathcal{T}})$ either returns the value of $\Sigma(\vec{\mathcal{T}})$ or $\mathcal{M}_b^{\vec{\mathcal{T}}}$ where $b = \Sigma(\vec{\mathcal{T}})$. From now on, we will use this notation even when Σ is a partial iteration strategy.

Notice the similarity with the short tree iterability for suitable mice in the context of core model induction or in the context of HOD analysis and Σ^{stc}. If $\mathcal{P}$ is a Σ_1^2-suitable premouse and Σ is fullness preserving iteration strategy for $\mathcal{P}$,[2] Σ^{stc} is just the short tree iterability strategy of $\mathcal{P}$.

3.2 The short tree game and short tree strategy mice

In order to define *short tree strategy mice*, we will need to define *short tree strategy* in a way that it is independent of a particular strategy. The short tree strategies are winning strategies for player II in the *short tree iteration game* introduced below. It will not be hard to see that if Σ is a strategy then Σ^{stc} is a short tree strategy.

[2] Here Σ_1^2 and fullness preservation are relative to an AD^+-model.

Definition 3.2.1 (The normal short tree game, $\mathcal{G}_k^{nst}(\mathcal{P},\lambda)$) *Suppose $\mathcal{P}$ is a hod-like lsa type ϕ-indexed lsp. Let $\kappa = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$ and $\delta = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$. Then the short tree game $\mathcal{G}_k^{nst}(\mathcal{P},\lambda)$ is a two player game on $\mathcal{P}$ played as follows. Just like in $\mathcal{G}_k(\mathcal{P},\lambda)$, I plays the successor steps in $\mathcal{G}_k^{nst}(\mathcal{P},\lambda)$ according to the rules of $\mathcal{G}_k(\mathcal{P},\lambda)$. Let then $\mathcal{T}$ be an iteration tree produced by a run of $\mathcal{G}_k^{nst}(\mathcal{P},\lambda)$. Suppose $\mathcal{T}$ has a limit length. Then II has the following two options:*

Option 1. *$\pi^{\mathcal{T},b}$ exists, $\pi^{\mathcal{T},b}(\kappa) < \delta(\mathcal{T})$ and there is $\mathcal{M}$ such that*

1. *$\mathcal{M}$ is ϕ-indexed,*
2. *$\mathcal{M}(\mathcal{T}) \trianglelefteq_{hod} \mathcal{M}$ and $\mathcal{M}$ is λ-iterable above $\delta(\mathcal{T})$ and*
3. *$\mathcal{M}$ is a hod-like lsa type lsp such that $\delta^{\mathcal{M}} = \delta(\mathcal{T})$.*

Option 2. *Otherwise.*

If Option 1 holds then II may choose, but is not required, to play $\mathcal{M}$ satisfying the above clauses. If II plays $\mathcal{M}$ then the game stops. In all other cases, II must play a cofinal branch b such that either $\pi_b^{\mathcal{T}}$ doesn't exist or $\pi_b^{\mathcal{T}}(\delta) > \delta(\mathcal{T})$.

Suppose p is a run of $\mathcal{G}_k^{nst}(\mathcal{P},\lambda)$. II wins p if all models in p are ϕ-indexed and well-founded.

If II plays according to Option 1 then we say that II plays a model (rather than a branch) or that II's move is a model and etc. Notice that if $\mathcal{T}$ is a tree satisfying hypothesis of Option 1 then for some node $\mathcal{Q}$ of $\mathcal{T}$, $\mathcal{P} \prec^{\mathcal{T},s} \mathcal{Q}$, $\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{Q}}$ exists, $\mathcal{Q}$ is a cutpoint of $\mathcal{T}$ and $\mathcal{T}_{\geq \mathcal{Q}}$ is a tree above $\pi^{\mathcal{T}}_{\mathcal{P},\mathcal{Q}}(\kappa)$.

Next, we introduce the version of the normal short tree game that has at most ω main rounds.

Definition 3.2.2 (The short tree game) *Suppose $\mathcal{P}$ is a hod-like lsa type ϕ-indexed lsp. The short tree game $\mathcal{G}_k^{st}(\mathcal{P},\lambda,\eta)$ is an iteration game that has at most ω main rounds each of which consists of a run of the usual (k,λ,η)-iteration game (see Definition 2.5.5) with the following exceptions.*

1. *Suppose $\mathcal{M}$ is a model at the beginning of the main round of some run of the game and $\vec{\mathcal{T}}$ is a run of (k,λ,η) on $\mathcal{M}$. Suppose $\mathcal{R}$ is a non-trivial terminal node in $\vec{\mathcal{T}}$. If $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}}$ exists then the largest irreducible initial segment of $\vec{\mathcal{T}}_{\geq\mathcal{R}}$ is played according to the rules of $\mathcal{G}^{nst}(\mathcal{R},\eta)$. If $\pi^{\vec{\mathcal{T}}}_{\mathcal{R}}$ doesn't exist then the largest irreducible initial segment of $\vec{\mathcal{T}}_{\geq\mathcal{R}}$ is played according to the rules of the usual iteration game.*

2. *If at any point during the run of a sub-round, II plays a model then I has to start a new main round on that model, and all main rounds are started in this fashion.*

Suppose p is a run of $\mathcal{G}^{st}_k(\mathcal{P}, \lambda, \eta)$. II wins p if all models in p are ϕ-indexed and well-founded. Additionally, if p has ω main rounds, then II wins.

Finally, we introduce the *un-dropping short tree game.*

Definition 3.2.3 (The un-dropping short tree game) *Suppose $\mathcal{P}$ is a hod-like lsa type ϕ-indexed lsp. The un-dropping short tree game on $\mathcal{P}$, $\mathcal{G}^{ust}_k(\mathcal{P}, \lambda, \eta, \alpha)$, is an iteration game that has at most ω main rounds each of which consists of a run of $\mathcal{G}^u_k(\mathcal{P}, \lambda, \eta, \alpha)$ (cf. Definition 2.7.3) with the following exceptions.*

1. *Suppose $\mathcal{M}$ is a model at the beginning of a main round of some play and $\vec{\mathcal{T}}$ is a run of $\mathcal{G}^u_k(\mathcal{P}, \lambda, \eta, \alpha)$. Suppose $\vec{\mathcal{T}} = (\mathcal{M}_\varsigma, \vec{\mathcal{T}}_\varsigma, \mathcal{Q}_\varsigma, E_\varsigma : \varsigma < \eta)$. Then for each $\varsigma < \eta$, $\vec{\mathcal{T}}_\varsigma$ is played according to the rules of $\mathcal{G}^{st}(\mathcal{M}_\varsigma, \eta, \alpha)$.*

2. *If at any point during the run of a sub-round, II plays a model then I has to start a new main round on that model, and all main rounds are started in this fashion.*

Suppose p is a run of $\mathcal{G}^{ust}_k(\mathcal{P}, \lambda, \eta, \alpha)$. II wins p if all models in p are ϕ-indexed and well-founded. Additionally, if p has ω main rounds, then II wins.

Definition 3.2.4 (Short tree strategy) *Suppose $\mathcal{P}$ is an lsa type ϕ-indexed lsp. We say Λ is a short tree (λ, η, α)-strategy for $\mathcal{P}$ if Λ is a wining strategy for II in $\mathcal{G}^{ust}_k(\mathcal{P}, \lambda, \eta, \alpha)$.*

Suppose now $\mathcal{P}$ and Λ are as in Definition 3.2.4. We let $b(\Lambda)$ be the set of all $\vec{\mathcal{T}} \in \mathrm{dom}(\Lambda)$ such that $\vec{\mathcal{T}}$ has a last normal component of limit length and $\Lambda(\vec{\mathcal{T}})$ is a cofinal wellfounded branch of $\vec{\mathcal{T}}$. Let $m(\Lambda) = \mathrm{dom}(\Lambda) - b(\Lambda)$. We call $m(\Lambda)$ the *model component of* Λ. Given $\vec{\mathcal{U}} \in \mathrm{dom}(\Lambda)$ such that the last component of $\vec{\mathcal{U}}$ has a limit length, we let

$$\mathcal{M}(\Lambda, \vec{\mathcal{U}}) = \begin{cases} \mathcal{M}^{\vec{\mathcal{U}}}_b & : \Lambda(\vec{\mathcal{U}}) = b \\ \Lambda(\vec{\mathcal{U}}) & : \text{otherwise.} \end{cases}$$

Remark 3.2.5 *In many situations, it is expected that winning* $\mathcal{G}_k^{st}(\mathcal{M}, \kappa, \lambda)$ *must be easy for II: II wins it as soon as she plays infinitely many models. However, we will be interested in strategies for II that have certain fullness preservation properties. For instance, suppose* $\mathcal{M}$ *is just a suitable mouse in the sense of* $L(\mathbb{R})$. *Suppose* Λ *is strategy for II in* $\mathcal{G}_k^{nst}(\mathcal{M}, \omega_1)$ *such that whenever* $\mathcal{T}$ *is a tree according to* Λ *then*

1. *if* $\mathcal{T} \in b(\Lambda)$, $b = \Lambda(\mathcal{T})$ *and* $\pi_b^{\mathcal{T}}$ *exists then* $\mathcal{M}_b^{\mathcal{T}}$ *is* $(\Sigma_1^2)^{L(\mathbb{R})}$*-full and*

2. *if* $\mathcal{T} \in m(\Lambda)$ *and* $\mathcal{N} = \Lambda(\mathcal{T})$ *then* $\mathcal{N}$ *is suitable in the sense of* $L(\mathbb{R})$

then Λ *is in fact a "short tree iterability strategy" in the sense of* $L(\mathbb{R})$, *it is* $L(\mathbb{R})$*-fullness preserving. Such strategies are difficult to construct, and in our current situation, we will be interested in a notion of fullness preservation with respect to a much more complicated pointclass than* $(\Sigma_1^2)^{L(\mathbb{R})}$.

3.3 Hull and branch condensation for short tree strategy

The goal of this section is to introduce *hull condensation* for short tree strategies. Hull condensation for iteration strategies was introduced in Definition 1.31 of [10]. It is an important property that is used to show that when doing hod pair constructions no discrepancies arise due to coring down. Thus if $\vec{\mathcal{T}}$ is according to a strategy with hull condensation and $\vec{\mathcal{U}}$ is a *hull* of $\vec{\mathcal{T}}$ (cf. Definition 3.3.3) then it is according to the strategy.

The difference between strategies and short tree strategies is that short tree strategies have a model component, and this difference causes some complications when trying to outright generalize hull condensation. The resulting definition is just simply too strong. Our solution is based on our indexing scheme Definition 3.6.2. In short tree strategy mice, we only index branches of a certain kind of iterations. We introduce such iterations.

First we define the *unambiguous stacks* which are essentially the stacks whose branches are easy to guess.

Definition 3.3.1 (Unambiguous stacks) *Suppose* $\mathcal{P}$ *is a hod-like lsa type lsp and* $\vec{\mathcal{T}}$ *is a run of* $\mathcal{G}_k^{st}(\mathcal{P}, \kappa, \lambda)$ *without any main rounds. We say* $\vec{\mathcal{T}}$ *is unambiguous if either, letting* $md^{\vec{\mathcal{T}}} = (\mathcal{R}_i, \vec{\mathcal{T}}_i : i \leq k)$ *be the sequence of main drops of* $\vec{\mathcal{T}}$, $k \geq 1$ *or one of the following holds:*

1. *There is a linear closed unbounded* $C \subseteq ntn(\vec{\mathcal{T}})$.

2. *$\vec{\mathcal{T}}$ has a last normal component $\mathcal{T}$ such that $lh(\mathcal{T})$ is a successor ordinal.*

3. *Clauses 1 and 2 above fail, and $\vec{\mathcal{T}}$ has a last normal component of limit length such that letting $\mathcal{T}$ be this normal component, one of the following conditions hold:*

   (a) *$\pi^{\mathcal{T},b}$ doesn't exist.*

   (b) *$\pi^{\mathcal{T},b}$ exists and for some cutpoint $\mathcal{S}$ of $\mathcal{T}$ and some $\eta < o(\mathcal{S})$ such that $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1} < \eta$, $\mathcal{T}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta}$ (see Definition 2.3.6) and is above η.*

   (c) *Clauses 3.a and 3.b fail, there is a cutpoint $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{T}_{\geq \mathcal{S}}$ is above $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}$, and there is $\mathcal{Q} \trianglelefteq \mathcal{J}(\mathcal{M}(\mathcal{T}))$ such that $\mathcal{Q} \vDash$ "$\delta(\mathcal{T})$ is Woodin" and $rud(\mathcal{Q}) \vDash$ "$\delta(\mathcal{T})$ isn't Woodin".*

Recall the notation $\mathcal{M}^+(\mathcal{T})$ from Notation 2.2.1.

Definition 3.3.2 (Finite stack) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp. We say $(\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \mathcal{T}_1, ..., \mathcal{P}_{n-1}, \mathcal{T}_{n-1}, \mathcal{P}_n)$ is a finite stack on $\mathcal{P}$ of length $n+1$ if*

1. *$n < \omega$ and $\mathcal{P}_0 = \mathcal{P}$,*

2. *For $i < n-1$, $\mathcal{T}_i$ is a normal ambiguous tree on $\mathcal{P}_i$ and $\mathcal{P}_{i+1} = \mathcal{M}^+(\mathcal{T}_i)$,*

3. *$\vec{\mathcal{U}}$, if it is defined, is a stack such that for some $\alpha + 1 < \lambda^{\mathcal{P}_n}$, $\vec{\mathcal{U}}$ is based on $\mathcal{P}_n(\alpha+1)$ and $\vec{\mathcal{U}}$ has a last normal component of limit length.*

4. *$\mathcal{T}_{n-1}$ is either a normal ambiguous tree on $\mathcal{P}_{n-1}$ and $\mathcal{P}_n = \mathcal{M}^+(\mathcal{T}_{n-1})$ or $\mathcal{P}_n$ is the last model of $\mathcal{T}_{n-1}$ and $\pi^{\mathcal{T}_{n-1}}$-exists.*

The iterations that we will consider in short tree strategy mice are stacks of length 2. We define hull condensation for such stacks.

Definition 3.3.3 (Hull of a stack) *Suppose $\mathcal{M}$ and $\mathcal{N}$ are hod-like lsa type lsp and $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \mathcal{T}_\alpha : \alpha < \eta)$ and $\vec{\mathcal{U}} = (\mathcal{M}_\beta, \mathcal{T}_\beta : \beta < \nu)$ are stacks on $\mathcal{M}$ and $\mathcal{N}$ respectively such that $\vec{\mathcal{T}}$ is based on $\mathcal{M}^b$ and $\vec{\mathcal{U}}$ is based on $\mathcal{N}^b$. We say $(\mathcal{N}, \vec{\mathcal{U}})$ is a hull of $(\mathcal{M}, \vec{\mathcal{T}})$ if there are (i) an embedding $\pi : \mathcal{N} \rightarrow_{\Sigma_1} \mathcal{M}$, (ii) an order preserving map $\sigma : lh(\vec{\mathcal{U}}) \rightarrow lh(\vec{\mathcal{T}})$ (iii) a sequence $\vec{\tau} = (\tau_\beta : \beta < \nu)$ of order preserving maps $\tau_\beta : lh(\mathcal{U}_\beta) \rightarrow lh(\mathcal{T}_{\sigma(\beta)})$ and (iv) a sequence of Σ_1-elementary embeddings $\vec{\pi} = (\pi^\beta_\alpha : \alpha < lh(\mathcal{U}_\beta) \wedge \beta < \nu)$ such that letting $\leq_{\vec{\mathcal{T}}}$ be the tree order of $\vec{\mathcal{T}}$ and $\leq_{\vec{\mathcal{U}}}$ be the tree order of $\vec{\mathcal{U}}$ then*

1. *for all* (γ,α) *and* (ξ,β) *such that* $\gamma,\xi<\nu$, $\alpha<lh(\mathcal{U}_\gamma)$ *and* $\xi<lh(\mathcal{U}_\xi)$, $(\gamma,\alpha)\leq_{\vec{\mathcal{U}}}(\xi,\beta)\leftrightarrow(\sigma(\gamma),\tau_\gamma(\alpha))\leq_{\vec{\mathcal{T}}}(\sigma(\xi),\tau_\xi(\beta))$ *and*

$$[(\gamma,\alpha),(\xi,\beta)]_{\vec{\mathcal{U}}}\cap D^{\vec{\mathcal{U}}}=\emptyset\leftrightarrow[(\sigma(\gamma),\tau_\gamma(\alpha)),(\sigma(\xi),\tau_\xi(\beta)]_{\vec{\mathcal{T}}}\cap D^{\vec{\mathcal{T}}}=\emptyset,$$

2. *for every* $\beta<\nu$ *and* $\alpha<lh(\mathcal{U}_\beta)$, $\pi^\beta_\alpha:\mathcal{M}^{\mathcal{U}_\beta}_\alpha\to\mathcal{M}^{\mathcal{T}_{\sigma(\beta)}}_{\tau_\beta(\alpha)}$ *and* $\pi^\beta_\alpha(E^{\mathcal{U}}_\alpha)=E^{\mathcal{T}_{\sigma(\beta)}}_{\tau_\beta(\alpha)}$,

3. *for every* $\gamma<\nu$ *and* $\beta<\alpha<lh(\mathcal{U}_\gamma)$, $\pi^\gamma_\alpha\restriction lh(E^{\mathcal{U}_\gamma}_\beta)+1=\pi^\gamma_\beta\restriction lh(E^{\mathcal{U}_\gamma}_\beta)+1$,

4. *for every* $\gamma<\nu$, *if* $\alpha\leq_U\beta$ *and* $[\alpha,\beta]_{U_\gamma}\cap D^{\mathcal{U}_\gamma}=\emptyset$ *then* $\pi^\gamma_\beta\circ\pi^{\mathcal{U}_\gamma}_{\alpha,\beta}=\pi^{\mathcal{T}_{\sigma(\gamma)}}_{\tau_\gamma(\alpha),\tau_\gamma(\beta)}\circ\pi^\gamma_\alpha$,

5. *for every* $\gamma<\nu$, *if* $\beta=pred_{U_\gamma}(\alpha+1)$ *then* $\tau_\gamma(\beta)=pred_{T_{\sigma(\gamma)}}(\tau_\gamma(\alpha+1))$ *and* $\pi^\gamma_{\alpha+1}([a,f]_{E^{\mathcal{U}_\gamma}_\alpha})=[\pi^\gamma_\alpha(a),\pi_\beta(f)]_{E^{\mathcal{T}_{\sigma(\gamma)}}_{\tau_\gamma(\alpha)}}$,

6. $(0,0)\leq_{\vec{\mathcal{T}}}(\sigma(0),\tau_0(0))$, $[(0,0),(\sigma(0),\tau_0(0))]\cap D^{\vec{\mathcal{T}}}=\emptyset$, *and* $\pi^0_0=\pi^{\vec{\mathcal{T}}}_{(0,0),(\sigma(0),\tau_0(0))}\circ\pi$.

We say $(\pi,\sigma,\vec{\tau},\vec{\pi})$ *witnesses that* $(\mathcal{N},\vec{\mathcal{U}})$ *is a hull of* $(\mathcal{M},\vec{\mathcal{T}})$.

Definition 3.3.4 (Hull of a stack of length 2) *Suppose* $\mathcal{M}$ *is a hod-like lsp and* $u=(\mathcal{M},\mathcal{U},\mathcal{M}_1,\vec{\mathcal{W}})$ *and* $t=(\mathcal{M},\mathcal{T},\mathcal{M}_2,\vec{\mathcal{S}})$ *are two stacks of length 2. We say* $(\mathcal{M},u)$ *is a hull of* $(\mathcal{M},t)$ *if there are (i) a pair* $(\pi,\vec{\pi})$ *witnessing that* $(\mathcal{M},\mathcal{U})$ *is a hull of* $(\mathcal{M},\mathcal{T})$ *and (iii) a sequence* $(\sigma,\tau,\vec{k},\vec{j})$ *witnessing that* $(\mathcal{M}_1,\vec{\mathcal{W}})$ *is a hull of* $(\mathcal{M}_2,\vec{\mathcal{S}})$ *such that if* $\pi^{\mathcal{U}}$ *exists then* $(\pi,\vec{\pi})^\frown(\sigma,\tau,\vec{k},\vec{j})$ *witnesses that* $(\mathcal{M},\mathcal{U}^\frown\vec{\mathcal{W}})$ *is a hull of* $(\mathcal{M},\mathcal{T}^\frown\vec{\mathcal{S}})$ *and if* $\pi^{\mathcal{U}}$ *doesn't exist then, letting*

1. $\sigma^b=\sigma\restriction(\mathcal{M}_1)^b$ *and*

2. *for* $(\gamma,\alpha)\in lh(\vec{\mathcal{W}})\times lh(\mathcal{W}_\gamma)$,

$$j^{\gamma,b}_\alpha=\begin{cases}j^\gamma_\alpha\restriction(\mathcal{M}^{\mathcal{W}_\gamma}_\alpha)^b & :\pi^{\vec{\mathcal{W}}_{\leq\mathcal{M}^{\vec{\mathcal{W}}}_{\gamma,\alpha}},b}\text{ exists}\\ j^\gamma_\alpha & :\text{otherwise}\end{cases}$$

then for any $(\gamma,\beta)\in lh(\vec{\mathcal{W}})\times\mathrm{lh}(\mathcal{W}_\gamma)$ *letting* $m=\pi^{\vec{\mathcal{W}}_{\leq\mathcal{M}^{\vec{\mathcal{W}}}_{\gamma,\alpha}},b}$ *and* $n=\pi^{\vec{\mathcal{S}}_{\leq\mathcal{M}^{\vec{\mathcal{S}}}_{\tau(\gamma),k_\gamma(\alpha)}},b}$ *(if they exist)*

$$n\circ\sigma^b\circ\pi^{\mathcal{U},b}=j^{\gamma,b}_\alpha\circ m\circ\pi^{\mathcal{U},b}$$

To finally define hull condensation for short tree strategy, we need to introduce a few more definitions. Suppose $(\mathcal{P}, \Sigma)$ is a pair such that $\mathcal{P}$ is a hod-like lsa type lsp and Σ is a short tree strategy for $\mathcal{P}$. First we introduce two sorts of iterates of $(\mathcal{P}, \Sigma)$, $I^b(\mathcal{P}, \Sigma)$ and $I(\mathcal{P}, \Sigma)$. To start, we let $max(\mathcal{P}, \Sigma)$ consist of pairs $(\vec{\mathcal{T}}, \mathcal{Q})$ such that $\vec{\mathcal{T}}$ is according to Σ, $\mathcal{Q}$ is the last model of $\vec{\mathcal{T}}$ and if $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq n)$ then for some ordinal γ, $\vec{\mathcal{T}}_n$ has a last normal component of length $\gamma + 1$ and $\vec{\mathcal{T}}_n^- \in m(\Sigma_{\mathcal{M}_n, \oplus_{i<n} \vec{\mathcal{T}}_i})$. Thus, $max(\mathcal{P}, \Sigma)$ is the set of maximal Σ-iterates of $\mathcal{P}$.

Definition 3.3.5 ($I^b(\mathcal{P}, \Sigma)$ and $I(\mathcal{P}, \Sigma)$) *Suppose $(\mathcal{P}, \Sigma)$ is a pair such that $\mathcal{P}$ is an lsa type lsp and Σ is a short tree strategy for $\mathcal{P}$. We then let*

$$I^b(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \vec{\mathcal{T}} \textit{ is according to } \Sigma, \mathcal{Q} \textit{ is the last model of } \vec{\mathcal{T}} \textit{ and } \pi^{\vec{\mathcal{T}}, b} \textit{ exists}\},$$

$$I(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \vec{\mathcal{T}} \textit{ is according to } \Sigma, \mathcal{Q} \textit{ is the last model of } \vec{\mathcal{T}} \textit{ and if } \vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq n) \textit{ then either } \pi^{\vec{\mathcal{T}}_n} \textit{ exists or } \vec{\mathcal{T}} \in max(\mathcal{P}, \Sigma)\},$$

From now on, we fix a natural coding of subsets of HC by sets of reals. We call such a coding $Code$.

Definition 3.3.6 *Suppose $(\mathcal{P}, \Sigma)$ is a pair such that $\mathcal{P}$ is a hod-like lsa type lsp and Σ is a short tree strategy for $\mathcal{P}$. We then let*

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \exists \mathcal{R}((\vec{\mathcal{T}}, \mathcal{R}) \in I^b(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \trianglelefteq_{hod} \mathcal{R}^b)\},$$

and

$$\Gamma^b(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists (\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}))\}.$$

Definition 3.3.7 (Hull condensation) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp and Σ is a short tree strategy for $\mathcal{P}$. We say Σ has hull condensation if*

1. *for all $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has hull condensation, and*

2. *whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $u = (\mathcal{Q}, \mathcal{U}, \mathcal{Q}_1, \vec{\mathcal{W}})$ and $t = (\mathcal{Q}, \mathcal{T}, \mathcal{Q}_2, \vec{\mathcal{W}}')$ are two stacks of length 2 on $\mathcal{Q}$ such that t is according to $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ and $(\mathcal{Q}, u)$ is a hull of $(\mathcal{Q}, t)$ then u is according to $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$.*

Next we introduce branch condensation for short tree strategies. We will need this notion in the definition of hod mice (see Definition 3.9.3).

Definition 3.3.8 (Branch condensation for short tree strategies) *Suppose* $(\mathcal{P}, \Sigma)$ *is such that* $\mathcal{P}$ *is a hod-like lsa type lsp and* Σ *is a short tree strategy for* $\mathcal{P}$*. We say* Σ *has branch condensation if whenever* $(\vec{\mathcal{T}}, \mathcal{Q}, \vec{\mathcal{U}}, \mathcal{R}, \pi, \mathcal{S}, c, \alpha, \beta)$ *is such that*

1. $(\vec{\mathcal{T}}, \mathcal{Q}), (\vec{\mathcal{U}}, \mathcal{R}) \in I^b(\mathcal{P}, \Sigma)$,
2. $\alpha < \lambda^{\mathcal{R}^b}$ *and* $\mathcal{S}$ *is a tree according to* $\Sigma_{\mathcal{R},\vec{\mathcal{U}}}$ *on* $\mathcal{R}^b$ *based on* $\mathcal{R}(\alpha+1)$ *such that it has limit length and is above* $\delta^{\mathcal{R}}_\alpha$,
3. c *is a branch of* $\mathcal{S}$ *such that* $\pi^{\mathcal{S}}_c$ *exists, and*
4. $\pi : \mathcal{S} \to \mathcal{Q}(\beta)$ *and* $\pi^{\vec{\mathcal{T}},b} = \pi \circ \pi^{\mathcal{S}}_c \circ \pi^{\vec{\mathcal{U}},b}$

then $c = \Sigma_{\mathcal{R},\vec{\mathcal{U}}}(\mathcal{S})$.

3.4 Lsa type pair

Suppose $\mathcal{P}$ is a hod-like lsa type lsp and suppose Λ is a short tree strategy for $\mathcal{P}$. We would like to introduce the notion of a short tree premouse and in particular, Λ-premouse. The main technical problem is that we do not have a reasonable notion of condensation for short tree strategies. In particular, if $\Lambda = \Sigma^{stc}$ for some strategy Σ, then it may well be that there is a tree $\mathcal{T}$ on $\mathcal{P}$ such that if $b = \Sigma(\mathcal{T})$ then b is non-dropping and $\pi^{\mathcal{T}}_b(\delta) = \delta(\mathcal{T})$ yet there is a hull $\mathcal{U}$ of $\mathcal{T}$ such that if $c = \Sigma(\mathcal{U})$ then in fact $\pi^{\mathcal{U}}_c(\delta) > \delta(\mathcal{U})$. Thus, $\Lambda(\mathcal{T}) = \mathcal{M}^{\mathcal{T}}_b$ while $\Lambda(\mathcal{U}) = c$.

The above scenario is the main difficulty with defining short tree strategy mice. We have to find a particular indexing of short tree strategies, or rather carefully skip over "bad trees", in a way that when $\mathcal{T}$ above is "cored down" to $\mathcal{U}$ above then our indexing is still preserved. In particular, the branch of $\mathcal{U}$ cannot be added too early. The idea is to wait until branches or rather the $\mathcal{Q}$-structures are *certified*. Before we define short tree hybrids, however, we have to make a few definitions that will be useful to us in the future.

We will only consider short tree strategies Λ with the property that whenever $\vec{\mathcal{T}} \in \mathrm{dom}(\Lambda)$ is an unambiguous stack then $\Lambda(\vec{\mathcal{T}})$ is a branch. If Λ is a short tree strategy for $\mathcal{P}$ and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ according to Λ with last model $\mathcal{N}$ then we let $\Lambda_{\mathcal{N},\vec{\mathcal{T}}}$ be the short tree strategy of $\mathcal{N}$ induced by Λ, i.e., for every $\vec{\mathcal{U}}$ on $\mathcal{N}$, $\Lambda_{\mathcal{N},\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Lambda(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

Definition 3.4.1 (Faithful short tree strategy) *Suppose* $\mathcal{P}$ *is a hod-like lsa type lsp and* Λ *is a short tree* (κ, λ, η)*-strategy for* $\mathcal{P}$*. We say* Λ *is a faithful short tree*

(κ, λ, η)-strategy if whenever $\vec{\mathcal{T}} = (\mathcal{M}_i, \vec{\mathcal{T}}_i : i \leq k < \omega) \in \mathrm{dom}(\Lambda)$, and $\mathcal{R} \in tn(\vec{\mathcal{T}})$ then, letting $\vec{\mathcal{U}}$ be the largest initial segment of $\vec{\mathcal{T}}$ that is based on $\mathcal{R}$ and has no main rounds, then

1. *if $\vec{\mathcal{U}}$ is unambiguous then $\vec{\mathcal{U}} \in b(\Lambda_{\mathcal{R}, \vec{\mathcal{T}}_{\leq \mathcal{R}}})$,*
2. *if clause 3.c of Definition 3.3.1 holds for $\vec{\mathcal{U}}$ then letting $\mathcal{S}$ be the cutpoint node of $\vec{\mathcal{U}}$ witnessing clause 3.c of Definition 3.3.1 then $\Lambda_{\mathcal{S}, \vec{\mathcal{U}}_{\leq \mathcal{S}}}(\vec{\mathcal{U}}_{\geq \mathcal{S}})$ is a branch of $\vec{\mathcal{U}}$ such that $\mathcal{Q}(b, \vec{\mathcal{U}}_{\geq \mathcal{S}})$ exists and $\mathcal{Q}(b, \vec{\mathcal{U}}_{\geq \mathcal{S}}) \trianglelefteq \mathcal{J}(\mathcal{M}(\vec{\mathcal{U}}_{\geq \mathcal{S}}))$.*

In the next section we will need to consider short tree iteration strategies that are partial and their range consists of branches. The next definition introduces this notion.

Definition 3.4.2 (Short tree strategy without a model component) *Suppose $\mathcal{P}$ is a hod-like lsa type lsp. We say Λ is a partial short tree strategy for $\mathcal{P}$ if it is a partial winning strategy in $\mathcal{G}_k^{st}(\mathcal{P}, \omega_1, \omega_1, \omega_1)$. If Λ is a partial short tree strategy for $\mathcal{P}$ then we say it is without model component if $m(\Lambda) = \emptyset$.*

We can then also define faithful short tree strategies without model component.

Definition 3.4.3 (Lsa type pair) *We say $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair if $\mathcal{P}$ is a hod-like lsa type lsp and Λ is an $(\omega_1, \omega_1, \omega_1)$ faithful short tree strategy with hull condensation. We say $(\mathcal{P}, \Lambda)$ is a hod-like lsa type pair without model component if $\mathcal{P}$ is a hod-like lsa type lsp and Λ is an $(\omega_1, \omega_1, \omega_1)$ faithful short tree strategy without model component.*

3.5 $(\mathcal{P}, \Sigma)$-hod pair construction

Suppose that $(\mathcal{P}, \Sigma)$ is a hod-like lsa type pair. Below we describe a fully backgrounded construction that, if successful, constructs a Σ-iterate of $\mathcal{P}$. We say a (κ, λ)-extender E coheres Σ if $\mathcal{P} \in V_\kappa$, $V_\lambda \subseteq Ult(V, E)$ and $\pi_E(\Sigma) \restriction V_\lambda = \Sigma \restriction V_\lambda$.

Definition 3.5.1 ($(\mathcal{P}, \Sigma)$-coherent fully backgrounded constructions) *Suppose κ is an inaccessible cardinal and $(\mathcal{P}, \Sigma)$ is a hod-like lsa type pair such that Σ is a (κ, κ, κ)-short tree strategy. Then for $\eta < \kappa$, we say $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma \leq \eta))$ is the output of the $(\mathcal{P}, \Sigma)$-coherent fully backgrounded construction if the following holds.*

1. $\mathcal{M}_0 = \emptyset$.

2. $\mathcal{M}_\gamma$ *is a hod-like lsp such that in the comparison of* $\mathcal{P}$ *with* $\mathcal{M}_\gamma$, $\mathcal{M}_\gamma$ *doesn't move and the comparison results in a tree* $\mathcal{T}_\gamma$ *on* $\mathcal{P}$ *according to* Σ *such that either* $\mathcal{T}_\gamma$ *has a last model* $\mathcal{M}$ *such that* $\mathcal{M}_\gamma \trianglelefteq_{hod} \mathcal{M}$ *or* $\mathcal{M}_\gamma = \mathcal{M}(\mathcal{T}_\gamma)$.

3. *Suppose* $\gamma \leq \eta$ *is such that either* $\mathcal{T}_\gamma$ *has a last model or* $\mathcal{T}_\gamma \in b(\Sigma)$. *Let* $\mathcal{M}$ *be the last model of* $\mathcal{T}_\gamma$ *if it exists and otherwise, letting* $b = \Sigma(\mathcal{T}_\gamma)$, *let* $\mathcal{M} = \mathcal{M}_b^{\mathcal{T}_\gamma}$. *Let* ς *be such that* $\mathcal{M}_\gamma = \mathcal{M}|\varsigma$ *and suppose* $\mathcal{M}_\gamma = \mathcal{J}_\xi^{\vec{E},f}$. *Then the following statements hold.*

   (a) *If* $\mathcal{M}_\gamma = \mathcal{M}$ *then* $\gamma = \eta$.

   (b) *Suppose* $\mathcal{M}_\gamma \triangleleft \mathcal{M}$. *Suppose there is no pair* (F^*, F) *and an ordinal* $\zeta < \xi$ *such that* $F^* \in V_\kappa$ *is an extender over* V *cohering* Σ, F *is an extender over* $\mathcal{M}_\gamma$, $V_{\zeta+\omega} \subseteq Ult(V, F^*)$ *and*
   $$F = F^* \cap ([\zeta]^\omega \times \mathcal{J}_\xi^{\vec{E},f})$$
   such that $(\mathcal{J}_\xi^{\vec{E},f}, \in, \vec{E}, f, \tilde{F})$ *is a hod-like lsp (here* $\tilde{F}$ *is the amenable code of* F*). Then* $\mathcal{N}_\gamma = \mathcal{J}_1(\mathcal{M}_\gamma)$ *and* $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$.

   (c) *Again suppose* $\mathcal{M}_\gamma \triangleleft \mathcal{M}$ *but there is a pair* (F^*, F) *and an ordinal* ζ *satisfying the above conditions. Then if* $F \in \vec{E}^{\mathcal{M}}$ *then we let*
   $$\mathcal{N}_\gamma = (\mathcal{J}_\xi^{\vec{E},f}, \in, \vec{E}, f, \tilde{F})$$
   where $\tilde{F}$ *is the amenable code of* F. *Also,* $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$. *If* $F \notin \vec{E}^{\mathcal{M}}$ *then* $\gamma = \eta$ *and we stop the construction.*

   (d) *Again suppose* $\mathcal{M}_\gamma \triangleleft \mathcal{M}$ *and that* $\mathcal{M}|\varsigma$ *is an active* $\mathcal{J}$*-structure such that its last predicate codes a set* A *that is not an extender. Let then* $\mathcal{N}_\gamma = (\mathcal{M}_\gamma, A, \in)$ *and* $\mathcal{M}_{\gamma+1} = \mathcal{C}_\omega(\mathcal{N}_\gamma)$.

4. *Suppose* $\gamma \leq \eta$ *is such that* $\mathcal{T}_\gamma$ *is of limit length and* $\mathcal{T}_\gamma \notin b(\Sigma)$. *Then* $\gamma = \eta$.

Remark 3.5.2 *Notice that the constructions introduced in Definition 3.5.1 can be carried out even when* $(\mathcal{P}, \Sigma)$ *is a hod-like lsa type pair without model component. It can also be carried out when* Σ *is a partial strategy. Also, if the background universe has a distinguished extender sequence then we tacitly assume that the extenders appearing in the* $(\mathcal{P}, \Sigma)$*-coherent fully background construction come from this distinguished extender sequence.*

3.6 A short tree strategy indexing scheme

Our goal here is to introduce the notion of a *short tree strategy premouse* (*sts premouse*). As we mentioned in the previous section, the difficulty with doing this lies in the fact that maximal trees might "core down" to short trees and thus, creating indexing issues. The idea behind the solution presented here is to add a branch for a tree as soon as we see a certificate, which in our case will be a $\mathcal{Q}$-structure, that it is short. As the $\mathcal{Q}$-structures that we will be looking for are themselves sts premice, this inevitably leads to an induction.

Technically speaking $\mathcal{M}$ in Definition 3.6.1 should not be sp (strategy premouse) as $f^{\mathcal{N}}$ doesn't quite code an iteration strategy. Its domain consist of finite stacks of length 2. But recall the abuse of terminology proposed by Remark 2.2.2

Definition 3.6.1 (Unambiguous sp) *Suppose $\mathcal{M}$ is an sp over some self-well-ordered set X based on a hod-like lsa type lsp $\mathcal{P}$. We say $\mathcal{M}$ is unambiguous if $\mathcal{M}$ is closed under sharps and whenever $t = (\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \vec{\mathcal{U}}) \in \mathcal{M}$ is a finite stack on $\mathcal{P}$ of length 2 according to $\Sigma^{\mathcal{M}}$ such that either*

1. *$\vec{\mathcal{U}} = \emptyset$ and $\mathcal{M} \vDash$ "$\mathcal{T}_0$ is an unambiguous tree of limit length" or*
2. *$\vec{\mathcal{U}}$ is a nonempty stack of limit length*

then $t \in \mathrm{dom}(\Sigma^{\mathcal{M}})$. We say $\mathcal{M}$ is ambiguous if it is not unambiguous.

The next definition introduces an indexing scheme that we will use to define short tree premice. The indexing scheme only defines the strategy on certain carefully chosen stacks. It turns out that this much information is enough to extend the strategy on all stacks (see Chapter 6). In the next two definitions, instead of explicitly writing what ψ says, we indicate the impact that it has on the structures satisfying it. We leave it to the reader to extract the actual formula from our description.

Definition 3.6.2 (ϕ-sts indexing scheme) *Suppose $\psi(x)$ and $\phi(x, y)$ are two formulas in the language of sp. We say ψ is a ϕ-sts indexing scheme for ϕ if whenever X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp and $\mathcal{N}$ is an sp over X based on $\mathcal{P}$ then $\mathcal{N} \vDash \psi[c]$ if and only if*

1. *$\mathcal{N}$ is closed under sharps,*
2. *$\mathcal{N} \vDash$ "$\Sigma^{\mathcal{N}}$ is a partial faithful short tree strategy without model component",*

3. *for some finite sequence* $t = (\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \vec{\mathcal{U}}) \in \mathcal{N}$ *on* $\mathcal{P}$ *of length 2 such that* t *is according to* $\Sigma^{\mathcal{N}}$*,* $lh(\mathcal{T}_0)$ *is not of measurable cofinality, and* $lh(\vec{\mathcal{U}})$[3] *is not of measurable cofinality,* $c = \mathcal{J}_\omega(t)$*,*

4. *letting* $t = (\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \vec{\mathcal{U}})$ *be as in clause 3 above, the following conditions hold.*

   (a) *There is* (ν, ξ) *such that letting* $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{U}_\gamma : \gamma < \eta))$ *be the output of the* $(\mathcal{P}, \Sigma^{\mathcal{N}})$*-coherent fully backgrounded construction of* $\mathcal{N}$ *in which extenders used have critical points* $> \nu$ *(see Definition 3.5.1),* $\mathcal{U}_\xi = \mathcal{T}_0$*.*

   (b) *If* $\mathcal{N}$ *is ambiguous*[4] *then* t *is the* $\mathcal{N}$*-least stack on* $\mathcal{P}$ *satisfying clause 4.a and witnessing that* $\mathcal{N}$ *is ambiguous.*

   (c) *If* $\mathcal{N}$ *is unambiguous, then* $\Sigma^{\mathcal{N}}(\mathcal{T}_0)$ *is undefined and letting* (ν, ξ) *be the least witnessing clause 4.a above,* $\mathcal{N} \models$ *"there is a unique cofinal well-founded branch* $b \in \mathcal{N}$ *of* $\mathcal{T}_0$ *such that* $\phi[\mathcal{T}_0, b]$ *holds"*

Notice that ψ is uniquely determined by ϕ. The meaning of clause 4 is as follows. Clause 4a implies that the domain of the strategy consist of stacks $(\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \vec{\mathcal{U}})$ of length 2 such that $\mathcal{T}_0$ is a tree appearing in the $(\mathcal{P}, \Sigma^{\mathcal{N}})$-coherent fully backgrounded construction. It is then required that $\vec{\mathcal{U}}$ be based on $\mathcal{P}_1^b$. Clause 4b says that for unambiguous stacks we use the standard indexing scheme. Clause 4c says that for ambiguous stacks indexing branches that have property ϕ, which we want to say that "there is a certified $\mathcal{Q}$-structure". This is done in Definition 3.8.2.

Definition 3.6.3 (Sts ϕ-premouse) *Suppose* X *is a self-well-ordered set,* $\mathcal{P} \in X$ *is a hod-like lsa type lsp and* $\phi(x, y)$ *is a formula in the language of sp. Then* $\mathcal{M}$ *is an sts* ϕ*-premouse over* X *based on* $\mathcal{P}$ *if* $\mathcal{M}$ *is an sp over* $\mathcal{P}$ *with an indexing scheme* ψ *where* ψ *is the* ϕ*-sts indexing scheme.*

If $\phi(x, y) =$ "$0 = 1$" then we say $\mathcal{M}$ has a trivial indexing scheme and also say that $\mathcal{M}$ is a trivial sts premouse. Notice that in a trivial sts ambiguous trees do not have branches.

[3] By this, we mean the sum of the lengths of the normal components of $\vec{\mathcal{U}}$.

[4] This implies that $\vec{\mathcal{U}} = \emptyset$ and $\mathcal{T}_0$ is $\mathcal{N}$-ambiguous.

3.7 Authentic finite stacks

Suppose $(\mathcal{P}, \Sigma)$ is a hod-like lsa type pair. Suppose $\mathcal{T}$ is a tree on $\mathcal{P}$ according to Σ such that $\pi^{\mathcal{T},b}$ exists and $\mathcal{M}^+(\mathcal{T}) \vDash$ "$\delta(\mathcal{T})$ is a Woodin cardinal". When defining short tree strategy mice, we will be faced with the following question? How can we guess the iterations of $\mathcal{M}^+(\mathcal{T})$ that are according to $\Sigma_{\mathcal{M}^+(\mathcal{T}),\mathcal{T}}$? In this section, we present an authentication process that allows us to guess the correct iterations of $\mathcal{M}^+(\mathcal{T})$.

The main technical object used in our authentication process is $s(\vec{\mathcal{T}}, \xi)$ introduced in Definition 2.6.7. We start by recalling it. Suppose $\mathcal{P}$ is a non-meek hod premouse and $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ such that $\pi^{\vec{\mathcal{T}},b}$ exists. Let $\mathcal{Q} = \pi^{\vec{\mathcal{T}},b}(\mathcal{P}^b)$. For $\xi + 1 \leq \lambda^{\mathcal{Q}}$ and $X \subseteq \mathcal{P}^b$, we let

$$s(\vec{\mathcal{T}}, X, \xi) = \{\alpha : \exists a \in (\delta^{\mathcal{Q}}_{\xi} + 1)^{<\omega} \exists f \in X(\alpha = \pi^{\vec{\mathcal{T}},b}(f)(a))\} \cap \delta^{\mathcal{Q}}_{\xi+1}$$

When $X = \mathcal{P}^b$ then we just write $s(\vec{\mathcal{T}}, \xi)$.

Definition 3.7.1 (Authentic hod-like lsp) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod-like pair, $\mathcal{T}$ is a normal tree on $\mathcal{P}$ according to Σ such that $\pi^{\mathcal{T},b}$ exists and $X \subseteq \mathcal{P}^b$. Let $\mathcal{S} = \pi^{\mathcal{T},b}(\mathcal{P}^b)$. Suppose $\mathcal{R}$ is a hod-like lsp. We say $(\mathcal{T}, X)$ authenticates $\mathcal{R}$ if for some $\alpha < \lambda^{\mathcal{S}}$ and some $\xi \leq o(\mathcal{S}(\alpha))$, there is a normal tree $\mathcal{U}$ on $\mathcal{R}$ with last model $\mathcal{S}||\xi$[5] and such that*

1. *$(\mathcal{S}(\alpha))^b = Hull^{\mathcal{S}^b}(\pi^{\mathcal{T},b}[X] \cup \delta^{(\mathcal{S}(\alpha))^b})$,*
2. *whenever $\gamma < lh(\mathcal{U})$ is a limit ordinal such that there is $\beta + 1 \leq \alpha$ with the property that $\mathcal{S} \vDash$ "$\delta^{\mathcal{S}}_{\beta+1}$ is a Woodin cardinal" and $\mathcal{M}(\mathcal{U} \restriction \gamma) = \mathcal{S}|\delta^{\mathcal{S}}_{\beta+1}$ then the branch b of $\mathcal{U} \restriction \gamma$ is such that for some $\tau \in b$,*

$$s(\mathcal{T}, X, \beta) \subseteq rng(\pi^{\mathcal{U}}_{\tau,b})$$

and

3. *if $\mathcal{R}$ is of limit type then $\{x \in (\mathcal{S}(\alpha))^b : x \in Hull^{\mathcal{S}}(\pi^{\mathcal{T},b}[X])\} \subseteq rng(\pi^{\mathcal{U},b})$.*

We say $\mathcal{R}$ is $(\mathcal{P}, \Sigma, X)$-authentic if there is $\mathcal{T}$ on $\mathcal{P}$ according to Σ such that $(\mathcal{T}, X)$ authenticates $\mathcal{R}$. We also say that $\mathcal{R}$ is $(\mathcal{P}, \Sigma, X, \mathcal{T})$-authentic.

Notice that there is only one tree $\mathcal{U}$ with the above properties. We say $\mathcal{U}$ is the $(\mathcal{T}, X)$-authentication tree on $\mathcal{R}$, and (α, ξ) are the $(\mathcal{T}, X)$-authentication ordinals. When $X = \mathcal{P}^b$ we simply omit it from terminology.

[5] Recall that $\mathcal{M}||\xi$ is $\mathcal{M}$ up to ξ with the last predicate

Clearly the tree $\mathcal{U}$ in Definition 3.7.1 is a tree built via a comparison process in which $\mathcal{S}$ doesn't move. A typical $\mathcal{R}$ that we would like to authenticate will be an iterate of $\mathcal{P}$. When Σ has nice properties, such as *strong branch condensation* (see Definition 4.7.1 and Section 5.5), the clauses 2 and 3 of Definition 3.7.1 can be satisfied. Next, we would like to define *authentic iterations*.

Definition 3.7.2 (Authentic iterations) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod-like pair, $\mathcal{T}$ is a normal tree on $\mathcal{P}$ according to Σ such that $\pi^{\mathcal{T},b}$ exists and $X \subseteq \mathcal{P}^b$. Let $\mathcal{S} = \pi^{\mathcal{T},b}(\mathcal{P}^b)$. Suppose $\mathcal{R}$ is a hod-like lsp and $\vec{\mathcal{W}}$ is a stack on $\mathcal{R}$. We say $(\mathcal{T}, X)$ authenticates $(\mathcal{R}, \vec{\mathcal{W}})$ if $(\mathcal{T}, X)$ authenticates $\mathcal{R}$ and, letting $\mathcal{U}$ be the $(\mathcal{T}, X)$-authentication tree on $\mathcal{R}$ and (α, ξ) be the $\mathcal{T}$-authentication ordinals, $\vec{\mathcal{W}}$ is according to $\pi^{\mathcal{U}}$-pullback of $\Sigma_{\mathcal{S}||\xi}$.*

Again we omit X when $X = \mathcal{P}^b$. We say $(\mathcal{R}, \vec{\mathcal{W}})$ is a $(\mathcal{P}, \Sigma, X)$-authenticated iteration if there is a tree $\mathcal{T}$ on $\mathcal{P}$ according to Σ such that $(\mathcal{T}, X)$ authenticates $(\mathcal{R}, \vec{\mathcal{W}})$. We also say that $(\mathcal{R}, \vec{\mathcal{W}})$ is $(\mathcal{P}, \Sigma, X, \mathcal{T})$-authentic. When $X = \mathcal{P}^b$ we simply omit it from terminology.

Next we define *authentic stacks of length 2*. These are stacks that will be important in our definition of short tree strategy mice in the next section.

Definition 3.7.3 (Authentic stacks of length 2) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, $X \subseteq \mathcal{P}^b$ and $\mathcal{R}$ is an lsa type hod premouse. Suppose $t = (\mathcal{R}_0, \mathcal{U}, \mathcal{R}_1, \vec{\mathcal{W}})$ is a stack on $\mathcal{R}$ of length 2. We say t is a $(\mathcal{P}, \Sigma, X)$-authenticated if the following conditions hold.*

1. *Suppose $\mathcal{S}$ is a cutpoint of $\mathcal{U}$, $\pi^{\mathcal{U}_{\leq \mathcal{S}},b}$ exists and some initial segment of $\mathcal{U}_{\geq \mathcal{S}}$ is based on $\mathcal{S}^b$. Then $(\mathcal{S}^b, \mathcal{K})$ is $(\mathcal{P}, \Sigma, X)$-authenticated iteration, where $\mathcal{K}$ is the longest component of $\mathcal{U}_{\geq \mathcal{S}}$ that is based on $\mathcal{S}^b$.*
2. *Suppose $\mathcal{S}$ is a cutpoint of $\mathcal{U}$ such that $\pi^{\mathcal{U}_{\leq \mathcal{S}},b}$ exists and some initial segment of $\mathcal{U}_{\geq \mathcal{S}}$ is above $\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}$. Let $\mathcal{K}$ be the longest such initial segment. Then the following conditions hold.*
   - (a) *Suppose $\mathcal{K}$ doesn't have any fatal drops. Then for any limit $\alpha < lh(\mathcal{K})$, if b is the branch of $\mathcal{K} \restriction \alpha$ then $\mathcal{Q}(b, \mathcal{K} \restriction \alpha)$ exists and is $(\mathcal{P}, \Sigma, X)$-authentic.*
   - (b) *Suppose $\mathcal{K}$ has a fatal drop at (α, η). Let $\mathcal{Q} = \mathcal{O}^{\mathcal{M}^{\mathcal{K}}_{\alpha}}_{\eta}$. Then $(\mathcal{Q}, \mathcal{K}_{\geq \mathcal{Q}})$ is a $(\mathcal{P}, \Sigma, X)$-authenticated iteration.*
3. *$((\mathcal{R}_1)^b, \vec{\mathcal{W}})$ is a $(\mathcal{P}, \Sigma, X)$-authenticated iteration.*

When $X = \mathcal{P}^b$ we simply omit it from terminology.

It is of course desirable that $(\mathcal{P}, \Sigma)$-authenticated stacks are according to Σ. We will show this in Section 5.5. In the next section, we will use our authentication idea to define certified stacks.

3.8 Short tree strategy mice

We now have developed enough terminology and tools to define sts premice. We use the following notation below. Suppose $\mathcal{M}$ is a transitive model of some fragment of set theory and λ is a limit of Woodin cardinals. Let $g \subseteq Coll(\omega, <\lambda)$ be $\mathcal{M}$-generic. Then we let $D(\mathcal{M}, \lambda, g)$ stand for the derived model of $\mathcal{M}$ at λ computed using g. More precisely, letting $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R}^{\mathcal{M}[g\cap Coll(\omega,<\alpha)]}$, $D(\mathcal{M}, \lambda, g)$ is defined in $\mathcal{M}(\mathbb{R}^*)$ by first letting $\Gamma = \{A \subseteq \mathbb{R}^* : L(A, \mathbb{R}^*) \vDash \mathsf{AD}^+\}$ and then letting $D(\mathcal{M}, \lambda, g) = L(\Gamma, \mathbb{R}^*)$. Woodin's derived model theorem says that $D(\mathcal{M}, \lambda, g) \vDash \mathsf{AD}^+$ (see [27]).

Before we introduce the notion of short tree strategy premouse, we take a moment to describe the intuition behind the definition. Suppose $\mathcal{P}$ is a hod-like lsa type lsp and $\mathcal{T}$ is a normal ambiguous tree on $\mathcal{P}$. We would like to find the correct $\mathcal{Q}$-structure for $\mathcal{T}$. We first attempt to find this $\mathcal{Q}$-structure among sp that have a trivial indexing scheme ψ_0, i.e., no ambiguous tree has an indexed branch. However, there may never be such an sp that can be used as $\mathcal{Q}$-structure. Assume then that this is the case. We then immediately encounter two problems.

The first problem is to know exactly when to stop looking for a $\mathcal{Q}$-structure among trivial sp's. We will do this as soon as we reach a sufficiently closed $\mathcal{Q}$. To know that we have reached such a level, we need to address the second problem.

The second problem is to describe the next type of gadgets that can be used as $\mathcal{Q}$-structures. A natural choice is the collection of sp's over $\mathcal{M}(\mathcal{T})$ in which all ambiguous trees have $\mathcal{Q}$-structures with the trivial indexing scheme. This is our second indexing scheme. Let us call it ψ_1. One wrinkle is that we need a certification method for the $\mathcal{Q}$-structures that are used in a ψ_1-sts premouse. This is done by using the ideas from Definition 3.7.3.

The way we put the two ideas together is as follows. We first search for a $\mathcal{Q}$-structure among sp's with trival indexing scheme ψ_0. If we reach a level $\mathcal{Q}_0$ that has a ψ_1-sts $\mathcal{Q}_1 \in \mathcal{Q}_0$ that can be used as $\mathcal{Q}$-structure then we stop and see if $\mathcal{Q}_0$ certifies $\mathcal{Q}_1$ (see Definition 3.8.2). If yes, then we declare success. If no, then we continue with trivial indexing. This naturally leads to an induction, in which we define more and more complex indexing schemes. To show that we indeed reach the desired $\mathcal{Q}$-structure we have to use an appropriate notion of fullness preservation.

Before we begin, we make the following useful definition.

Definition 3.8.1 (Terminal tree) *Suppose X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp, $\phi(x, y)$ is a formula in the language of sp and $\mathcal{N}$ is an sts ϕ-premouse over X based on $\mathcal{P}$. Given $\mathcal{T} \in \mathcal{N}$ on $\mathcal{P}$, we say $\mathcal{T}$ is $\mathcal{N}$-terminal if $\mathcal{T}$ is according to $\Sigma^{\mathcal{N}}$ and $\mathcal{N} \vDash$ "$\mathcal{T}$ is ambiguous".*

We now by induction define a sequence of indexing schemes $(\psi_\beta : \beta \in Ord)$. To start we let ψ_0 be the trivial indexing scheme, i.e., ψ_0 is just "0=1". Thus, if $\mathcal{M}$ is an sts ψ_0-premouse then $\mathcal{M}$ does not have branches for ambiguous trees.

Definition 3.8.2 (Sts indexing scheme) *Suppose $(\psi_\beta : \beta < \alpha)$ have been defined. We let ψ_α be the following formula in the language of sp. Suppose X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp and $\mathcal{M}$ is an unambiguous sp over X based on $\mathcal{P}$. Then $\mathcal{M} \vDash \psi_\alpha[\mathcal{T}, b]$ if and only if $(\mathcal{T}, b)$ is the $\mathcal{M}$-lexicographically least[6] pair such that $\mathcal{T}$ is an $\mathcal{M}$-terminal tree on $\mathcal{P}$, $lh(\mathcal{T})$ is not of measurable cofinality, and b is a cofinal branch through $\mathcal{T}$ such that for some pair (β, γ) such that $\gamma < \alpha$ and $\beta < o(\mathcal{M})$,*

1. *$\mathcal{M}|\beta$ is unambiguous (see Definition 3.6.1) and $\mathcal{M}|\beta \vDash \mathsf{ZFC}$ + "there are infinitely many Woodin cardinals $> \delta(\mathcal{T})$",*
2. *$b \in \mathcal{M}|\beta$ and $\mathcal{M}|\beta \vDash$ "b is well-founded branch",*
3. *$\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b, \mathcal{T})$ exists and is an sts ψ_γ-premouse over $\mathcal{M}(\mathcal{T})$" and*
4. *letting $(\delta_i : i < \omega)$ be the first ω Woodin cardinals $> \delta(\mathcal{T})$ of $\mathcal{M}|\beta$, $\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b, \mathcal{T})$ is $<Ord$-iterable above $\delta(\mathcal{T})$ via a strategy Σ such that letting $\lambda = \sup_{i<\omega} \delta_i$, for every generic $g \subseteq Coll(\omega, <\lambda)$, Σ has an extension $\Sigma^+ \in D(\mathcal{M}|\beta, \lambda, g)$ such that $D(\mathcal{M}, \lambda, g) \vDash$ "Σ^+ is an ω_1-iteration strategy" and whenever $\mathcal{R} \in D(\mathcal{M}|\beta, \lambda, g)$ is a Σ^+-iterate of $\mathcal{Q}(b, \mathcal{T})$ (above $\delta(\mathcal{T})$) and $t \in \mathcal{R}$ is a stack on $\mathcal{M}^+(\mathcal{T})$ of length 2 then t is $(\mathcal{P}, \Sigma^{\mathcal{M}|\beta})$-authenticated".*

The lexicographically least pair (β, γ) satisfying the above conditions is called the least $(\mathcal{M}, \psi_\alpha)$-witness for $(\mathcal{T}, b)$. We also say that (β, γ, b) is an $\mathcal{M}$-minimal shortness witness for $\mathcal{T}$. We also say that $\mathcal{T}$ has an $\mathcal{M}$-shortness witness.

[6] This is just the order defined by: first order the first coordinate by $<_{\mathcal{M}}$, the canonical well-order of $\mathcal{M}$, then order the second coordinate by $<_{\mathcal{M}}$.

Notice that $\mathcal{M}$ has at most one $\mathcal{M}$-shortness witnesses for $\mathcal{T}$. We set $\psi_{Ord} =_{def} \psi_{sts}$ and refer to ψ_{sts} as the sts indexing scheme. Notice that in Definition 3.8.2 we tacitly assumed the following absoluteness lemma that can be proved by an easy induction.

Lemma 3.8.3 *Suppose M is a transitive model of* ZFC*, $X \in M$ is a sellf-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp, $\mathcal{M} \in M$ is an sp over X based on $\mathcal{P}$ and $\gamma < o(M)$. Then $M \vDash$ "$\mathcal{M}$ is an sts ψ_γ-premouse" if and only if $\mathcal{M}$ is an sts ψ_γ-premouse.*

The next few definitions introduce sts premice.

Definition 3.8.4 (α-sts premouse) *Suppose $\alpha \in Ord$ or $\alpha = Ord$. Suppose X is a sellf-well-ordered set and $\mathcal{P} \in X$ is a hod-like lsa type lsp. We say $\mathcal{M}$ is an α-sts premouse over X based on $\mathcal{P}$ if $\mathcal{M}$ is a ψ_α-sts premouse over X based on $\mathcal{P}$. When $\alpha = Ord$ we just say that $\mathcal{M}$ is an sts premouse over X based on $\mathcal{P}$.*

Definition 3.8.5 (Sts mouse) *Suppose X is a self-well-ordered set and $\mathcal{P} \in X$ is a hod-like lsa type lsp. We say $\mathcal{M}$ is an sts mouse over X based on $\mathcal{P}$ if $\mathcal{M}$ is an sts premouse over X based on $\mathcal{P}$ which is $\omega_1 + 1$-iterable.*[7]

Definition 3.8.6 (Λ-sts premouse) *Suppose X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp, Λ is an short tree strategy for $\mathcal{P}$ and $\mathcal{M}$ is an sts premouse over $\mathcal{P}$. Then we say $\mathcal{M}$ is a Λ-sts premouse over $\mathcal{P}$ if $\Sigma^M \subseteq \Lambda \restriction \mathcal{M}$.*

Definition 3.8.7 (Λ-sts mouse) *Suppose X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp, Λ is an short tree strategy for $\mathcal{P}$ and $\mathcal{M}$ is a Λ-sts premouse over $\mathcal{P}$. Then we say $\mathcal{M}$ is a Λ-sts mouse over $\mathcal{P}$ if $\mathcal{M}$ has an $\omega_1 + 1$-iteration strategy Σ such that whenever $\mathcal{N}$ is a Σ-iterate of $\mathcal{M}$ via Σ, $\mathcal{N}$ is a Λ-sts premouse over $\mathcal{P}$.*

3.9 Hod mice

The main goal of this section is to introduce *lsa small hod premice*. We start by isolating the types of points in $Y^{\mathcal{P}}$ where $\mathcal{P}$ is hod-like lsp.

Notation 3.9.1 (Meek and lsa points) *Suppose $\mathcal{P}$ is a hod-like lsp.*

[7] Here implicit in this is the demand that iterates of $\mathcal{P}$ according to the strategy are sts premice.

1. *meek*$(\mathcal{P}) = \{\mathcal{Q} \in Y^{\mathcal{P}} : \mathcal{Q}$ *is of meek type*$\}$.
2. *lsa*$(\mathcal{P}) = \{\mathcal{Q} \in Y^{\mathcal{P}} : \mathcal{Q}$ *is of lsa type*$\}$.
3. $\mathcal{Q}^{\mathcal{P}} = \bigcup Y^{\mathcal{P}}$.

Definition 3.9.2 *Suppose $\mathcal{P}$ is a hod-like lsp and $\mathcal{Q} \triangleleft \mathcal{R}$ are either layers of $\mathcal{P}$ or $\mathcal{Q} = \mathcal{M}^+(\mathcal{P}|\delta^{\mathcal{P}})$ and $\mathcal{R} = \mathcal{P}$.*

1. *We say $\mathcal{R}$ is the $\mathcal{P}$-successor of $\mathcal{Q}$ if there is no $\mathcal{S} \in Y^{\mathcal{P}}$ such that $\mathcal{Q} \triangleleft \mathcal{S} \triangleleft \mathcal{R}$.*
2. *We say $\mathcal{Q}$ is a cutpoint of $\mathcal{P}$ if $\mathcal{Q} \triangleleft \mathcal{P}|\delta^{\mathcal{P}}$ and if $\mathcal{S} \in Y^{\mathcal{P}}$ is the $\mathcal{P}$-successor of $\mathcal{Q}$ then $\mathcal{S}$ is of successor type.*
3. *Suppose $\mathcal{Q}$ is of lsa type. We say $\mathcal{R}$ witnesses that $\mathcal{Q}$ is not of lsa type if $\mathcal{R}$ is least such layer of $\mathcal{P}$ such that $\mathcal{R} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal" but $\mathcal{J}_1(\mathcal{R}) \vDash$ "$\delta^{\mathcal{Q}}$ is not a Woodin cardinal".*

Definition 3.9.3 (Hod premouse) *Suppose $\mathcal{P} = \mathcal{J}_{\beta}^{\vec{E},f}$ is a hod-like lsp. We say $\mathcal{P}$ is an lsa small hod premouse if the following holds:*

1. *Suppose $\mathcal{P}$ is meek. Then $\mathcal{P} = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}},\delta^{\mathcal{P}}}$.*
2. *Suppose $\mathcal{P}$ is of lsa type I. Then $\mathcal{P} = \mathcal{O}^{\mathcal{P},\omega}_{\delta^{\mathcal{P}},((\delta^{\mathcal{P}})^+)^{\mathcal{P}}}$. Moreover, for every $n \in [1,\omega)$, $\mathcal{P}|((\delta^{\mathcal{P}})^{+n+1})^{\mathcal{P}}$ is an sts premouse over $\mathcal{P}|((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}}$ based on $\mathcal{P}|((\delta^{\mathcal{P}})^{+})^{\mathcal{P}}$.*
3. *(Lsa smallness) For every α such that $\alpha + 1 < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1)$ isn't of lsa type.*
4. *For all $\mathcal{Q} \in Y^{\mathcal{P}} - lsa(\mathcal{P})$, $\mathcal{P} \vDash$ "$\Sigma^{\mathcal{P}}_{\mathcal{Q}}$ is an (Ord, Ord, Ord)-strategy with hull condensation and strong branch condensation" and if $\mathcal{Q}$ has a successor $\mathcal{R}$ then $\mathcal{P} \vDash$ "$\mathcal{R}$ is a $\Sigma^{\mathcal{P}}_{\mathcal{Q}}$-premouse over $\mathcal{Q}$".*
5. *For all $\mathcal{Q} \in lsa(\mathcal{P})$, letting $\mathcal{R} \in Y^{\mathcal{P}}$ be the successor of $\mathcal{Q}$[8], $\mathcal{R} \vDash$ "$\Sigma^{\mathcal{R}}_{\mathcal{Q}}$ is a partial short tree strategy with hull condensation that acts on stacks of length 2" and $\mathcal{R}$ is a $\Sigma^{\mathcal{R}}_{\mathcal{Q}}$-sts premouse over $\mathcal{Q}$.*
6. *For all $\mathcal{Q} \in lsa(\mathcal{P})$, if $\mathcal{R} \in Y^{\mathcal{P}}$ witnesses that $\mathcal{Q}$ is not of lsa type, then letting Λ be the id-pullback of $\Sigma^{\mathcal{P}}_{\mathcal{R}}$, $\Sigma^{\mathcal{P}}_{\mathcal{Q}} = \Lambda^{sts}$.*
7. *Suppose η is a cutpoint of $\mathcal{P}$. Then the following hold.*

[8] Which exists because of close 3 above.

(a) *If $\mathcal{P}$ is meek or lsa type I then $\mathcal{P} \vDash$ "$\mathcal{O}^{\mathcal{P}}_{\eta,\eta}$ has an Ord-strategy acting on trees that are above η".*

(b) *If $\mathcal{P}$ is non-meek but not of lsa type I and $\eta < \delta^{\mathcal{P}}$ then $\mathcal{P} \vDash$ "$\mathcal{O}^{\mathcal{P}}_{\eta,\eta}$ has a $\delta^{\mathcal{P}}$-strategy acting on trees that are above η".*

8. *If $\lambda^{\mathcal{P}}$ is a successor ordinal and $\mathcal{P}$ is either of successor type or of lsa type I then for any $\mathcal{P}$-cardinal $\eta \in (\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \delta^{\mathcal{P}})$, $\mathcal{P} \vDash$ "$\mathcal{P}|(\eta^+)$ is (Ord, Ord)-iterable for stacks that are above $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$"*

Definition 3.9.3 implicitly introduces an indexing scheme ϕ such that whenever $\mathcal{P}$ is a ϕ-indexed lsp then $\mathcal{P}$ is a hod premouse. Next we define hod pairs.

Definition 3.9.4 (Hod pairs) *We say $(\mathcal{P}, \Sigma)$ is a hod pair if $\mathcal{P}$ is a hod premouse and Σ is an $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ with hull condensation and such that whenever $\mathcal{Q}$ is a Σ-iterate of $\mathcal{P}$ via $\vec{\mathcal{T}}$ and $\mathcal{S} \in Y^{\mathcal{Q}}$, either*

1. *$\mathcal{R} \notin lsa(\mathcal{Q})$ and $\Sigma^{\mathcal{Q}}_{\mathcal{R}} \subseteq \Sigma_{\mathcal{R},\vec{\mathcal{T}}} \restriction \mathcal{Q}$ or*

2. *$\mathcal{R} \in lsa(\mathcal{Q})$ and $\Sigma^{\mathcal{Q}}_{\mathcal{R}} \subseteq \Sigma^{stc}_{\mathcal{R},\vec{\mathcal{T}}} \restriction \mathcal{Q}$.*

Next we introduce the collection of sets generated by hod pairs.

Definition 3.9.5 ($\Gamma(\mathcal{P}, \Sigma)$ and $B(\mathcal{P}, \Sigma)$) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair of limit type. We then let*

$$B(\mathcal{P}, \Sigma) = \{(\vec{\mathcal{T}}, \mathcal{Q}) : \exists \mathcal{R}((\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \lhd_{hod} \mathcal{R}^b)\}$$

and

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists (\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q},\vec{\mathcal{T}}})\}.$$

Definition 3.9.6 (Pre-sts hod pairs) *We say $(\mathcal{P}, \Sigma)$ is a pre-sts hod pair if $(\mathcal{P}, \Sigma)$ is lsa type pair (see Definition 3.4.3), $\mathcal{P}$ is an lsa type hod premouse and Σ is a short tree $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ with hull condensation such that whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\mathcal{Q}$ is an lsa type hod premouse and for all $\mathcal{R} \in Y^{\mathcal{Q}}$, $\Sigma^{\mathcal{Q}}_{\mathcal{R}} = \Sigma_{\mathcal{R},\vec{\mathcal{T}}} \restriction \mathcal{Q}$.*

To define sts hod pairs, we will make use of the notation introduced in Definition 3.3.6. Recall that in Definition 3.3.6, we introduced $\Gamma^b(\mathcal{P}, \Sigma)$ but not $\Gamma(\mathcal{P}, \Sigma)$. We will define $\Gamma(\mathcal{P}, \Sigma)$ for sts hod pairs in Section 8.1.

Suppose now that X is a self-well-ordered set, $(\mathcal{P}, \Sigma)$ is a pre-sts pair such that $\mathcal{P} \in X$ and $\mathcal{Q}$ is a Σ-sts mouse over X based on $\mathcal{P}$. Let Λ be the strategy of $\mathcal{Q}$. We then let $\Gamma(\mathcal{Q}, \Lambda)$ be the collection of all sets of reals A such that for some Λ-iterate $\mathcal{R}$ of $\mathcal{Q}$, there is $(\vec{\mathcal{T}}, \mathcal{S}) \in B(\mathcal{P}, \Sigma^{\mathcal{R}})$ such that $A \leq_w \Sigma_{\mathcal{S},\vec{\mathcal{T}}}$.

Definition 3.9.7 (Sts hod pairs) *We say $(\mathcal{P}, \Sigma)$ is an sts hod pair if $(\mathcal{P}, \Sigma)$ is a pre-sts pair such that whenever*

1. $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$,
2. $\eta \in (\delta^{\mathcal{R}}_{\lambda^{\mathcal{R}}-1}, \delta^{\mathcal{R}}_{\lambda^{\mathcal{R}}}]$ *is such that* $\mathcal{J}_1(\mathcal{M}^+(\mathcal{R}|\eta)) \vDash$ *"η is a Woodin cardinal"*[9],
3. $\nu \geq \eta$ *is a $\mathcal{P}$-cardinal, and*
4. $\mathcal{Q} \trianglelefteq \mathcal{R}$ *is a $\Sigma_{\mathcal{M}^+(\mathcal{R}|\eta), \vec{\mathcal{T}}}$-sts mouse over $\mathcal{R}|\nu$ based on $\mathcal{R}|\eta$*

then $\mathcal{Q}$ has an iteration strategy $\Phi \in \Gamma^b(\mathcal{P}, \Sigma)$ witnessing that $\mathcal{Q}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{R}|\eta), \vec{\mathcal{T}}}$-sts mouse over $\mathcal{R}|\nu$ based on $\mathcal{R}|\eta$ and such that $\Gamma(\mathcal{Q}, \Phi) <_w \Gamma^b(\mathcal{P}, \Sigma)$.

Definition 3.9.7 imposes conditions on sts hod pairs that may seem unnatural. However, these conditions are needed to prove that sts hod pairs behave nicely. For instance, we will use these clauses in the proof of Lemma 5.5.1, which is an important lemma showing that our indexing scheme doesn't index incorrect branches. Our sts indexing scheme is such that when indexing a branch of an ambiguous tree we do not consult the strategy but instead look at the sts mouse itself. Lemma 5.5.1 shows that indexed branches are according to the strategy.

We finish this section by introducing the minimal component of a short tree strategy. Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair and suppose $\vec{\mathcal{T}} = (\mathcal{P}_i, \vec{\mathcal{T}}_i : i < \omega)$ is a stack on $\mathcal{P}$ according to Σ. We then let $\vec{\mathcal{T}}^{min}$ be the same as $\vec{\mathcal{T}}$ except that whenever $\mathcal{S}$ is a cutpoint of $\vec{\mathcal{T}}$ such that $(\vec{\mathcal{T}}_{\leq \mathcal{S}}, \mathcal{S}) \in I(\mathcal{P}, \Sigma)$ and if $\mathcal{W}$ is the largest normal component of $\vec{\mathcal{T}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}}$ then, letting $\mathcal{W}^-$ be $\mathcal{W}$ without its last model if it exists and otherwise just $\mathcal{W}$, $\mathcal{W}^-$ is of limit length, $\vec{\mathcal{T}}_{\leq \mathcal{S}}{}^\frown \mathcal{W} \in m(\Sigma)$ and in $\vec{\mathcal{T}}^{min}$, II plays $\mathcal{M}^+(\mathcal{W}^-)$. Thus, in $\vec{\mathcal{T}}^{min}$, when II plays a model she always plays $\mathcal{M}^+(\mathcal{W}^-)$.

Definition 3.9.8 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair. We say Ψ is the minimal component of Σ if Ψ is a short tree strategy for $\mathcal{P}$ such that*

1. $\vec{\mathcal{T}} \in dom(\Sigma)$ *if and only if* $\vec{\mathcal{T}}^{min} \in dom(\Psi)$,
2. $\vec{\mathcal{T}} \in b(\Sigma)$ *if and only if* $\vec{\mathcal{T}}^{min} \in b(\Psi)$, *and*
3. *if* $\vec{\mathcal{T}} \in m(\Psi)$ *then* $\Psi(\vec{\mathcal{T}}) = \mathcal{M}^+(\mathcal{W})$ *where $\mathcal{W}$ is the last normal component of* $\vec{\mathcal{T}}$.

[9] Recall Notation 2.2.1.

Chapter 4

Comparison theory of hod mice

4.1 Background triples and Suslin capturing

The goal of this section is to introduce *background triples* and *Suslin, co-Suslin capturing*. We will use these notions to build hod pairs with desired properties, such as *fullness preservation* and *branch condensation*.

Definition 4.1.1 (Background triple, Definition 2.24 of [10]) *We say*

$$\mathbb{M} = (M, \delta, \Sigma)$$

is a weak background triple if $M \vDash ZFC +$ "δ *is a Woodin cardinal" and* $\Sigma \in M$ *is a* $(\delta, \delta+1)$*-iteration strategy for* V_δ^M *with hull condensation acting on stacks that are in* $\mathcal{J}_\omega(V_\delta^M)$*. We say* (M, δ, Σ) *is a background triple if* Σ *is an* (ω_1, ω_1)*-strategy for* M *and* $(M, \delta, \Sigma_{V_\delta} \restriction \mathcal{J}_\omega(V_\delta^M))$ *is a weak background triple.*

Suppose $\mathbb{M} = (M, \delta, \Sigma)$ is a a background triple and $A \subseteq \mathbb{R}$. We review the standard capturing notions. We say $\mathbb{M}$ Suslin captures A at η if there is a tree $T \in M$ such that whenever N is a Σ iterate of M and $i : M \to N$ and whenever g is $< i(\eta)$-generic over N, $(p[i(T)])^{N[g]} = A \cap N[g]$. We say $\mathbb{M}$ Suslin, co-Suslin captures A at η if it Suslin captures both A and A^c.

Suppose Γ is a good pointclass.[1] For $x \in \mathbb{R}$, we let $C_\Gamma(x)$ be the largest countable $\Gamma(x)$-set of reals. For transitive $a \in HC$ and surjection $g : \omega \to a$, we let a_g be the real coding $(a, \in)$ via g. More precisely, letting mE_gn if and only if $g(m) \in g(n)$, $a_g = \{(m,n) : g(m) \in g(n)\}$. Let $\pi_g : (\omega, E_g) \to (a, \in)$ be the transitive collapse of

[1] See [26, Definition 9.12].

(ω, E_g). If also $b \subseteq a$, the we let $b_g = \{m : \pi_g(m) \in b\}$. We then let $C_\Gamma(a) = \{b \subseteq a :$ for comeager many $g : \omega \to a$, $b_g \in C_\Gamma(a_g)\}$.

Continuing with Γ, we say P is a Γ-Woodin if

1. P is countable,
2. for some P-cardinal δ, $P = C_\Gamma(C_\Gamma(V_\delta^P))$,
3. $P \vDash$ "δ is the only Woodin cardinal" and
4. for every $\eta < \delta$, $C_\Gamma(V_\eta) \vDash$ "η is not a Woodin cardinal".

We say (P, Ψ) is a Γ-*Woodin pair* if

1. Ψ is an ω_1-iteration strategy for P and
2. for every Ψ-iterate Q of P, $\mathcal{Q}$ is a Γ-Woodin.

Woodin showed under AD^+ that for any good, scaled pointclass Γ not closed under $\forall^{\mathbb{R}}$, there are Γ-Woodin pairs (see [26, Theorem 10.3]). Given a Γ-Woodin pair (P, Ψ), we let $\mathcal{M}_n^{\#,\Psi}$ be the minimal active Ψ-mouse with n Woodin cardinals and Ψ_n be the unique ω_1-iteration strategy of $\mathcal{M}_n^{\#,\Psi}$.[2]

Definition 4.1.2 *Suppose Γ is any pointclass and Γ^* is the least good, scaled pointclass such that $\Gamma \subseteq \utilde{\Delta}_{\Gamma^*}$. We say a background triple $\mathbb{M}$ Suslin, co-Suslin captures Γ if for some Γ^*-Woodin pair (P, Ψ), $\mathbb{M}$ Suslin, co-Suslin captures the sequence $(Code(\Psi_n) : n < \omega)$. We also say that $\mathbb{M}$ captures Γ via the pair (P, Ψ).*

The following is an important yet straightforward lemma that we will use throughout this book. See [19, Section 1.5] for a proof.

Lemma 4.1.3 (Correctness of background triples) *Suppose $\mathbb{M} = (M, \delta, \Sigma)$ is a background triple that captures a good, scaled pointclass Γ via the pair (P, Ψ) and suppose $x \in \mathbb{R} \cap M$. Then $\mathbb{M}$ Suslin, co-Suslin captures any set of reals that is lightface definable over $(HC, Code(\Psi), x, \in)$.*

Suppose Γ is a pointclass and $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $Code(\Sigma) \in \Gamma$. Recall the definition of $Lp^{\Gamma,\Sigma}(X)$. In the case Σ is an iteration strategy, $Lp^{\Gamma,\Sigma}(X)$ is the stack of all sound Σ-mice $\mathcal{M}$ over X[3] such that $\rho(\mathcal{M}) = X$

[2] Under AD^+, this is equivalent to Ψ_n being the unique $\omega_1 + 1$-iteration strategy of $\mathcal{M}_n^{\#,\Psi}$

[3] In case X isn't transitive or $\mathcal{P} \notin X$, "over X" means "over $Tc(\{X, \mathcal{P}\})$".

and $\mathcal{M}$ has a strategy in Γ. In the case Σ is a short tree strategy $Lp^{\Gamma,\Sigma}(X)$ is the stack of all sound Σ-sts mice $\mathcal{M}$ over X based on $\mathcal{P}$ such that $\rho(\mathcal{M}) = X$ and $\mathcal{M}$ has a strategy in Γ.[4] Below if Ψ is an iteration strategy or short tree strategy then we let M_Ψ be the structure that it iterates.

Notation 4.1.4 *Suppose Γ is some pointclass. Following Section 2.5 of [10] we let*

$$HP^\Gamma = \{(\mathcal{P},\Sigma) : (\mathcal{P},\Sigma) \textit{ is a hod pair or an sts hod pair such that } Code(\Sigma) \in \Gamma\}$$
$$Mice^\Gamma = \{(a,\Sigma,\mathcal{M}) : a \in HC \wedge a \textit{ is a swo} \wedge (M_\Sigma,\Sigma) \in HP^\Gamma \wedge \mathcal{M}_\Sigma \in a \wedge \mathcal{M} \trianglelefteq Lp^{\Gamma,\Sigma}(a) \wedge \rho(\mathcal{M}) = a\}$$

and given $(\mathcal{P},\Sigma) \in HP^\Gamma$,

$$Mice^\Gamma_\Sigma = \{(a,\mathcal{M}) : a \in HC \wedge a \textit{ is a swo} \wedge \mathcal{P} \in a \wedge \mathcal{M} \trianglelefteq Lp^{\Gamma,\Sigma}(a) \wedge \rho(\mathcal{M}) = a\}$$

When $\Gamma = \wp(\mathbb{R})$, we omit it from our notation.

Given a set $A \subseteq \mathbb{R}$ with $w(\Gamma) \leq w(A)$, we let A_Γ be the set of triples of continuous functions $(\sigma_0,\sigma_1,\sigma_2)$ such that $\sigma_0^{-1}[A]$ is a code for some $(\mathcal{P},\Sigma) \in HP^\Gamma$, $\sigma_1^{-1}[A]$ is a code for a triple $(a,\Sigma,\mathcal{M}) \in Mice^\Gamma$ and $\sigma_2^{-1}[A]$ is a code for the unique ω_1-strategy of $\mathcal{M}$.

The following is an easy consequence of Lemma 4.1.3.

Corollary 4.1.5 *Suppose $\mathbb{M} = (M,\delta,\Sigma)$ is a background triple that captures a pointclass Γ via the pair (P,Ψ). Then $\mathbb{M}$ Suslin, co-Suslin captures $Code(\Psi)_\Gamma$, $Code(HP^\Gamma)$ and $Code(Mice^\Gamma)$.*

We finish by recalling the notion of self-capturing background triple (Definition 2.24 of [10]). Suppose $\mathbb{M} = (M,\delta,\Sigma)$ is a background triple. We say $\mathbb{M}$ is *self-capturing* if for every M-inaccessible cardinal $\lambda < \delta$ there is a set $X \in M$ such that for any M-generic $g \subseteq Coll(\omega,\lambda)$ and for every $M[g]$-cardinal η which is countable in V, $(M[g],\Sigma)$ Suslin, co-Suslin captures $Code(\Sigma_{V_\lambda^M})$ at η as witnessed by a pair $(T,S) \in OD_X^{M[g]}$. Self-capturing background triples are very useful for building hod pairs and proving comparison. The following theorem of Woodin shows that under AD^+, self-capturing background triples are abundant.

Theorem 4.1.6 (Woodin, Theorem 10.3 of [26]) *Assume AD^+. Suppose Γ is a good, scaled pointclass and there is a good, scaled pointclass Γ^* such that $\Gamma \subsetneq \Delta_{\Gamma^*}$. Suppose (N,Ψ) Suslin, co-Suslin capture Γ. There is then a function F defined on $\mathbb{R}$ such that for a Turing cone of x, $F(x) = (\mathcal{N}^*_x,\mathcal{M}_x,\delta_x,\Sigma_x)$ such that*

[4] From here on, "Lp" means "g-organized Lp" as defined in [20] unless explicitly stated otherwise. We will occasionally remind the reader of this convention. The reason we need to use g-organization is so that S-constructions go through.

1. $N \in L_1[x]$,

2. $\mathcal{N}_x^*|\delta_x = \mathcal{M}_x|\delta_x$,

3. $\mathcal{M}_x$ *is a* Ψ*-mouse: in fact,* $\mathcal{M}_x = \mathcal{M}_1^{\Psi,\#}(x)|\kappa_x$ *where* κ_x *is the least inaccessible cardinal of* $\mathcal{M}_1^{\Psi,\#}$,

4. $\mathcal{N}_x^* \vDash$ *"δ_x is the only Woodin cardinal"*,

5. Σ_x *is the unique iteration strategy of* $\mathcal{M}_x$,

6. $\mathcal{N}_x^* = L(\mathcal{M}_x, \Lambda)$ *where* Λ *is the restriction of* Σ_x *to stacks* $\vec{\mathcal{T}} \in \mathcal{M}_x$ *that have finite length and are based on* $\mathcal{M}_x \restriction \delta_x$,

7. $(\mathcal{N}_x^*, \Sigma_x)$ *Suslin, co-Suslin captures* $Code(\Psi)$ *and hence,* $(\mathcal{N}_x^*, \Sigma_x)$ *Suslin, co-Suslin captures* Γ,

8. $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$ *is a self-capturing background triple.*

Suppose next that Γ is a pointclass and $\mathbb{M} = (M, \delta, \Sigma)$ is a self-capturing background triple capturing Γ via a pair (P, Ψ). In Section 4.3.9, we will describe the Γ-hod pair construction of $\mathbb{M}$ that produces a hod pair in HP^Γ. When describing this construction, we will use the following simple observations.

Remark 4.1.7 *It follows from Corollary 4.1.5 that the statement* $(\mathcal{P}, \Lambda) \in HP^\Gamma$ *is absolute between* V *and* M*. Indeed, given a hod pair* $(\mathcal{P}, \Lambda)$ *such that for some* $< \delta$*-generic* g, $\mathcal{P} \in HC^{M[g]}$ *and* $\Lambda \restriction HC^{M[g]} \in M[g]$*, we write* $M[g] \vDash (\mathcal{P}, \Lambda) \in HP^\Gamma$ *if there is a continuous function* $\sigma \in \mathbb{R}^{M[g]}$ *such that* $Code(\Lambda) = \sigma^{-1}Code(\Psi)$*. Notice that because of Lemma 4.1.3 if* $\sigma_1, \sigma_2 \in \mathbb{R}^{M[g]}$ *are two continuous functions then*

$$M[g] \vDash \text{“}\sigma_1^{-1}[Code(\Psi)] = \sigma_2^{-1}[Code(\Psi)]\text{” if and only if}$$
$$\sigma_1^{-1}[Code(\Psi)] = \sigma_2^{-1}[Code(\Psi)].$$

Remark 4.1.8 *Similarly mouse operators are definable over background triples. Indeed, suppose* $(\mathcal{P}, \Lambda) \in HP^\Gamma$ *is such that for some* g, $M[g] \vDash (\mathcal{P}, \Lambda) \in HP^\Gamma$*. Suppose further that* $F : HC \to HC$ *is given by* $F(a) = Lp^{\Gamma,\Lambda}(a)$*. Then for any* $M[g]$*-generic* h*, the function* $F \restriction M[g][h]$ *is uniformly definable from* h, $Code(\Psi)$ *and any continuous functions* $\sigma_0, \sigma_1 \in \mathbb{R}^{M[g]}$ *such that* $\sigma_0^{-1}[Code(\Psi)] = Code(\Lambda)$ *and* $Code(F) = \sigma_1^{-1}[Code(\Psi)]$.

4.2 Fully backgrounded constructions relative to short tree strategy

Suppose (M, δ, Σ) is a weak background triple and $\mathcal{P} \in V_\delta^M$ is an lsa type hod like lsp. Suppose $\Lambda \in M$ is a short tree (δ, δ, δ)-strategy for $\mathcal{P}$ and $X \in V_\delta^M$ is a transitive self-well-ordered set such that $\mathcal{P} \in X$. We can then define the model $\mathcal{J}^{\vec{E},\Lambda}(X)$ exactly like in the case Λ is an iteration strategy. The construction will ensure that the model $\mathcal{J}^{\vec{E},\Lambda}(X)$ is an sts premouse over X based on $\mathcal{P}$. Here is the precise definition.

Recall that if $(\mathcal{M}_\alpha : \alpha < \xi)$ is a sequence of $\mathcal{J}$-structures and ξ is a limit ordinal then $\mathcal{M} = lim_{\alpha \to \xi}\mathcal{M}_\alpha$ is the $\mathcal{J}$-structure with the property that for each β such that $\mathcal{J}_\beta^{\mathcal{M}}$ is defined, there is $\gamma < \xi$ such that for all $\alpha \in (\gamma, \xi)$, $\mathcal{J}_\beta^{\mathcal{M}_\alpha} = \mathcal{J}_\beta^{\mathcal{M}}$.

Suppose (M, δ, Σ) is a weak background triple and $E \in V_\delta^M$ is an extender. Then we say E coheres Λ if $\nu(E)$ is an inaccessible cardinal of M, $V_{\nu(E)}^M \subseteq Ult(\mathcal{M}, E)$ and $\Lambda \cap V_{\nu(E)}^M = \pi_E(\Lambda) \cap V_{\nu(E)}^M$. Recall that an lhp $\mathcal{M}$ is reliable if for all k, $\mathcal{C}_k(\mathcal{M})$ exists and is k-iterable, where $\mathcal{C}_k(\mathcal{M})$ is the kth core of $\mathcal{M}$ (see [8, Chapter 11]).

Definition 4.2.1 *Suppose (M, δ, Σ) is a weak background triple and $\mathcal{P} \in V_\delta^M$ is an lsa type hod like lsp. Suppose $\Lambda \in M$ is a short tree (δ, δ, δ)-strategy for $\mathcal{P}$ and $X \in V_\delta^M$ is a transitive self-well-ordered set such that $\mathcal{P} \in X$. Suppose further that Λ has hull condensation. Then for $\eta \leq \delta$, $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ is the ηth initial segment of the output of the fully backgrounded construction relative to Λ if the following is true.*

1. *$\mathcal{M}_0 = \mathcal{J}_1(X)$, and for all $\xi < \eta$, $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ are Λ-sts premice.*
2. *Suppose $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \xi), (F_\gamma : \gamma < \xi))$ has been defined for $\xi < \eta$. Then we define $\mathcal{M}_{\xi+1}$, $\mathcal{N}_{\xi+1}$ and F_ξ as follows.*
   (a) *Suppose $\mathcal{M}_\xi = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, i.e., with no last predicate, and there is an extender F^* such that F^* coheres Λ and reflects $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \xi), (F_\gamma : \gamma < \xi))$, an extender F over $\mathcal{M}_\xi$, and an ordinal $\nu < \alpha$ such that $V_{\nu+\omega} \subseteq Ult(V, F^*)$ and*

   $$F \restriction \nu = F^* \cap ([\nu]^\omega \times \mathcal{J}_\alpha^{\vec{E},f}).$$

   Then

   $$\mathcal{N}_{\xi+1} = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f, \tilde{F})$$

and $\nu = \nu^{\mathcal{N}_{\xi+1}}$ where $\tilde{F}$ is the amenable code of F[5]*. Also, if $\mathcal{N}_{\xi+1}$ is reliable then $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$*[6] *and $F_\xi = F$. If $\mathcal{N}_{\xi+1}$ is not reliable then we stop the construction.*

(b) *Suppose $\mathcal{M}_\xi = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, the hypothesis of item 2.a above doesn't hold, $\mathcal{M}_\xi \vDash$ ZFC-Replacement, and $\mathcal{M}_\xi$ is ambiguous. Let $t = (\mathcal{P}_0, \mathcal{T}, \mathcal{P}_1, \vec{\mathcal{U}}) \in \mathcal{J}_\alpha^{\vec{E},f} \cap dom(\Lambda)$ on $\mathcal{P}$ be the $\mathcal{M}_\xi$-least stack of length 2 witnessing that $\mathcal{M}_\xi$ is ambiguous and such that $lh(\mathcal{T})$ is not of measurable cofinality in $\mathcal{M}_\xi$ and $lh(\vec{\mathcal{U}})$ is not of measurable cofinality in $\mathcal{M}_\xi$. Set $b = \Lambda(t)$, $\beta = \sup b$ and $\mathcal{N} = \mathcal{J}_\beta(\mathcal{M}_\xi)$. If $\rho(\mathcal{N}) \geq \alpha$ then*

$$\mathcal{N}_\xi = (\mathcal{J}_\beta^{\vec{E},f^+}, \in, \vec{E}, f^+)$$

where $f^+ = f \cup (\mathcal{J}_\omega(t), \tilde{b})$ where $\tilde{b} \subseteq \alpha+\beta$ is defined by $\alpha+\nu \in \tilde{b} \leftrightarrow \nu \in b$. If $\rho(\mathcal{N}) < \alpha$ then let $\gamma \in (\alpha, \beta]$ be least such that $\rho(\mathcal{J}_\gamma(\mathcal{M}_\xi)) < \alpha$ and let $\mathcal{N}_{\xi+1} = \mathcal{C}(\mathcal{J}_\gamma(\mathcal{M}_\xi))$. Also, if $\mathcal{N}_{\xi+1}$ is reliable then $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$ and $F_\xi = \emptyset$. If $\mathcal{N}_{\xi+1}$ is not reliable then we stop the construction.

(c) *Suppose $\mathcal{M}_\xi = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, the hypothesis of item 2.a and 2.b above don't hold, $\mathcal{M}_\xi \vDash$ ZFC, $\mathcal{M}_\xi$ is unambiguous and there is a normal terminal $\mathcal{T} \in \mathcal{J}_\alpha^{\vec{E},f} \cap dom(\Lambda)$ such that $\mathcal{M}_\xi \vDash$ "$\mathcal{T}$ is ambiguous and $lh(\mathcal{T})$ is not of measurable cofinality", $f^{\mathcal{M}_\xi}(\mathcal{T})$ isn't defined and there is an $\mathcal{M}_\xi$-minimal shortness witness for $\mathcal{T}$. Let $\mathcal{U}$ be the $\mathcal{M}_\xi$-least such tree, (ϕ, ζ, e) be a shortness witness for $\mathcal{U}$, $b = \Lambda(\mathcal{U})$, $\beta = \sup b$ and $\mathcal{N} = \mathcal{J}_\beta(\mathcal{M}_\xi)$.*

Important Anomaly: *If $e \neq b$ then stop the construction.*

Assume then that $e = b$. If $\rho(\mathcal{N}) \geq \alpha$ then

$$\mathcal{N}_\xi = (\mathcal{J}_\beta^{\vec{E},f^+}, \in, \vec{E}, f^+)$$

where $f^+ = f \cup \{(\mathcal{J}_\omega(\mathcal{U}), \tilde{b})\}$ where $\tilde{b} \subseteq \alpha+\beta$ is defined by $\alpha+\nu \in \tilde{b} \leftrightarrow \nu \in b$. If $\rho(\mathcal{N}) < \alpha$ then let $\gamma \in (\alpha, \beta]$ be least such that $\rho(\mathcal{J}_\gamma(\mathcal{M}_\xi)) < \alpha$ and let $\mathcal{N}_{\xi+1} = \mathcal{C}(\mathcal{J}_\gamma(\mathcal{M}_\xi))$. Also, if $\mathcal{N}_{\xi+1}$ is reliable then $\mathcal{M}_{\xi+1} = \mathcal{C}(\mathcal{N}_{\xi+1})$ and $F_\xi = \emptyset$. If $\mathcal{N}_{\xi+1}$ is not reliable then we stop the construction.

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[5] For the definition of the "amenable code" see the last paragraph on page 14 of [28].

[6] Recall that $\mathcal{C}(\mathcal{M})$ is the core of $\mathcal{M}$.

3. *Suppose $\xi \leq \eta$ is a limit ordinal and $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma < \xi), (F_\gamma : \gamma < \xi))$ has been defined. Then we define $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ as follows[7]. Let $\nu = limsup_{\lambda\to\xi}(\rho^+)^{\mathcal{M}_\xi}$. Then we let $\mathcal{N}_\xi$ be the passive lhp $\mathcal{P} = \mathcal{J}_\nu^{\mathcal{P}}$, where for all $\beta < \nu$ we set $\mathcal{J}_\beta^{\mathcal{P}}$ be the eventual value of $\mathcal{J}_\beta^{\mathcal{M}_\lambda}$ as $\lambda \to \xi$. Also if $\mathcal{N}_\xi$ is reliable then $\mathcal{M}_\xi = \mathcal{C}(\mathcal{N}_\xi)$. If $\mathcal{N}_\xi$ is not reliable then we stop the construction.*

The important comment in clause 2.c is a non-trivial matter. Recall that according to our sts indexing scheme (see Definition 3.8.2), the branch we have to index at stage ξ in clause 2.c is e not b. However, if $e \neq b$ then the resulting structure cannot be a Λ-sts mouse. Thus, if $e \neq b$ then we have to halt the construction. When Λ has nice properties such *strong branch condensation* (see Definition 4.7.3) then such anomaly will never arise, as shown in Corollary 5.5.2. See Remark 5.5.3 for an in-depth discussion of this issue.

4.3 Hod pair constructions

Next we define Γ-hod pair constructions. Unlike in [10], here we view these constructions in a somewhat different yet equivalent way. For us a hod pair construction is a procedure that builds four types of operations E, B, J and Lim. We call them respectively *the extender operator*, *the branch operator*, *the constructibility operator* and *the limit operator*. We also refer to these operators as the hpc-operators. We start by describing three auxiliary sets.

Suppose Γ is a pointclass and $\mathbb{M} = (M, \delta, \Sigma)$ is a background triple Suslin, co-Suslin capturing Γ. We will work with $\mathbb{M}$ and Γ, but we will omit both from our notations. For instance, E below should really be $\mathsf{E}^{\mathbb{M}}$. Also, all the fully backgrounded constructions that we will use are fully backgrounded constructions in the sense of V_δ^M, and if M is equipped with a distinguished extender sequence then we tacitly assume that all the backgounded constructions use extenders from this particular extender sequence.

Definition 4.3.1 (E^0, B^0 and J^0) *Below we define three sets E^0, B^0 and J^0.*

1. *$\mathcal{Q} \in dom(\mathsf{E}^0)$ if $\mathcal{Q} \in V_\delta^M$ is a passive lhp and there is an extender $F^* \in M$, an extender $F \in M$ over $\mathcal{Q}$ and an ordinal ν such that $M \vDash$ "$\nu(F^*)$ is an inaccessible cardinal", $F = F^* \cap [\nu]^{<\omega} \times \mathcal{Q}$, and $(\mathcal{Q}, \tilde{F})$ is a reliable lhp where $\tilde{F}$ is the amenable code of F and $\nu^{(\mathcal{Q},\tilde{F})} = \nu$.*

[7] F_ξ will be defined at the next stage of the induction as in clause 2.

2. *$\mathcal{Q} \in dom(\mathsf{B}^0)$ if $\mathcal{Q} = \mathcal{J}^{E,f}_{\alpha+\beta} \in V^M_\delta$ is a passive lhp such that for some $\mathcal{R} \in Y^{\mathcal{Q}}$ such that $\mathcal{R}$ is a hod premouse and there is a stack $\vec{\mathcal{T}} \in \mathcal{Q} - dom(\Sigma^{\mathcal{Q}}_{\mathcal{R}})$ based on $\mathcal{R}$ such that $\vec{\mathcal{T}}$ is according to $\Sigma^{\mathcal{Q}}_{\mathcal{R}}$, $lh(\vec{\mathcal{T}})$ is not of measurable cofinality in $\mathcal{Q}$, and there is some cofinal well-founded branch $b \in M$ of $\vec{\mathcal{T}}$ such that $\beta = \sup b$ and if $\tilde{b}$ is such that $\alpha + \gamma \in \tilde{b}$ if and only if $\gamma \in b$ then $(\mathcal{Q}, \in, \vec{E}, f^+)$ is an lhp where $f^+ = f \cup \{(\mathcal{J}_\omega(\vec{\mathcal{T}}), b)\}$.*

3. *$\mathcal{Q} \in dom(\mathsf{J}^0)$ if $\mathcal{Q}$ is an lhp and $\mathcal{Q} \in V^M_\delta - (dom(\mathsf{E}^0) \cup dom(\mathsf{B}^0))$.*

The next definition introduces the *bad lhps.*

(Bad) Suppose $\mathcal{M}$ is an lhp such that every $\mathcal{R} \in Y^{\mathcal{M}}$ is a hod premouse. We say $\mathsf{Bad}(\mathcal{M})$ holds if one of the following conditions hold.

1. $\mathcal{M}$ is unreliable (i.e, for some $k < \omega$, $\mathcal{C}_k(\mathcal{M})$ doesn't exist).

2. There is $\mathcal{R} \in Y^{\mathcal{M}}$ such that $\mathcal{R}$ is of successor type and $\rho(\mathcal{M}) < \delta^{\mathcal{R}}$.

3. There is $\mathcal{R} \in Y^{\mathcal{M}}$ of limit type such that $\rho(\mathcal{M}) < (\nu^+)^{\mathcal{R}}$ where $\nu = \delta^{\mathcal{R}^b}$.

We will have that $dom(\mathsf{E}) \subseteq dom(\mathsf{E}^0)$ and $dom(\mathsf{B}) \subseteq dom(\mathsf{B}^0)$. All four functions E, B, J and Lim will be defined by induction.

Definition 4.3.2 (Stage 0) *We set.*

1. *$\mathsf{J}(0) = \emptyset$.*

2. *$\mathsf{E}(0) = \mathsf{B}(0) = \mathsf{Lim}(0) = \emptyset$.*

When defining J, E, B and Lim, we will maintain the following requirements.

Requirements

1. $dom(\mathsf{J})$, $dom(\mathsf{E})$, $dom(\mathsf{B})$ and $dom(\mathsf{Lim})$ are subsets of δ.

2. If $\alpha^{\mathbb{M}} =_{def} \sup\{\xi + 1 : \xi \in dom(\mathsf{J}) \cup dom(\mathsf{E}) \cup dom(\mathsf{B}) \cup dom(\mathsf{Lim}))\}$ then the four sets $dom(\mathsf{J})$, $dom(\mathsf{E})$, $dom(\mathsf{B})$ and $dom(\mathsf{Lim})$ form a partition of $\alpha^{\mathbb{M}}$ and $\alpha^{\mathbb{M}} \leq \delta$.

3. $\{\beta < \alpha^{\mathbb{M}} : \beta \text{ is a successor ordinal}\} \subseteq dom(\mathsf{J})$.

4. For all $\beta < \alpha^{\mathbb{M}}$, the value of the hpc-operators at β is either undefined or is an lhp $\mathcal{Q}$ such that for every $\mathcal{R} \in Y^{\mathcal{Q}}$, $\mathcal{R}$ is a hod premouse.

5. Given any $\mathcal{Q}$ and $\mathcal{R}$ as in clause 4, Σ induces, via the construction described in [8, Chapter 12], a strategy $\Lambda_{\mathcal{R}}$ for $\mathcal{R}$.

6. If $\beta \in dom(\mathsf{E}) \cup dom(\mathsf{B})$ then β is a successor ordinal and $\beta - 1 \in dom(\mathsf{Lim})$.

We start by describing how the operator E works.

Definition 4.3.3 (The extender operator) *Suppose* $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ *and* $\mathsf{Lim} \restriction \beta$ *have been defined and* $\beta = \gamma + 1$. *Let* $\mathcal{Q} = \mathsf{Lim}(\gamma)$.

1. *Suppose* $\mathcal{Q} \notin \mathsf{E}^0$. *Then let* $\mathsf{E}(\beta)$ *be undefined.*

2. *Suppose then that* $\mathcal{Q} \in \mathsf{E}^0$.

   (a) *Suppose there is no triple* (F^*, F, ν) *witnessing that* $\mathcal{Q} \in \mathsf{E}^0$ *with the additional property that* F^* *coheres* $(\mathsf{J} \restriction \beta, \mathsf{E} \restriction \beta, \mathsf{B} \restriction \beta, \mathsf{Lim} \restriction \beta)$. *Then we let* $\mathsf{E}(\beta)$ *be undefined.*

   (b) *Otherwise let* (F^*, F, ν) *witness that* $\mathcal{Q} \in \mathsf{E}^0$ *with the additional property that* F^* *coheres* $(\mathsf{J} \restriction \beta, \mathsf{E} \restriction \beta, \mathsf{B} \restriction \beta, \mathsf{Lim} \restriction \beta)$. *Letting* $\tilde{F}$ *be the amenable code of* F *and* $\mathcal{M} = (\mathcal{Q}, \tilde{F})$, *set*

$$\mathsf{E}(\beta) = \begin{cases} \textit{undefined} & : \mathsf{Bad}(\mathcal{M}) \textit{ holds} \\ \mathcal{C}(\mathcal{M}) & : \textit{otherwise.} \end{cases}$$

We split the branch operator into three pieces $\mathsf{B}_{\mathsf{nlsa}}$, $\mathsf{B}_{\mathsf{ualsa}}$ and $\mathsf{B}_{\mathsf{alsa}}$. These respectively stand for *non lsa*, *unambiguous lsa* and *ambiguous lsa*. We then let $\mathsf{B} = \mathsf{B}_{\mathsf{nlsa}} \cup \mathsf{B}_{\mathsf{ualsa}} \cup \mathsf{B}_{\mathsf{alsa}}$. Suppose $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ and $\mathsf{Lim} \restriction \beta$ have been defined and $\beta = \gamma + 1$. Let $\mathcal{Q} = \mathsf{Lim}(\gamma)$. The folowing condition is part of the definition of B.

(B1) Suppose $\mathcal{Q} \notin \mathsf{B}^0$. Then let $\mathsf{B}(\beta)$ be undefined.

Suppose then that $\mathcal{Q} = \mathcal{J}^{\vec{E}, f}_{\xi+\nu} \in \mathsf{B}^0$ and let $\mathcal{R} \in Y^{\mathcal{Q}}$ be the least member of $Y^{\mathcal{Q}}$ witnessing that $\mathcal{Q} \in \mathsf{B}^0$. Let Λ be the strategy of $\mathcal{R}$ induced by Σ. We say $\mathcal{R}$ is layerable if one of the following conditions holds:

1. $\mathcal{R}$ is of successor type and $\mathcal{R} = Lp_\omega^{\Gamma, \Lambda_{\mathcal{R}(\lambda^{\mathcal{R}}-1)}}(\mathcal{R}|\delta^{\mathcal{R}})$.

2. $\mathcal{R}$ is of limit type but not of lsa type and $\mathcal{R}^b = Lp_\omega^{\Gamma, \oplus_{\alpha<\lambda^{\mathcal{R}^b}} \Lambda_{\mathcal{R}(\alpha)}}(\mathcal{R}^b|\delta^{\mathcal{R}^b})$.

3. $\mathcal{R}$ is of lsa type and $\mathcal{R}^b = Lp_\omega^{\Gamma,\oplus_{\alpha<\lambda^{\mathcal{R}^b}}\Lambda_{\mathcal{R}(\alpha)}}(\mathcal{R}^b|\delta^{\mathcal{R}^b})$ and $\mathcal{J}_1(\mathcal{Q}) \vDash$ "$\delta^{\mathcal{R}}$ is not a Woodin cardinal".

The next three definitions will use the notation introduced above. In all three definitions, we will isolate a stack $\vec{\mathcal{T}}$ based on $\mathcal{R}$ and a branch b of $\vec{\mathcal{T}}$. Then letting $\tilde{b} \subseteq \xi + \nu$ be given by $\xi + \zeta \in \tilde{b} \leftrightarrow \zeta \in b$, set $f^+ = f \cup \{(trc(\vec{\mathcal{T}}), \tilde{b})\}$. If one of the following conditions is satisfied then we will let $\mathsf{B}(\beta)$ be undefined.

($\mathsf{B2}$) $M \vDash$ "for some $\kappa \in [|\mathcal{R}|, \delta) \Vdash_{Coll(\omega,\kappa)} (\mathcal{R}, \Lambda) \in HP^{\Gamma}$"[8], $\sup(b) \neq \nu$ or $\mathsf{Bad}(\mathcal{Q}, f^+)$.

Definition 4.3.4 (The non lsa branch operator) *Suppose $\mathcal{R}$ is layerable and let $\vec{\mathcal{T}} \in \mathcal{Q} - dom(\Sigma_{\mathcal{R}}^{\mathcal{Q}})$ be the $\mathcal{Q}$-least stack that is according to $\Sigma_{\mathcal{R}}^{\mathcal{Q}}$, $lh(\vec{\mathcal{T}})$ is not of measurable cofinality in $\mathcal{Q}$,[9] and $\Sigma_{\mathcal{R}}^{\mathcal{Q}}(\vec{\mathcal{T}})$ is not defined. Set $b = \Lambda(\vec{\mathcal{T}})$. If $\mathsf{B2}$ holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{nlsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{nlsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

The following condition is also part of the definition of B.

($\mathsf{B3}$) Suppose $\mathcal{R}$ is of lsa type and $\mathcal{J}_1(\mathcal{Q}) \vDash$ "$\delta^{\mathcal{R}}$ is a Woodin cardinal". If $\mathcal{Q}$ is not an sts premouse over $\mathcal{R}$ based on $\mathcal{M}^+(\mathcal{R}|\delta^{\mathcal{R}})$ or it is but it is not closed under sharps then let $\mathsf{B}(\beta)$ be undefined.

Suppose then $\mathcal{Q}$ is an sts premouse over $\mathcal{R}$ based on $\mathcal{M}^+(\mathcal{R}|\delta^{\mathcal{R}})$ and $\mathcal{Q}$ is closed under sharps.

Definition 4.3.5 (The ambiguous branch operator) *Suppose $\mathcal{Q}$ is ambiguous and let $t \in \mathcal{Q}$ be the $\mathcal{Q}$-least stack of length 2 witnessing this. Again since $\mathcal{Q} \in \mathsf{B}^0$, we can require $lh(t)$ is not of measurable cofinality in $\mathcal{Q}$. Let $\Lambda^{stc}(t) = b$. If $\mathsf{B2}$ holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{alsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{alsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

Definition 4.3.6 (The unambiguous branch operator) *Suppose $\mathcal{Q}$ is unambiguous. Suppose there is no $\mathcal{Q}$-terminal $\mathcal{T}$ that has a $\mathcal{Q}$-shortness witness. Then let $\mathsf{B}(\beta)$ be undefined. Suppose then that there is a $\mathcal{Q}$-terminal $\mathcal{T}$ that has a $\mathcal{Q}$-shortness witness and $\mathcal{T}$ is chosen as in the definition of $\mathcal{Q} \in \mathsf{B}^0$. Let $(\mathcal{T}, b) \in \mathcal{Q}$ be the lexicographically $\mathcal{Q}$-least pair such that for some (ξ, ν), $\mathcal{T}$ is $\mathcal{Q}$-terminal and (ξ, ν, b) is a minimal $\mathcal{Q}$-shortness witness. If $\mathsf{B2}$ holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{ualsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{ualsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

[8] See the discussion after Definition 4.3.9

[9] $\vec{\mathcal{T}}$ exists because $\mathcal{Q} \in \mathsf{B}^0$.

Finally set $\mathsf{B}(\beta) = \mathsf{B}_{\mathsf{nlsa}}(\beta) \cup \mathsf{B}_{\mathsf{ualsa}}(\beta) \cup \mathsf{B}_{\mathsf{alsa}}(\beta)$. Next we define the constructibility operator.

Definition 4.3.7 (The constructibility operator) *Suppose* $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ *and* $\mathsf{Lim} \restriction \beta$ *have been defined and* $\beta = \gamma + 1$. *Let*

$$\mathcal{Q} = \begin{cases} \mathsf{J}(\gamma) & : \gamma \in dom(\mathsf{J}) \\ \mathsf{B}(\gamma) & : \gamma \in dom(\mathsf{B}) \\ \mathsf{Lim}(\gamma) & : \gamma \in dom(\mathsf{Lim}) \end{cases}$$

Then

$$\mathsf{J}(\beta) = \begin{cases} undefined & : \beta \in dom(\mathsf{E}) \cup dom(\mathsf{B}) \\ undefined & : \beta \notin dom(\mathsf{E}) \cup dom(\mathsf{B}) \text{ and } \mathsf{Bad}(\mathcal{Q}) \text{ holds} \\ \mathcal{J}_1(\mathcal{Q}) & : otherwise \end{cases}$$

Finally we define the limit operator.

Definition 4.3.8 (The limit operator) *Suppose* $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ *and* $\mathsf{Lim} \restriction \beta$ *have been defined and* β *is a limit ordinal. For* $\gamma < \beta$, *let*

$$\mathcal{Q}_\gamma = \begin{cases} \mathsf{J}(\gamma) & : \gamma \in dom(\mathsf{J}) \\ \mathsf{B}(\gamma) & : \gamma \in dom(\mathsf{B}) \\ \mathsf{Lim}(\gamma) & : \gamma \in dom(\mathsf{Lim}) \end{cases}$$

Given an ordinal ξ, *we let* $\mathcal{Q}^\xi$ *be the eventual value of* $\mathcal{Q}_\gamma||\xi$ *as* γ *approaches* β *provided this eventual value exists. Then*

$$\mathsf{Lim}(\beta) = \begin{cases} undefined & : \text{ for some } \xi,\ \mathcal{Q}^\xi \text{ is undefined} \\ undefined & : \mathsf{Bad}(\cup_{\xi \in \mathsf{Ord}} \mathcal{Q}^\xi) \text{ holds} \\ \cup_{\xi \in Ord} \mathcal{Q}^\xi & : \text{ otherwise.} \end{cases}$$

We say $\mathcal{Q}$ appears at stage β if $\mathcal{Q}$ is the value of one of the construction operators at β. We let $\mathcal{Q}_\beta$ be this model and Σ_β be the strategy of $\mathcal{Q}_\beta$ induced by Σ. We then say that $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ are the models and strategies of the hod pair constructions of $\mathbb{M}$. When $\Gamma = \wp(\mathbb{R})$ we omit it from our terminology. The following is the final condition signaling the halt of the construction.

(LSA) If for some limit β, $\mathcal{Q}_\beta$ is of lsa type such that $\mathcal{Q}_\beta = Lp^{\Gamma, \Sigma_\beta^{stc}}(\mathcal{M}^+(\mathcal{Q}_\beta | \delta^{\mathcal{Q}_\beta}))$ then stop the construction.

Definition 4.3.9 (Hod pair constructions) *The Γ-hod pair construction of $\mathbb{M}$ below δ is the quadruple $(\mathsf{E}^{\mathbb{M}}, \mathsf{B}^{\mathbb{M}}, \mathsf{J}^{\mathbb{M}}, \mathsf{Lim}^{\mathbb{M}})$. We say that the hod pair construction is successful if $\alpha^{\mathbb{M}} = \delta$. We say $\mathcal{Q}$ is a model appearing in the hod pair construction of $\mathbb{M}$ if for some $\beta < \alpha^{\mathbb{M}}$,*

$$\mathcal{Q} = \begin{cases} \mathsf{E}^{\mathbb{M}}(\beta) & : \beta \in dom(\mathsf{E}^{\mathbb{M}}) \\ \mathsf{B}^{\mathbb{M}}(\beta) & : \beta \in dom(\mathsf{B}^{\mathbb{M}}) \\ \mathsf{J}^{\mathbb{M}}(\beta) & : \beta \in dom(\mathsf{J}^{\mathbb{M}}) \\ \mathsf{Lim}^{\mathbb{M}}(\beta) & : \beta \in dom(\mathsf{Lim}^{\mathbb{M}}) \end{cases}$$

Our statement of $\mathsf{B2}$ is somewhat ambiguous. We now explain the notation $M \vDash$ "for some $\kappa \in [|\mathcal{R}|, \delta) \Vdash_{Coll(\omega,\kappa)} (\mathcal{R}, \Lambda) \in HP^{\Gamma}$". Provided $\mathcal{R}$ is countable, the meaning of $M \vDash$ "$(\mathcal{R}, \Lambda) \in HP^{\Gamma}$" was explained in Remark 4.1.7. The following discussion and lemma makes the notation meaningful.

We assume that $\mathbb{M}$ Suslin, co-Suslin captures Γ via the pair (P, Ψ). Because $\mathbb{M}$ is self-capturing, we have that whenever g is $Coll(\omega, \mathcal{R})$-generic and η is an $M[g]$-cardinal, $\Lambda \restriction V_\eta^{M[g]}$ has a uniform definition in η and parameters from M (recall that Λ is the induced strategy of $\mathcal{R}$, it is build according to the procedure described in [8, Chapter 12]). The following lemma is an easy consequence of genericity iterations.

Lemma 4.3.10 *Suppose $g \subseteq Coll(\omega, \kappa)$ is M-generic and $\sigma \in \mathbb{R}^{M[g]}$ is a continuous function such that $M \vDash \sigma^{-1}[Code(\Psi)] = Code(\Lambda \restriction HC^{M[g]})$. Then $Code(\Lambda) = \sigma^{-1}[Code(\Psi)]$. In particular, if $M[g] \vDash$ "$(\mathcal{R}, \Lambda \restriction HC^{M[g]}) \in HP^{\Gamma}$" then $(\mathcal{R}, \Lambda) \in HP^{\Gamma}$.*

4.4 Iterability of backgrounded constructions

Our first definition is a game that we will use to show that hod pair constructions inherit an $(\omega_1, \omega_1, \omega_1)$-iteration strategies. As is customary, unless it is specifically mentioned that a transitive set M is fine structural, all iteration trees on M are coarse, meaning that extenders used to build the tree are all total (there are no drops in such iterations).

Definition 4.4.1 ($\mathcal{G}(M, \kappa, \lambda, \nu)$) *Suppose M is a transitive model of some fragment of ZFC. Then $\mathcal{G}(M, \kappa, \lambda, \nu)$ is an iteration game on M with following rules.*

1. *$\mathcal{G}(M, \kappa, \lambda, \nu)$ has at most κ main rounds.*

2. *If M_α is the model at the beginning of the αth main round then the αth main round is a run of $\mathcal{G}(M_\alpha, \lambda, \nu)$.*

3. *Suppose p is a run of $\mathcal{G}(M, \kappa, \lambda, \nu)$ with α main rounds and $(M_\gamma : \gamma < \alpha)$ are the models at the beginning of the main rounds of p. Then if $\beta < \gamma < \alpha$ and $\gamma + 1 < \alpha$ then the iteration embedding $\pi : M_\beta \to M_\gamma$ exists.*

4. *I is the player starting the main rounds. She does it as follows. Suppose p is a run of $\mathcal{G}(M, \kappa, \lambda, \nu)$ with α main rounds. Let $(M_\gamma : \gamma < \alpha)$ be the models at the beginning of the main rounds of p and let $\pi_{\beta,\gamma} : M_\beta \to M_\gamma$ be the iteration embeddings. Then there are two cases.*

   (a) *Suppose α is limit. Letting M_α be the direct limit of $(M_\gamma : \gamma < \alpha)$ under the iteration maps $\pi_{\beta,\gamma}$, I can start the αth main round on M_α.*

   (b) *Suppose $\alpha = \beta + 1$. Let $\vec{\mathcal{T}}$ be the stack of iteration trees produced during the βth main round. I can start a new main round only if $\vec{\mathcal{T}}$ has a last model. Suppose then this is the case and let Q be the last model of $\vec{\mathcal{T}}$. Suppose $\xi < o(Q)$. For $\gamma \leq \beta$ let $\xi_\gamma \in M_\gamma$ be such that $\pi^{\vec{\mathcal{T}}} \circ \pi_{\gamma,\beta}(\xi_\gamma) \geq \xi$ and let E_γ be the (ξ_γ, ξ)-extender derived from $\pi^{\vec{\mathcal{T}}} \circ \pi_{\gamma,\beta}$. Then I may choose any $\gamma \leq \beta$, set $M_\alpha = Ult(M_\gamma, E_\gamma)$ and start the main new round on M_α.*[10]

5. *II wins the game if all the models produced in the iteration game are well-founded.*

We say M is (κ, λ, ν)-iterable if II has a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$. We also say Σ is a (κ, λ, ν)-strategy for M if Σ is a winning strategy for II in $\mathcal{G}(M, \kappa, \lambda, \nu)$. As is usual, when M has a distinguished extender sequence then player I can only play extenders from the images of the distinguished extender sequence of M.

As we show below a winning strategy in $\mathcal{G}(M, \kappa, \lambda)$ induces a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$. We will use the following notation. Given an iteration strategy Σ let $dom^+(\Sigma) = \{\vec{\mathcal{T}} : \vec{\mathcal{T}}$ is according to $\Sigma\}$.

Definition 4.4.2 (Certified strategy) *Suppose M and N are two transitive models of $\mathsf{ZFC} - \mathsf{Powerset}$. Suppose Σ and Λ are iteration strategies for M and N respectively (in one of the iteration games that we have defined, not necessarily the*

[10] As in clause 5 of Definition 2.7.3, player I can choose $\gamma < \beta$ only once in a run of the game.

same). We say Σ is certified by Λ if there is a set X and a function $F : dom^+(\Sigma) \to dom^+(\Lambda) \times X$ such that the following holds:

1. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$, $\vec{\mathcal{U}}$ has a last model iff $(F(\vec{\mathcal{U}}))_0$ has a last model.*

2. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$, if $\vec{\mathcal{U}}$ has a last model then letting Q and R be the last models of $\vec{\mathcal{U}}$ and $(F(\vec{\mathcal{U}}))_0$, $(F(\vec{\mathcal{U}}))_1 = \sigma$ such that $\sigma : Q \to_{\Sigma_1} R$.*

3. *For all $\vec{\mathcal{U}} \in dom^+(\Sigma)$ if $\alpha < lh(\vec{\mathcal{U}})$ then letting $\vec{\mathcal{T}} = (F(\vec{\mathcal{U}}))_0$ and $\vec{\mathcal{T}}^* = (F(\vec{\mathcal{U}} \restriction \alpha))_0$ then $\vec{\mathcal{T}}^*$ is an initial segment of $\vec{\mathcal{T}}$.*

4. *If $\vec{\mathcal{T}}$ is a stack on M according to Σ with last model Q and $\mathcal{U}$ is a normal tree on Q then letting R be the last model $(F(\vec{\mathcal{T}}))_0$ and $\mathcal{W}$ be such that $(F(\vec{\mathcal{T}}))_0{}^\frown\mathcal{W} = F(\vec{\mathcal{T}}{}^\frown\mathcal{U})_0$ then $\mathcal{W}$ is a normal tree such that $lh(\mathcal{U}) = lh(\mathcal{W})$, and for every $\alpha_0, \alpha_1 < lh(\mathcal{U})$, letting $\beta_0, \beta_1 < lh(\mathcal{W})$ be such that for $i = 0, 1$, $F(\vec{\mathcal{T}}{}^\frown\mathcal{U} \restriction \alpha_i + 1)_0 = (F(\vec{\mathcal{T}}))_0{}^\frown\mathcal{W} \restriction \beta_i + 1$,*
   1. *(a) $\alpha_0 <_U \alpha_1 \leftrightarrow \beta_0 <_W \beta_1$*
   2. *(b) letting for $i = 0, 1$, $\sigma_i = (F(\vec{\mathcal{T}}{}^\frown\mathcal{U} \restriction \alpha_i + 1))_1$, if $\alpha_0 <_U \alpha_1$ then $\pi^{\mathcal{W}}_{\beta_0,\beta_1} \circ \sigma_0 = \sigma_1 \circ \pi^{\mathcal{U}}_{\alpha_0,\alpha_1}$.*

Clearly pullback constructions produce certified strategies.

Theorem 4.4.3 *Suppose M is a transitive model of some fragment of* ZFC *and $\kappa \leq \lambda$. Then if II has a winning strategy Λ in $\mathcal{G}(M, \lambda, \nu)$ then II has a winning strategy in $\mathcal{G}(M, \kappa, \lambda, \nu)$ certified by Λ.*

Proof. Suppose we have defined F as in Definition 4.4.2 on $\vec{\mathcal{T}} \in \text{dom}(\Sigma)$ which have $< \alpha$-many main rounds. We want to define F on $\vec{\mathcal{T}}$ with exactly α-many main rounds. We assume that α is a successor and leave the rest to the reader. Let $\alpha = \beta + 1$. Thus, we need to extend Σ to act on $\beta + 1$st round of $\mathcal{G}(M, \kappa, \lambda, \nu)$. Let then $\vec{\mathcal{T}} \in \text{dom}(\Sigma)$ be such that $\text{lh}(\vec{\mathcal{T}}) = \beta + 1$ and $\vec{\mathcal{T}}$ has a last model Q. Let R be the last model $\vec{\mathcal{U}} = (F(\vec{\mathcal{T}}))_0$ and let $\sigma = (F(\vec{\mathcal{T}}))_1$.

Suppose that I wants to start a new main round. Suppose then $(M_\gamma : \gamma \leq \beta)$ are the models at beginning of the main rounds of $\vec{\mathcal{T}}$. Suppose $\xi < o(Q)$. For $\gamma \leq \beta$ let $\xi_\gamma \in M_\gamma$ be such that $\pi^{\vec{\mathcal{T}}} \circ \pi_{\gamma,\beta}(\xi_\gamma) \geq \xi$ and let E_γ be the (ξ_γ, ξ)-extender derived from $\pi^{\vec{\mathcal{T}}} \circ \pi_{\gamma,\beta}$. Suppose then I sets $M_\alpha = Ult(M_\gamma, E_\gamma)$ where $\gamma \leq \beta$. Let $k : M_\alpha \to Q$ be the factor embedding. Thus

$$\pi^{\vec{\mathcal{T}}} = k \circ \pi_{E_\gamma}.$$

Let $\pi = \sigma \circ k \circ \pi_{E_\gamma}$. We then let $F(\vec{\mathcal{T}}^\frown\{M_{\beta+1}\}) = (\vec{\mathcal{U}}, \pi)$ which clearly has the desired properties. Next we require that II plays the $\beta+1$st round on $M_{\beta+1}$ according to π-pullback of $\Lambda_{R,\vec{\mathcal{U}}}$. □

4.5 Fullness preservation

Throughout this section we assume AD^+. Suppose that $\Gamma \subseteq \wp(\mathbb{R})$ is a pointclass, $X \in HC$ is a self-well-ordered set (swo),[11] $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $\mathcal{P} \in HC$ and $Code(\Sigma) \in \Gamma$.[12] Below, we use $\mathcal{R}^*$ to denote the $*$-translation of $\mathcal{R}$ (cf. [18]).

Definition 4.5.1 (Γ-Fullness preservation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $\mathcal{P} \in HC$ and Γ is a pointclass. We say Σ is Γ-fullness preserving if the following holds for all $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma)$.*

1. *For all limit type $\mathcal{R} \in Y^{\mathcal{Q}}$, $\mathcal{R}^b = Lp_\omega^{\Gamma, \oplus_{\mathcal{S}\in Y^{\mathcal{R}^b}} \Sigma_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\delta^{\mathcal{R}})$.*

2. *For all successor type $\mathcal{R} \in Y^{\mathcal{Q}}$,*
$$\mathcal{R} = Lp_\omega^{\Gamma, \oplus_{\mathcal{S}\in Y^{\mathcal{R}^b}} \Sigma_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\delta^{\mathcal{R}}).$$

3. *If $\mathcal{Q}$ is of lsa type then $\mathcal{Q} = Lp_\omega^{\Gamma, \Sigma^{stc}_{\mathcal{M}^+(\mathcal{Q}|\delta^{\mathcal{Q}}),\vec{\mathcal{T}}}}(\mathcal{Q}|\delta^{\mathcal{Q}})$[13],*

4. *If η is a cardinal cutpoint of $\mathcal{Q}$ such that for some $\mathcal{R}_1, \mathcal{R}_2 \in Y^{\mathcal{Q}}$ such that $\mathcal{R}_2$ is the $\mathcal{Q}$-successor of $\mathcal{R}_1$ (see Definition 3.9.2), $\mathcal{R}_1$ is a cutpoint of $\mathcal{Q}$ and $\eta \in (\delta^{\mathcal{R}_1}, \delta^{\mathcal{R}_2})$ then*
$$(\mathcal{Q}|(\eta^+)^{\mathcal{Q}})^* = Lp^{\Gamma, \Sigma_{\mathcal{R}_1}, \vec{\mathcal{T}}}(\mathcal{Q}|\eta).$$

The next lemma follows from clause 4 above. Below $\mathcal{S}^*(\mathcal{R})$ is the $*$-transform of $\mathcal{S}$ into a mouse over $\mathcal{R}$, it is defined when $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ (cf. [18]).

[11] Recall that X is a self-well-ordered set if $\mathcal{J}_\omega(X) \vDash$ "X is well-orderable".
[12] Recall that $Code(\Sigma)$ is the set of reals coding Σ.
[13] Here, if Σ is a short tree strategy then $\Sigma^{sts} = \Sigma$.

Lemma 4.5.2 *Let $(\vec{\mathcal{T}}, \mathcal{T})$ be a countable tree via Σ, consisting of a stack $\vec{\mathcal{T}}$ followed by a normal tree $\mathcal{T}$, such that $\mathcal{T}$ has successor length and $b^{\mathcal{T}}$ drops. Let $\mathcal{Q} = \mathcal{M}^{\mathcal{T}}_{\infty}$ and $\lambda = \lambda^{\mathcal{Q}}$. Suppose $\mathcal{Q}(\lambda)$ is a cutpoint of $\mathcal{Q}$.*[14] *Let γ be least such that $o(\mathcal{Q}(\lambda)) < \mathrm{lh}(E^{\mathcal{T}}_{\gamma})$ and let $\mathcal{U} = \vec{\mathcal{T}}^\frown(\mathcal{T} \restriction (\gamma+1))$. (Note $b^{\mathcal{U}}$ does not drop.) Let $\mathcal{R}, \mathcal{S}$ be such that $\mathcal{Q}(\lambda) \trianglelefteq \mathcal{R} \triangleleft \mathcal{S} \trianglelefteq \mathcal{Q}$ and $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ and $\mathcal{S}$ projects $\leq o(\mathcal{R})$ and is $o(\mathcal{R})$-sound (so either $\mathcal{S} \triangleleft \mathcal{Q}$ or all generators of $\mathcal{T}$ are $< o(\mathcal{R})$). Then letting $\Lambda = \Sigma^{stc}_{\mathcal{Q}(\lambda),\mathcal{U}}$ if $\mathcal{Q}(\lambda)$ is of lsa-type and $\Lambda = \Sigma_{\mathcal{Q}(\lambda),\mathcal{U}}$ otherwise,*

$$\mathcal{S}^*(\mathcal{R}) \triangleleft \mathrm{Lp}^{\Gamma,\Lambda}(\mathcal{R}).$$

Theorem 4.5.3 (Fullness preservation of induced strategies) *Assume AD^+. Suppose for some α such that $\theta_\alpha < \Theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$ and $\mathbb{M} = (M, \delta, \Sigma)$ is a self-capturing background triple that Suslin, co-Suslin captures Γ via (P, Ψ). Let $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ be the models and strategies of the Γ-hod pair construction of $\mathbb{M}$. Suppose β is such that $(\mathcal{Q}_\beta, \Sigma_\beta) \in HP^\Gamma$ and that for some $<\delta$-generic g, there is a continuous function $\sigma \in M[g] \cap \mathbb{R}$ such that $\sigma^{-1}[Code(\Psi)] = Code(\Sigma_\beta)$. Then Σ_β is Γ-fullness preserving.*

Proof. Let $\mathcal{P} = \mathcal{Q}_\beta$ and $\Lambda = \Sigma_\beta$. Towards a contradiction, assume Λ is not Γ-fullness preserving. It follows by absoluteness (see Lemma 4.1.3 and Corollary 4.1.5) that there is a counterexample in $M[g]$ where g is $<\delta$-generic. Fix a $<\delta$-generic g such that there $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Lambda) \cap M[g]$ witnessing that Λ isn't Γ-fulness preserving. All the clauses of Γ-fullness preservation are very similar and follow from the universality of background constructions. Below we derive a contradiction from the failure of clause 1 of Definition 4.5.1 and leave the rest to the reader.

Fix $\mathcal{R}^* \in Y^{\mathcal{Q}}$ witnessing the failure of clause 1 of Definition 4.5.1. Let $\mathcal{R} = (\mathcal{R}^*)^b$ and $\kappa = \delta^{\mathcal{R}}$. We need to see that

$$\mathcal{R} = Lp_\omega^{\Gamma, \oplus_{\mathcal{S} \in Y^{\mathcal{R}}} \Lambda_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

We only show that

$$\mathcal{R}|(\kappa^+)^{\mathcal{R}} = Lp^{\Gamma, \oplus_{\mathcal{S} \in Y^{\mathcal{R}}} \Lambda_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

and leave the rest to the reader.

Suppose first that $\mathcal{M} \trianglelefteq \mathcal{R}|(\kappa^+)^{\mathcal{R}}$ is a $\oplus_{\mathcal{S} \in Y^{\mathcal{R}}} \Lambda_{\mathcal{S},\vec{\mathcal{T}}}$-mouse over $\mathcal{R}|\kappa$ such that $\rho(\mathcal{M}) \leq \kappa$. Because $\mathcal{P}$ is constructed via backgrounded construction, it follows that $\mathcal{M}$ is ω_1-iterable as a $\oplus_{\mathcal{S} \in Y^{\mathcal{R}}} \Lambda_{\mathcal{S},\vec{\mathcal{T}}}$-mouse and therefore,

[14] So $\mathcal{Q}$ is a $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}^\frown\mathcal{T}}$ (or $\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}^\frown\mathcal{T}}$)-mouse over $\mathcal{Q}(\lambda)$ and $\mathcal{Q}(\lambda)$ is the largest hod premouse $\mathcal{R} \triangleleft \mathcal{Q}$.

$$\mathcal{M} \trianglelefteq Lp^{\Gamma, \oplus_{\mathcal{S}\in Y^{\mathcal{R}}}\Lambda_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\kappa).$$

Fix now $\mathcal{M} \trianglelefteq Lp^{\Gamma, \oplus_{\mathcal{S}\in Y^{\mathcal{R}}}\Lambda_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{R}|\kappa)$ and let Φ be its ω_1-strategy. We let $\pi = \pi^{\vec{\mathcal{T}}}$. Let $\mathcal{N} = (L[\oplus_{\mathcal{S}\in Y^{\mathcal{P}^b}}\Lambda_{\mathcal{S}}][\mathcal{P}^b])^{V_\delta^M}$ and notice that if $E = E_\pi \restriction \pi(\delta^{\mathcal{P}})$ then $M \models$ "$Ult(\mathcal{N}, E)$ is δ-iterable"[15]. Let then $\pi^+ = \pi_E^{\mathcal{N}}$ and let

$$\mathcal{N}^* = (\mathcal{J}^{\vec{E}, \oplus_{\mathcal{S}\in Y^{\mathcal{R}}}\Lambda_{\mathcal{S},\vec{\mathcal{T}}}})^{\pi^+(\mathcal{N})}.$$

It then follows that $\mathcal{N}^*$ too is δ-iterable and so we can compare $\mathcal{N}^*$ with $\mathcal{M}$. By universality of backgrounded constructions, $\mathcal{M}$ has to lose the comparison implying that $\mathcal{M} \trianglelefteq \mathcal{N}^*$. Therefore, $\mathcal{M} \in \pi^+(\mathcal{N})$. Since $\mathcal{M}$ is ω_1-iterable, it follows that $\mathcal{M} \trianglelefteq \mathcal{R}$. □

The proof actually gives more.

Definition 4.5.4 (Strongly Γ-fullness preserving) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair and Γ is a pointclass. We say Σ is strongly Γ-fullness preserving if Σ is Γ-fullness preserving and whenever*

1. *$\vec{\mathcal{T}}$ is a tree according to Σ with last model $\mathcal{S}$ such that if $\mathcal{P}$ is of limit type then $\pi^{\vec{\mathcal{T}},b}$ exists and otherwise $\pi^{\vec{\mathcal{T}}}$ exists, and*
2. *$\mathcal{R}$ is such that there are (σ, τ) with the property that*
   - *(a) if $\mathcal{P}$ is of limit type then $\sigma : \mathcal{P}^b \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}^b$ and $\pi^{\vec{\mathcal{T}},b} = \tau \circ \sigma$, and*
   - *(b) if $\mathcal{P}$ is of successor type then $\sigma : \mathcal{P} \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}$ and $\pi^{\vec{\mathcal{T}}} = \tau \circ \sigma$,*

then the τ-pullback strategy of $\Sigma_{\mathcal{S}^b,\vec{\mathcal{T}}}$ if 2(a) holds and of $\Sigma_{\mathcal{S},\vec{\mathcal{T}}}$ if 2(b) holds is Γ-fullness preserving.

The following is then a corollary to the proof of Theorem 4.5.3 and we leave it to the reader.

Theorem 4.5.5 (Strong fullness preservation of induced strategies) *Assume* AD^+*. Suppose for some α such that $\theta_\alpha < \Theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$*

[15]The proof of the iterability of $\mathcal{N}$ shows that $Ult(\mathcal{N}, E)$ is well-founded. To see this let $\vec{\mathcal{W}}$ be the tree on M, according to Σ obtained by lifting $\vec{\mathcal{T}}$ to M. We then have $\sigma : \mathcal{Q} \to \pi^{\vec{\mathcal{W}}}(\mathcal{P})$ such that $\pi^{\vec{\mathcal{W}}} \restriction \mathcal{P} = \sigma \circ \pi$. It is now not hard to see that σ extends to $\sigma^+ : Ult(\mathcal{N}, E) \to \pi^{\vec{\mathcal{W}}}(\mathcal{N})$. The same argument shows that $Ult(\mathcal{N}, E)$ is δ-iterable in M.

and $\mathbb{M} = (M, \delta, \Sigma)$ is a self-capturing background triple that Suslin, co-Suslin captures Γ via (P, Ψ). Let $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ be the models and strategies of the hod pair construction of $\mathbb{M}$. Suppose β is such that $(\mathcal{Q}_\beta, \Sigma_\beta) \in HP^\Gamma$ and that for some $< \delta$-generic g, there is a continuous function $\sigma \in M[g] \cap \mathbb{R}$ such that $\sigma^{-1}[Code(\Psi)] = Code(\Sigma_\beta)$. Then Σ_β is strongly Γ-fullness preserving.

The following is an easy yet useful consequence of strong fullness preservation.

Lemma 4.5.6 *Assume AD^+ and suppose Γ is a pointclass. Suppose further that $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that Σ is strongly Γ-fullness preserving. Let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$ according to Σ with last model $\mathcal{S}$ such that if $\mathcal{P}$ is of limit type then $\pi^{\vec{\mathcal{T}},b}$ exists and otherwise $\pi^{\vec{\mathcal{T}}}$ exists. Suppose $(\mathcal{R}, \sigma, \tau)$ is such that*

1. *if $\mathcal{P}$ is of limit type then $\sigma : \mathcal{P}^b \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}^b$ and $\pi^{\vec{\mathcal{T}},b} = \tau \circ \sigma$, and*
2. *if $\mathcal{P}$ is of successor type then $\sigma : \mathcal{P} \to \mathcal{R}$, $\tau : \mathcal{R} \to \mathcal{S}$ and $\pi^{\vec{\mathcal{T}}} = \tau \circ \sigma$.*

Let E be such that

1. *if $\mathcal{P}$ is of limit type then E is the $(\delta^{\mathcal{P}^b}, \delta^{\mathcal{R}})$-extender derived from σ, and*
2. *if $\mathcal{P}$ is of successor type then E is the $(\delta^{\mathcal{P}}, \delta^{\mathcal{R}})$-extender derived from σ*

Then $\mathcal{R} = Ult(\mathcal{P}, E)$. In particular, $\mathcal{R} = \{\pi_E(f)(a) : f \in \mathcal{P}$ and $\alpha \in (\delta^{\mathcal{R}})^{<\omega}\}$.

Proof. Let $k : Ult(\mathcal{P}, E) \to \mathcal{R}$ be the factor map, i.e., $k(\pi(f)(a)) = \sigma(f)(a)$. Then if $\mathcal{P}$ is of limit type then $\pi^{\vec{\mathcal{T}},b} = \tau \circ k \circ \pi_E$ and if $\mathcal{P}$ is of successor type then $\pi^{\vec{\mathcal{T}}} = \tau \circ k \circ \pi_E$. Notice that $\mathrm{crit}(k) > \delta^{\mathcal{R}}$. It now follows from strong Γ-fullness preservation of Σ that $\Sigma^{\tau \circ k}_{\mathcal{S}, \vec{\mathcal{T}}}$ is Γ-fullness preserving. But because $k \restriction \delta^{\mathcal{R}} = id$, we have that for every $\alpha + 1 \leq \lambda^{\mathcal{R}}$,

$$(\Sigma^{\tau \circ k}_{\mathcal{S}, \vec{\mathcal{T}}})_{\mathcal{R}(\alpha+1)} = (\Sigma^{\tau}_{\mathcal{S}, \vec{\mathcal{T}}})_{\mathcal{R}(\alpha+1)}.$$

It then follows that $\mathcal{R} = Ult(\mathcal{P}, E)$. $\square$

4.6 The normal-tree comparison theory

As in Theorem 2.2.2 of [10], under AD^+ and in several other contexts, we can prove a comparison theorem where comparison is achieved via normal trees. In this section we state a comparison theorem for hod pairs that can be applied inside models of AD^+ and also, inside models satisfying sufficiently rich extensions of ZFC, like hod mice themselves. Such comparison arguments, among other things, are useful in core model induction arguments and in the analysis of HOD of models of AD^+.

We start with some general definitions and facts. One warning is that our exposition differs from the one in [10] mainly because we would like to set up our arguments here in a more general setting than the ones stated in [10].

Definition 4.6.1 (Comparison) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then we say comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there is $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ such that one of the following holds:*

1. *$\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Lambda_{\mathcal{R},\vec{\mathcal{U}}} = \Sigma_{\mathcal{R},\vec{\mathcal{T}}}$.*
2. *$\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Sigma_{\mathcal{S},\vec{\mathcal{T}}} = \Lambda_{\mathcal{S},\vec{\mathcal{U}}}$.*

We say normal comparison for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ holds if we can take $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ to be normal.

As in [10], we can prove comparison for pairs whose corresponding strategies are fullness preserving.

4.6.1 Tracking disagreements

Here we introduce terminology that we will use to track the disagreements between strategies. Given a stack $\vec{\mathcal{T}}$ on a hod premouse $\mathcal{P}$, we let $\delta(\vec{\mathcal{T}})$ be the sup of the generators of $\vec{\mathcal{T}}$ (see Definition 1.15 of [10]).

Definition 4.6.2 (Low level disagreement between strategies) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs. Suppose there is $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{Q}) \in B(\mathcal{P}, \Lambda)$ such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} \neq \Lambda_{\mathcal{Q},\vec{\mathcal{U}}}$. Then we say that there is a low level disagreement between Σ and Λ. We say $(\vec{\mathcal{T}}, \mathcal{Q})$ constitutes a minimal low level disagreement if*

1. *$\mathcal{Q}$ is of successor type and $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma) \cap B(\mathcal{P}, \Lambda)$,*
2. *for every $\alpha < \lambda^{\mathcal{Q}} - 1$, $\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}} = \Lambda_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}$,*

3. *if $\mathcal{P}$ is meek (see Definition 2.4.8) then* $\delta(\vec{\mathcal{T}}), \delta(\vec{\mathcal{U}}) \subseteq \mathcal{Q}(\lambda^{\mathcal{Q}} - 1)$*,*

4. *if $\mathcal{P}$ is non-meek then letting E be the un-dropping extender of $\vec{\mathcal{T}}$ then*

$$\delta^{\mathcal{Q}} = \sup\{\pi_E(f)(a) : f \in \mathcal{P}^b \wedge a \in \mathcal{Q}(\lambda^{\mathcal{Q}} - 1)\}$$

Next we show that the existence of a disagreement translates into the existence of a minimal low level disagreement.

Lemma 4.6.3 (Disagreement implies low level disagreement) *Suppose Γ is a pointclass closed under Wadge reducibility, and $(\mathcal{P}, \Sigma)$ and $(\mathcal{P}, \Lambda)$ are two hod pairs such that both Σ and Λ are Γ-fullness preserving. Suppose that one of the following conditions holds:*

1. *$\mathcal{P}$ is of limit type and not of lsa type, and $\Sigma \neq \Lambda$.*

2. *$\mathcal{P}$ is of lsa type and $\Sigma^{stc} \neq \Lambda^{stc}$.*

Then there is a minimal low level disagreement between Σ and Λ.

Proof. We give the proof from clause 2 and leave the proof from clause 1, which is easier, to the reader (also, see Proposition 2.41 of [10]). Assume there is no low level disagreement between Σ and Λ. We can also assume without loss of generality that

(1) for any $(\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{Q})$ such that (i) $(\vec{\mathcal{T}}, \mathcal{Q}) \in I^b(\mathcal{P}, \Sigma)$, (ii) $(\vec{\mathcal{U}}, \mathcal{Q}) \in I^b(\mathcal{P}, \Lambda)$ and (iii) there is $\alpha \leq \lambda^{\mathcal{Q}^b}$ such that for every $\beta < \alpha$, $\Sigma_{\mathcal{Q}(\beta), \vec{\mathcal{T}}} = \Lambda_{\mathcal{Q}(\beta), \vec{\mathcal{U}}}$ but $\Sigma_{\mathcal{Q}(\alpha), \vec{\mathcal{T}}} \neq \Lambda_{\mathcal{Q}(\alpha), \vec{\mathcal{U}}}$, there is a minimal low level disagreement between $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ and $\Lambda_{\mathcal{Q}, \vec{\mathcal{U}}}$.

Let now $\vec{\mathcal{T}} = (\mathcal{M}_\alpha, \vec{\mathcal{T}}_\alpha, \mathcal{Q}_\alpha, E_\alpha : \alpha \leq \eta)$ be any disagreement between Σ^{stc} and Λ^{stc}. We must have that η is a successor ordinal, E_η is undefined, and the last normal component of $\vec{\mathcal{T}}_\eta$ is of limit length. Notice that if $\vec{\mathcal{T}}$ has main drops then, because we are assuming (1) above, the claim of the lemma follows. We then assume that $\vec{\mathcal{T}}$ has no main drops.

Notice that there cannot be a club $C \subseteq ntn(\vec{\mathcal{T}})$ as otherwise $\Sigma(\vec{\mathcal{T}}) = \Lambda(\vec{\mathcal{T}}) = b_C$. Let then $\mathcal{S} = \mathcal{S}_{\vec{\mathcal{T}}}$. Thus, $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{S}$. Notice that, because $\vec{\mathcal{T}}$ has no main drops, we must have that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists.

Let now $\mathcal{T} = \vec{\mathcal{T}}_{\geq \mathcal{S}}$. It then follows that $\mathcal{T}$ must be above $o(\mathcal{S}^b)$ as otherwise it will generate a low level disagreement, which then can be easily turned into a minimal low level disagreement[16]. Without loss of generality, we can further assume that $\Sigma_{\mathcal{S}^b,\vec{\mathcal{T}}_{\leq\mathcal{S}}} = \Lambda_{\mathcal{S}^b,\vec{\mathcal{T}}_{\leq\mathcal{S}}}$ as otherwise we get a low level disagreement using (1) and the argument given in the above footnote.

Claim. $\vec{\mathcal{T}} \in b(\Sigma^{stc}) \cap b(\Lambda^{stc})$, i.e., $\Sigma^{stc}(\vec{\mathcal{T}})$ and $\Lambda^{stc}(\vec{\mathcal{T}})$ are branches rather than models.

Proof. To see this suppose that $\vec{\mathcal{T}} \in m(\Sigma^{stc})$. Let $b = \Sigma(\vec{\mathcal{T}})$, $c = \Lambda(\vec{\mathcal{T}})$ and

$$\mathcal{Q} = \mathcal{M}^+(\mathcal{T})(=_{def} (\mathcal{M}(\mathcal{T}))^\#).$$

We now define two hybrid mice $\mathcal{M}_0$ and $\mathcal{M}_1$ and an ordinal ν. Suppose first that $\vec{\mathcal{T}} \in b(\Lambda^{stc})$. We then have that $\mathcal{M}_c^{\vec{\mathcal{T}}} \vDash$ "$\delta^{\mathcal{M}_b^{\vec{\mathcal{T}}}}$ isn't a Woodin cardinal". Let $\mathcal{M}_1 \trianglelefteq \mathcal{M}_c^{\vec{\mathcal{T}}}$ be the largest such that $\mathcal{M}_1 \vDash$ "$\delta^{\mathcal{M}_b^{\vec{\mathcal{T}}}}$ is a Woodin cardinal". Next suppose that $\mathcal{M}_b^{\vec{\mathcal{T}}} \ntrianglelefteq \mathcal{M}_1$. Then we let $\mathcal{M}_0 = \mathcal{M}_b^{\vec{\mathcal{T}}}$. Suppose now that $\mathcal{M}_b^{\vec{\mathcal{T}}} \trianglelefteq \mathcal{M}_1$. It follows from Γ-fullness preservation that for any η, $\mathcal{M}_1 \ntrianglelefteq Lp_\eta^{\Gamma,\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}}(\mathcal{Q})$. Let then η be the least such that $Lp_{\eta+1}^{\Gamma,\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}}(\mathcal{Q}) \ntrianglelefteq \mathcal{M}_1$ and let $\mathcal{M}_0 = Lp_{\eta+1}^{\Gamma,\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}}(\mathcal{Q})$. Finally let $\nu = o(Lp_\eta^{\Gamma,\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}}(\mathcal{Q}))$.

Suppose next that $\vec{\mathcal{T}} \in m(\Lambda^{stc})$. Because $\Lambda^{sts}(\vec{\mathcal{T}}) \neq \Sigma^{sts}(\vec{\mathcal{T}})$, we have that $\mathcal{M}_b^{\vec{\mathcal{T}}} \neq \mathcal{M}_c^{\vec{\mathcal{T}}}$. Set $\xi = \delta(\mathcal{T})$ and let n be least such that $\mathcal{M}_b^{\vec{\mathcal{T}}}|(\xi^{+n})^{\mathcal{M}_b^{\vec{\mathcal{T}}}} = \mathcal{M}_c^{\vec{\mathcal{T}}}|(\xi^{+n})^{\mathcal{M}_c^{\vec{\mathcal{T}}}}$ but $\mathcal{M}_b^{\vec{\mathcal{T}}}|(\xi^{+n+1})^{\mathcal{M}_b^{\vec{\mathcal{T}}}} \neq \mathcal{M}_c^{\vec{\mathcal{T}}}|(\xi^{+n+1})^{\mathcal{M}_c^{\vec{\mathcal{T}}}}$. Let $\nu = (\xi^{+n})^{\mathcal{M}_b^{\vec{\mathcal{T}}}}$, and let $\mathcal{M}_0 \trianglelefteq \mathcal{M}_b^{\vec{\mathcal{T}}}|(\xi^{+n+1})^{\mathcal{M}_b^{\vec{\mathcal{T}}}}$ and $\mathcal{M}_1 \trianglelefteq \mathcal{M}_c^{\vec{\mathcal{T}}}|(\xi^{+n+1})^{\mathcal{M}_c^{\vec{\mathcal{T}}}}$ be least such that $\rho(\mathcal{M}_0) = \rho(\mathcal{M}_1) = \nu$ and $\mathcal{M}_0 \neq \mathcal{M}_1$.

Notice that in both cases we have that

(2) $\mathcal{M}_0 \ntrianglelefteq \mathcal{M}_1$, $\mathcal{M}_1 \ntrianglelefteq \mathcal{M}_0$, $\mathcal{M}_0|\nu = \mathcal{M}_1|\nu$, and $\mathcal{M}_0$ and $\mathcal{M}_1$ are either ν-sound and project to ν or are limit of levels that are ν-sound and project to ν, and ν is a strong cutpoint of both $\mathcal{M}_0$ and $\mathcal{M}_1$.
(3) $\mathcal{M}_0$ is a $\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}$-mouse and $\mathcal{M}_1$ is a $\Lambda^{stc}_{\mathcal{Q},\vec{\mathcal{T}}}$-mouse.
(4) The comparison of $\mathcal{M}_0$ and $\mathcal{M}_1$ cannot halt.

(4) holds as otherwise its failure implies that either $\mathcal{M}_0 \trianglelefteq \mathcal{M}_1$ or $\mathcal{M}_1 \trianglelefteq \mathcal{M}_0$, both of which are impossible (because of (2)).

It follows that the comparison of $\mathcal{M}_0$ and $\mathcal{M}_1$ encounters disagreements involving strategies, as otherwise the usual comparison argument would imply that the comparison halts. Let Φ and Ψ be the canonical strategies of $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively.

[16] To see this suppose $\mathcal{T}$ is based on $\mathcal{S}^b(\alpha+1)$ for some $\alpha+1 < \lambda^{\mathcal{S}^b}$. Let $\mathcal{R}$ be $\preceq^{\mathcal{T}}$-least cutpoint of $\mathcal{T}$ such that for some $\beta+1$, $\pi^{\mathcal{T}}_{\mathcal{R},\mathcal{S}}(\beta+1) = \alpha+1$. Then $(\vec{\mathcal{T}}_{\leq\mathcal{R}}, \mathcal{R})$ constitutes a low level minimal disagreement.

Thus, Φ witnesses that $\mathcal{M}_0$ is a $\Sigma^{sts}_{\mathcal{Q},\vec{\mathcal{T}}}$-mouse, and Ψ witnesses that $\mathcal{M}_1$ is a $\Lambda^{sts}_{\mathcal{Q},\vec{\mathcal{T}}}$-sts mouse.

We can then find Φ-iterate $\mathcal{K}$ of $\mathcal{M}_0$ and Ψ-iterate $\mathcal{N}$ of $\mathcal{M}_1$ such that $\mathcal{K}$ and $\mathcal{N}$ are produced via the usual extender comparison procedure (this implies that both iterations are above ν) and for some α,

(2) $\mathcal{K}|\alpha = \mathcal{N}|\alpha$, $\mathcal{K}||\alpha \neq \mathcal{N}||\alpha$, $\alpha \notin \mathrm{dom}(\vec{E}^{\mathcal{K}})$ and $\alpha \notin \mathrm{dom}(\vec{E}^{\mathcal{N}})$.

Notice that it follows from our indexing scheme (see Definition 3.6.2) that there must be a branch indexed at α in both $\mathcal{K}$ and $\mathcal{N}$. Let then $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{W}, \mathcal{S}_1, \vec{\mathcal{U}}) \in \mathcal{K}||\alpha$ be such that its branch is indexed at α in both $\mathcal{K}$ and $\mathcal{N}$.

We now have to analyze exactly what kind of stack t is. Recall that our indexing scheme is so that we add branches for two kinds of stacks that we now list.

Case 1. $\mathcal{W}$ is an unambiguous normal tree and $\vec{\mathcal{U}}$ is undefined.
Case 2. $\vec{\mathcal{U}}$ is defined and is a stack on $(\mathcal{S}_1)^b$.

We can immediately rule out case 1 above: $\mathcal{K}|\alpha = \mathcal{N}|\alpha$ and the branch of $\mathcal{W}$ just depends on $\mathcal{K}|\beta$ (see Lemma 3.8.2). Case 2 immediately leads to a low level disagreement.

□

Let $b = \Sigma(\vec{\mathcal{T}})$ and $c = \Lambda(\vec{\mathcal{T}})$. Recall that just before the statement of the claim we set $\mathcal{T} = \vec{\mathcal{T}}_{\geq \mathcal{S}}$. It follows from the claim that both $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ exist. Because $b \neq c$, we have that $\mathcal{Q}(b, \mathcal{T}) \neq \mathcal{Q}(c, \mathcal{T})$. It follows that

$$\mathcal{M}^+(\mathcal{T}) \lhd (\mathcal{Q}(b, \mathcal{T}) \cap \mathcal{Q}(c, \mathcal{T})).$$

Let $\mathcal{P}_1 = \mathcal{M}^+(\mathcal{T})$. Notice that it follows from our smallness assumption on hod mice, namely that hod mice do not have lsa hod initial segments, that $\delta(\mathcal{T})$ is not overlapped in both $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$. We then have that $\mathcal{Q}(b, \mathcal{T})$ is a $\Sigma^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}}$-mouse over $\mathcal{P}_1$, $\mathcal{Q}(c, \mathcal{T})$ is a $\Lambda^{stc}_{\mathcal{P}_1,\vec{\mathcal{T}}}$-mouse over $\mathcal{P}_1$ and the comparison of $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ does not halt. Applying the proof of the claim to $\mathcal{M}_0 =_{def} \mathcal{Q}(b, \mathcal{T})$ and $\mathcal{M}_1 =_{def} \mathcal{Q}(c, \mathcal{T})$, we get a minimal low level disagreement.

□

Next we introduce several definitions that will be useful in the sequel.

Definition 4.6.4 (Comparison stack) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs or sts hod pairs. Then we say*

$(\vec{\mathcal{T}}, \mathcal{R}, \vec{\mathcal{U}}, \mathcal{S})$ are comparison stacks for $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ with last models $(\mathcal{R}, \mathcal{S})$

if $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$, $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$, and either

1. *$\mathcal{S} \in Y^{\mathcal{R}}$ and $\Sigma_{\mathcal{S}, \vec{\mathcal{T}}} = \Lambda_{\mathcal{S}, \vec{\mathcal{U}}}$.*

2. *$\mathcal{R} \in Y^{\mathcal{S}}$ and $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}} = \Lambda_{\mathcal{R}, \vec{\mathcal{U}}}$.*

Definition 4.6.5 (Agreement up to the top) *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice of limit type. Then we say $\mathcal{P}$ and $\mathcal{Q}$ agree up to the top if $\lambda^{\mathcal{P}} = \lambda^{\mathcal{Q}}$ and $\mathcal{P}^b = \mathcal{Q}^b$. Suppose further that Σ and Λ are such that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs or sts hod pairs. Then we say $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ agree up to the top if $\mathcal{P}$ and $\mathcal{Q}$ agree up to the top and $\Sigma_{\mathcal{P}^b} = \Lambda_{\mathcal{Q}^b}$.*

Definition 4.6.6 (Extender and strategy disagreement) *Given two hod premice $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \neq \mathcal{Q}$, we let $\beta(\mathcal{P}, \mathcal{Q})$ be the least ordinal γ such that $\mathcal{P}|\gamma = \mathcal{Q}|\gamma$ but $\mathcal{P}||\gamma \neq \mathcal{Q}||\gamma$. We say $\mathcal{P}$ and $\mathcal{Q}$ have an extender disagreement if $\beta(\mathcal{P}, \mathcal{Q}) \in \mathrm{dom}(\vec{E}^{\mathcal{R}}) \triangle \mathrm{dom}(\vec{E}^{\mathcal{Q}})$. We say $\mathcal{P}$ and $\mathcal{Q}$ have a strategy disagreement if $\beta(\mathcal{P}, \mathcal{Q}) \notin \mathrm{dom}(\vec{E}^{\mathcal{R}}) \cup \mathrm{dom}(\vec{E}^{\mathcal{Q}})$. In this case, we let $\mathcal{R}_{\mathcal{P},\mathcal{Q}} \in Y^{\mathcal{P}} \cap Y^{\mathcal{Q}}$ be the $\mathcal{P}|\beta(\mathcal{P}, \mathcal{Q})$-least such that if $\vec{\mathcal{T}} \in \mathcal{P} \cap \mathcal{Q}$ is the stack for which $\mathcal{P}$ and $\mathcal{Q}$ have a branch indexed at $\beta(\mathcal{P}, \mathcal{Q})$ then $\vec{\mathcal{T}}$ is a stack on $\mathcal{R}_{\mathcal{P},\mathcal{Q}}$. We say $\mathcal{R}_{\mathcal{P},\mathcal{Q}}$ is the disagreement layer of $\mathcal{P}$ and $\mathcal{Q}$.*

Definition 4.6.7 (Extender comparison) *Suppose that $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs which agree up to the top. Then we say $(\mathcal{T}, \mathcal{R}, \mathcal{U}, \mathcal{S})$ are the trees of the extender comparison of $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if*

1. *$\mathcal{T}$ is according to Σ and $\mathcal{R}$ is its last model,*

2. *$\mathcal{U}$ is according to Λ and $\mathcal{S}$ is its last model, and*

3. *$\mathcal{T}$ and $\mathcal{U}$ are obtained by using the usual extender comparison process (i.e., by removing the least extender disagreements) for comparing the top windows of $\mathcal{P}$ and $\mathcal{Q}$ until a strategy disagreement appears.*

It follows that if in Definition 4.6.7, $\mathcal{R} \neq \mathcal{S}$ then $\mathcal{R}$ and $\mathcal{S}$ have a strategy disagreement.

4.6.2 Universality of backgrounded constructions

Here we show that the fully backgrounded constructions are universal in a sense that they win the comparison with hod pairs. Suppose $\mathbb{M} = (M, \delta, \Sigma)$ is a weak background triple and $\Lambda \in (\wp(V_\delta^M))^M$. We say $(\mathsf{E}^{\mathbb{M}}, \mathsf{B}^{\mathbb{M}}, \mathsf{J}^{\mathbb{M}}, \mathsf{Lim}^{\mathbb{M}})$ are the construction functions of Λ-coherent hod pair construction of $\mathbb{M}$ if the extenders used during the construction cohere Λ.[17]

Our next theorem establishes that backgrounded constructions are universal. To establish it, we will use *the strategy absorption* argument. The strategy absorption argument was first presented in [10] (see the proof of Theorem 2.28 of [10]) and it is based on unpublished arguments of Steel. Because we will use the strategy absorption argument several times in this paper and in the next proof, it is important to understand how it works. The general form of the argument is as follows. We have a hod pair $(\mathcal{P}, \Lambda)$ captured by some *background triple* (M, δ, Σ). There is also an iteration tree $\mathcal{T}$ on $\mathcal{P}$ according to Λ with last model $\mathcal{Q}$ and $\mathcal{R} \trianglelefteq_{hod} \mathcal{Q}$ such that $\mathcal{R}$ is constructed via some fully backgrounded construction of M. It is additionally required that the certificates used to build $\mathcal{R}$ cohere Λ. The goal of the argument is to show that the strategy $\mathcal{R}$ inherits from the background universe is the same as $\Lambda_{\mathcal{R},\mathcal{T}}$. In many cases, this can be done by appealing to branch condensation and the existence of minimal disagreements. Here is how a typical argument works.

Let Φ be the iteration strategy of $\mathcal{R}$. Fix $\vec{\mathcal{U}}$ on $\mathcal{R}$ that is according to both $\Lambda_{\mathcal{R},\mathcal{T}}$ and Φ but $\Lambda_{\mathcal{R},\mathcal{T}}(\vec{\mathcal{U}}) \neq \Phi(\vec{\mathcal{U}})$. Let $\vec{\mathcal{U}}^*$ be the stack on M obtained from $\vec{\mathcal{U}}$ by lifting $\vec{\mathcal{U}}$ to M. Let $b = \Phi(\vec{\mathcal{U}}^*)$. We then have that $\pi_b^{\vec{\mathcal{U}}^*}(\mathcal{T})$ is according to Λ (this is where we use coherence). Then branch condensation is applied to the equality $\pi^{\pi_b^{\vec{\mathcal{U}}^*}(\mathcal{T})} = \sigma \circ \pi_b^{\vec{\mathcal{U}}} \circ \pi^{\mathcal{T}}$ where $\sigma : \mathcal{M}_b^{\vec{\mathcal{U}}} \to \pi_b^{\vec{\mathcal{U}}^*}(\mathcal{R})$ is the canonical factor map that the lifting process gives (in particular, $\pi_b^{\vec{\mathcal{U}}^*} \restriction \mathcal{R} = \sigma \circ \pi_b^{\vec{\mathcal{U}}}$).

Now we state our result on universality of background constructions.

Theorem 4.6.8 (Universality of backgrounded construction) *Assume* AD^+. *Suppose* $\mathbb{M} = (M, \delta, \Sigma)$ *is a self-capturing background triple,* $(\mathcal{P}, \Lambda)$ *is a hod pair or an sts hod pair and for some* α *such that* $\theta_\alpha < \Theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. *Suppose further that* Λ *is* Γ*-fullness preserving and* $\mathbb{M}$ *Suslin, co-Suslin captures* Γ *and* Λ. *Let* $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ *be the models and strategies of* Γ*-hod pair construction of* $\mathbb{M}$. *Then there is a* β *such that* $(\mathcal{Q}_\beta, \Sigma_\beta)$ *is a normal tail of* $(\mathcal{P}, \Lambda)$.

Proof. As in the proof of Lemma 2.10 of [10], in the comparison of $\mathcal{P}$ with the hod pair construction no extender disagreement appears on $\mathcal{Q}$ side. It is then enough to

[17] Where we say E coheres Λ if $\pi_E(\Lambda) = \Lambda \cap Ult(V, E)$. See Section 4.3.9 for the definition of $(\mathsf{E}^{\mathbb{M}}, \mathsf{B}^{\mathbb{M}}, \mathsf{J}^{\mathbb{M}}, \mathsf{Lim}^{\mathbb{M}})$.

show that

(1) for every $\beta < \alpha^{\mathbb{M}}$ if $\mathcal{T}$ is a tree on $\mathcal{P}$ according to Λ with last model $\mathcal{R}$ and for any $\mathcal{S} \in Y^{\mathcal{R}}$ such that $\mathcal{S} \trianglelefteq_{hod} \mathcal{Q}_\beta$,

$$(\Sigma_\beta)_{\mathcal{S}} = \Lambda_{\mathcal{S},\mathcal{T}}.$$

Towards a contradiction, we assume that (1) fails. Let $\beta < \alpha^{\mathbb{M}}$ and $(\mathcal{S}, \mathcal{T}, \mathcal{R})$ witness the failure of (1). We assume that $\mathcal{S} \in Y^{\mathcal{Q}_\beta}$ is the least layer for which (1) fails. Let $\Phi = \Sigma_\beta$ and $\mathcal{Q} = \mathcal{Q}_\beta$.

Suppose first that $\mathcal{S}$ is of successor type. Then we get a contradiction using branch condensation of Λ. Let $\vec{\mathcal{U}}$ be a stack on $\mathcal{S}$ such that it is according to both Φ and $\Lambda_{\mathcal{S},\mathcal{T}}$ but $\Phi(\vec{\mathcal{U}}) \neq \Lambda_{\mathcal{S},\mathcal{T}}(\vec{\mathcal{U}})$. Let $b = \Phi(\vec{\mathcal{U}})$ and $c = \Lambda_{\mathcal{S},\mathcal{T}}(\vec{\mathcal{U}})$. Let $\vec{\mathcal{U}}^*$ be the result of lifting $\vec{\mathcal{U}}$ to the background universe M. Then because extenders used to construct $\mathcal{Q}$ cohere Λ, we have that $\pi^{\vec{\mathcal{U}}^*}(\mathcal{T})$ is according to Λ. Let N be the last model of $\vec{\mathcal{U}}^*$.

Notice now that it follows from Γ-fullness preservation and the fact that $\Phi_{\mathcal{S}(\lambda^{\mathcal{S}}-1)} = \Lambda_{\mathcal{S}(\lambda^{\mathcal{S}}-1),\mathcal{T}}$ that $\pi^{\vec{\mathcal{U}}}_b$ exists. To see this, assume not. Suppose c drops. Then because $\mathcal{S}$ is of successor type, we can assume $\vec{\mathcal{U}}$ is above $\mathcal{S}(\lambda^{\mathcal{S}}-1)$. It is then not hard to see that neither $\mathcal{Q}(b,\vec{\mathcal{U}})$ nor $\mathcal{Q}(c,\vec{\mathcal{U}})$ has an extender E such that $\mathrm{crt}(E) \leq \delta(\vec{\mathcal{U}}) \leq \mathrm{lh}(E)$; but this implies $\mathcal{Q}(b,\vec{\mathcal{U}}) = \mathcal{Q}(c,\vec{\mathcal{U}})$.[18] Hence $b = c$. Contradiction. So c does not drop. We assume $\neg(\mathcal{Q}(\vec{\mathcal{U}}, b) \trianglelefteq \mathcal{M}^{\vec{\mathcal{U}}}_c)$ (otherwise, $b = c$). Let $\tau : \mathcal{Q}(\vec{\mathcal{U}}, b) \to \mathcal{Q}' \trianglelefteq \pi^{\vec{\mathcal{U}}^*_b}(\mathcal{R})$ be the lifting map and let $\delta' = \tau(\delta(\vec{\mathcal{U}}))$, $\mathcal{Y} = \pi^{\vec{\mathcal{U}}^*}_b(\mathcal{T})$. If δ' is a cutpoint of $\mathcal{M}^{\mathcal{Y}}_\infty = \pi^{\vec{\mathcal{U}}^*_b}(\mathcal{R})$, then since $\mathcal{Y}$ is according to Λ and Λ is Γ-fullness preserving, $\delta(\vec{\mathcal{U}})$ is a cutpoint of $\mathcal{Q}(\vec{\mathcal{U}}, b)$ and $\mathcal{Q}(\vec{\mathcal{U}}, b)$ has iteration strategy in Γ. This implies $\mathcal{Q}(\vec{\mathcal{U}}, b) \triangleleft \mathcal{M}^{\vec{\mathcal{U}}}_c$ (because c does not drop and Λ is Γ-fullness preserving). Contradiction. So δ' is not a cutpoint of $\mathcal{M}^{\mathcal{Y}}_\infty$. Let E be the least extender on the extender sequence of $\mathcal{M}^{\mathcal{Y}}_\infty$ such that $\mathrm{crt}(E) < \delta' < \mathrm{lh(E)}$. So $o(\mathcal{Q}') < \mathrm{lh}(E)$. Consider the tree $\mathcal{Z}$ on $\mathcal{M}^{\mathcal{Y}}_\infty$ using E. So $\mathcal{Q}' \trianglelefteq \mathcal{M}^{\mathcal{Z}}_1$ and δ' is a cutpoint of $\mathcal{Z}$. This again implies $\mathcal{Q}' \triangleleft \mathcal{M}^{\vec{\mathcal{U}}}_c$. Contradiciton. So $\pi^{\vec{\mathcal{U}}}_b$ exists.

Let then $\mathcal{R}^* = Ult(\mathcal{R}, E)$ where E is the $(\delta^{\mathcal{S}}, \pi^{\vec{\mathcal{U}}}_b(\delta^{\mathcal{S}}))$-extender derived from $\pi^{\vec{\mathcal{U}}}_b$. We then have $\sigma : \mathcal{R}^* \to \pi^{\vec{\mathcal{U}}^*}_b(\mathcal{R})$ such that

$$\pi^{\pi^{\vec{\mathcal{U}}^*}_b(\mathcal{T})} = \sigma \circ \pi_E \circ \pi^{\mathcal{T}}.$$

[18] $\mathrm{crt}(E)$ in fact must be $< \delta(\vec{\mathcal{U}})$. And if such an E exists then letting F be the least such and $\kappa = \mathrm{crt}(F)$, there is some model $\mathcal{W}$ of$\vec{\mathcal{U}}$ such that $\vec{\mathcal{U}}_{\geq \mathcal{W}}$ is a normal tree above κ and is on a strict initial segment $\mathcal{W}' \triangleleft \mathcal{W}$, where κ is a cutpoint of $\mathcal{W}'$. This easily implies that $\vec{\mathcal{U}}^\frown b$ and $\vec{\mathcal{U}}^\frown c$ are both according to $\Lambda_{\mathcal{S},\mathcal{T}}$.

Notice, however, that π_E is just an iteration embedding obtained by applying $\vec{\mathcal{U}}$ to $\mathcal{R}$. It then follows from branch condensation of Λ that $\vec{\mathcal{U}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{U}}}\}$ is according to Λ implying that $b = c$, contradiction! Thus, $\mathcal{S}$ cannot be of successor type.

Suppose next that $\mathcal{S}$ is of limit type. Then by appealing to Lemma 4.6.3, we can fix some $(\vec{\mathcal{U}}, \mathcal{S}_1)$ that constitutes a low level minimal disagreement between Φ and $\Lambda_{\mathcal{S},\mathcal{T}}$. Let

(2) $\vec{\mathcal{W}}$ be a stack on $\mathcal{S}_1$ which is according to both $\Phi_{\mathcal{S}_1,\vec{\mathcal{U}}}$ and $\Lambda_{\mathcal{S}_1,\mathcal{T}^\frown\vec{\mathcal{U}}}$ but letting $b = \Phi(\vec{\mathcal{T}}^\frown\vec{\mathcal{W}})$ and $c = \Lambda(\mathcal{T}^\frown\vec{\mathcal{U}}^\frown\vec{\mathcal{W}})$, $b \neq c$.

Notice that again it follows from Γ-fullness preservation and the minimality of $(\vec{\mathcal{U}}, \mathcal{S}_1)$ that $\pi_b^{\vec{\mathcal{W}}}$ and $\pi_c^{\vec{\mathcal{W}}}$ exists. Let then $\mathcal{R}^*$ be the result of applying $\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}$ and b to $\mathcal{R}$. Let $\vec{\mathcal{U}}^*$ be the result of resurrecting $\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}$ to M, and let $N = \mathcal{M}_b^{\vec{\mathcal{U}}^*}$. There is then $\sigma : \mathcal{R}^* \to \pi_b^{\vec{\mathcal{U}}^*}(\mathcal{R})$ such that

$$\pi^{\pi_b^{\vec{\mathcal{U}}^*}(\mathcal{T})} = \sigma \circ \pi_b^{\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}} \circ \pi^{\mathcal{T}}.$$

It then again follows from the branch condensation of Λ that $\Lambda(\mathcal{T}^\frown\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}) = b$, contradiction! □

As a corollary to Theorem 4.6.8 we get that the comparison holds.

Corollary 4.6.9 *Assume* AD^+ *and suppose* Γ *is a pointclass such that for some good pointclass* Γ_1, $\Gamma \subseteq \underset{\sim}{\Delta}_{\Gamma_1}$. *Suppose further that* $(\mathcal{P}, \Sigma)$ *and* $(\mathcal{Q}, \Lambda)$ *are two hod pairs such that both* Σ *and* Λ *are* Γ*-fullness preserving and have branch condensation. Suppose further that both* $Code(\Sigma)$ *and* $Code(\Lambda)$ *are Suslin, co-Suslin. Then the normal comparison holds for* $(\mathcal{P}, \Sigma)$ *and* $(\mathcal{Q}, \Lambda)$.

Proof. Fix a good pointclass Γ_2 such that $\Gamma_1 \cup \{Code(\Sigma), Code(\Lambda)\} \subseteq \underset{\sim}{\Delta}_{\Gamma_2}$. Let F be as in Theorem 4.1.6 for Γ_2 and let $x \in \mathrm{dom}(F)$ be such that $\mathbb{M} = (\mathcal{N}_x^*, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures Γ, $Code(\Sigma)$ and $Code(\Lambda)$. Let $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ be the models and strategies of the hod pair construction of $\mathbb{M}$. It follows from Theorem 4.6.8 that there are $\beta, \gamma < \alpha^{\mathbb{M}}$ and normal trees $\mathcal{T}$ and $\mathcal{U}$ such

1. $(\mathcal{T}, \mathcal{Q}_\beta) \in I(\mathcal{P}, \Sigma)$ and $\Sigma_\beta = \Sigma_{\mathcal{Q}_\beta,\mathcal{T}}$ and

2. $(\mathcal{U}, \mathcal{Q}_\gamma) \in I(\mathcal{R}, \Lambda)$ and $\Sigma_\gamma = \Lambda_{\mathcal{Q}_\gamma,\mathcal{T}}$.

If $\beta = \gamma$ then clearly the normal comparison for $(\mathcal{P}, \Sigma)$ and $(\mathcal{R}, \Lambda)$ holds. Suppose $\beta < \gamma$. Let $\mathcal{R}_1$ be the Λ-iterate of $\mathcal{R}$ via a normal tree $\mathcal{U}_1$ such that $\mathcal{Q}_\beta \in Y^{\mathcal{R}_1}$. It then follows from Theorem 4.6.8 that $\Sigma_\beta = \Lambda_{\mathcal{Q}_\beta,\mathcal{U}_1}$. Therefore, normal comparison for $(\mathcal{P},\Sigma)$ and $(\mathcal{R},\Lambda)$ holds. The case $\gamma<\beta$ is symmetrical. □

Using reflection, we can eliminate the extra assumptions on Γ and the two strategies.

Corollary 4.6.10 (Comparison) *Assume* AD^+ *and suppose* Γ *is a pointclass. Suppose further that* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$ *are two hod pairs such that both* Σ *and* Λ *are* Γ*-fullness preserving and have branch condensation. Then the normal comparison hold for* $(\mathcal{P},\Sigma)$ *and* $(\mathcal{Q},\Lambda)$.

Proof. Suppose not. Applying Σ^2_1-reflection, we can fine Γ^* and two hod pair $(\mathcal{P}_1,\Sigma_1)$ and $(\mathcal{Q}_1,\Lambda_1)$ such that $\Gamma^*\cup\{Code(\Sigma_1), Code(\Lambda_1)\}\subseteq \utilde{\Delta}^2_1$ and the claim of the corollary fails for $(\Gamma^*,(\mathcal{P}_1,\Sigma_1),(\mathcal{Q}_1,\Lambda_1))$. We then apply Corollary 4.6.9. □

4.7 Branch condensation

In this subsection we prove that the hod pair constructions produce strategies with branch condensation and in fact more. In order, however, to prove that hod pair constructions converge, we will need to establish the solidity and universality of the standard parameter of the models appearing in such constructions. Establishing such fine structural facts wasn't an issue in [10] as the fine structure for hod mice considered in that paper was a routine generalization of the fine structure theory developed in [8]. Here the matters are somewhat more complicated as the fine structure of non-meek hod mice cannot be viewed as a routine generalization of the fine structure of [8]. Nevertheless, the matter isn't too complicated as a simple generalization of branch condensation, *strong branch condensation*, allows us to reduce our case to the one in [8].

In this subsection, we will establish that hod pair constructions produce strategies with strong branch condensation. The reader is encouraged to concentrate on clause 1 of Definition 4.7.1. Clause 2 is a technical addition that will be used in the proof of Corollary 5.5.1.

Definition 4.7.1 (Strong branch condensation) *Suppose* $(\mathcal{P},\Sigma)$ *is a hod pair. We say* Σ *has strong branch condensation if* Σ *has branch condensation and*

1. *whenever* $(\vec{\mathcal{T}},\mathcal{Q},\pi,\mathcal{R},\alpha,\sigma)$ *is such that*

(a) $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\mathcal{R}$ is a hod premouse,

(b) $\pi : \mathcal{P} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{Q}$ and $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,

(c) $\alpha + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\alpha + 1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$

then letting $\Lambda = \sigma$*-pullback of* $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$*, whenever* $\vec{\mathcal{W}}$ *is such that* $(\vec{\mathcal{W}}, \mathcal{R}(\alpha+1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$*, if there is no low level disagreement between* $\Lambda_{\mathcal{R}(\alpha+1)}$ *and* $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ *then* $\Lambda_{\mathcal{R}(\alpha+1)} = \Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$.

2. *whenever* $(\vec{\mathcal{T}}, \mathcal{Q}, \pi, \mathcal{R}, \alpha, \beta, \xi, k, \sigma)$ *is such that*

(a) $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and $\mathcal{R}$ is a hod premouse of limit type,

(b) $\beta \leq \lambda^{\mathcal{P}}$ and $\xi \leq \lambda^{\mathcal{Q}}$ are limit ordinals such that $\pi^{\vec{\mathcal{T}}}(\beta) \geq \xi$ and

$$(\mathcal{Q}(\xi))^b = Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[(\mathcal{P}(\beta))^b] \cup \delta_\xi^{\mathcal{Q}}),$$

(c) $k : \mathcal{P}(\beta)^b \to (\mathcal{Q}(\xi))^b$ is $k_0 \circ k_1$, where $k_1 = \pi^{\vec{\mathcal{T}}} \restriction \mathcal{P}(\beta)^b$ and k_0 is the inverse of the collapse of $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[(\mathcal{P}(\beta))^b] \cup \delta_\xi^{\mathcal{Q}})$,

(d) $\pi : (\mathcal{P}(\beta))^b \to \mathcal{R}^b$, $\sigma : \mathcal{R}^b \to (\mathcal{Q}(\xi))^b$ and $k = \sigma \circ \pi$,

(e) $\alpha + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\alpha + 1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$

then letting $\Lambda = \sigma$*-pullback of* $\Sigma_{(\mathcal{Q}(\xi))^b,\vec{\mathcal{T}}}$*, whenever* $\vec{\mathcal{W}}$ *is such that* $(\vec{\mathcal{W}}, \mathcal{R}(\alpha + 1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$*, if there is no low level disagreement between* $\Lambda_{\mathcal{R}(\alpha+1)}$ *and* $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ *then* $\Lambda_{\mathcal{R}(\alpha+1)} = \Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$.

Theorem 4.7.2 *Assume* AD^+. *Suppose for some* α *such that* $\theta_\alpha < \Theta$, $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$ *and* $\mathbb{M} = (M, \delta, \Sigma)$ *is a self-capturing background triple that Suslin, co-Suslin captures* Γ *via* (P, Ψ). *Let* $(\mathcal{Q}_\beta, \Sigma_\beta : \beta < \alpha^{\mathbb{M}})$ *be the models and strategies of the hod pair construction of* $\mathbb{M}$. *Suppose* ξ *is such that* $(\mathcal{Q}_\xi, \Sigma_\xi) \in HP^\Gamma$ *is a hod pair and that for some* $<\delta$*-generic* g*, there is a continuous function* $\sigma \in M[g] \cap \mathbb{R}$ *such that* $\sigma^{-1}[Code(\Psi)] = Code(\Sigma_\xi)$. *Then* Σ_ξ *has strong branch condensation.*

Proof. The proof of clause 2 of Definition 4.7.1 is only notationally more involved than the proof of clause 1 of Definition 4.7.1. Because of this we only present the proof of clause 1.

Towards a contradiction, suppose that for some $\xi < \alpha^{\mathbb{M}}$, $\mathcal{Q}_\xi$ is a hod premouse and Σ_ξ doesn't have strong branch condensation. Just like in the proof of fullness preservation (see Theorem 4.5.3), if Σ_ξ does not have strong branch condensation then the witness can be found in $M[g]$ where g is $<\delta$-generic over M.

Let $\mathcal{Q} = \mathcal{P}_\xi$ and $\Lambda = \Sigma_\xi$. We start working in $M[g]$. What we need to show is that whenever $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$ is such that

1. $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ and $\mathcal{R}$ is a hod premouse,
2. $\pi : \mathcal{Q} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{S}$ and $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,
3. $\beta + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$,

then letting $\Phi = \Lambda^\sigma_{\mathcal{S},\vec{\mathcal{T}}}$, whenever $\vec{\mathcal{U}}^*$ is such that $(\vec{\mathcal{U}}^*, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$, if there is no low level disagreement between $\Phi_{\mathcal{R}(\beta+1)}$ and $\Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{W}}}$ then $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{W}}}$.

Fix then such a sequence $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$. Let $(\vec{\mathcal{U}}^*, \mathcal{W}) \in I(\mathcal{Q}, \Lambda)$ be such that $\mathcal{R}(\beta+1) = \mathcal{W}(\beta+1)$. Let $\Phi = \Lambda^\sigma_{\mathcal{S},\vec{\mathcal{T}}}$. It follows from strong fullness preservation of Λ (see Theorem 4.5.5) that $\Phi_{\mathcal{R}^b}$ is fullness preserving.

We assume that there is no low level disagreement between $\Phi_{\mathcal{R}(\beta+1)}$ and $\Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{U}}^*}$ and want to show that $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{U}}^*}$. Towards a contradiction assume that $\Phi_{\mathcal{R}(\beta+1)} \neq \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{U}}^*}$.

It follows from Lemma 4.6.3 that either $\mathcal{R}(\beta+1)$ is of successor type or of lsa type. We then have two cases. Suppose first that $\mathcal{R} = \mathcal{W} = \mathcal{R}(\beta+1)$. Letting

$$\Lambda^* = \begin{cases} \Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)} & : \mathcal{Q} \text{ is of successor type} \\ \Lambda^{stc} & : \text{otherwise} \end{cases}$$

and

$$\Phi^* = \begin{cases} \Phi_{\mathcal{R}(\beta)} & : \mathcal{Q} \text{ is of successor type} \\ \Phi^{stc} & : \text{otherwise} \end{cases}$$

set

$$\mathcal{Q}^* = (\mathcal{J}^{\vec{E},\Lambda^*})^{V_\delta^M} \text{ and } \mathcal{Q}^+ = \mathcal{S}^{\Lambda^*}(\mathcal{Q}^*)^{19}$$

Let E be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{R}})$-extender derived from π, F be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{S}})$-extender derived from $\pi^{\vec{\mathcal{T}}}$ and H be the $(\delta^{\mathcal{Q}}, \delta^{\mathcal{W}})$-extender derived from $\pi^{\vec{\mathcal{U}}^*}$. We let

$$\mathcal{R}^+ = Ult(\mathcal{Q}^+, E),\ \mathcal{S}^+ = Ult(\mathcal{Q}^+, F) \text{ and } \mathcal{W}^+ = Ult(\mathcal{Q}^+, H).$$

We then have that (see Lemma 5.3 of [10])

[19] $\mathcal{S}$ here denotes the stack. See Section 5.1 of [10].

$$\mathcal{R}^+ = \mathcal{S}^{\Phi^*}(\mathcal{R}^+|\delta),\ \mathcal{S}^+ = \mathcal{S}^{\Lambda^*_{\mathcal{S},\vec{\mathcal{T}}}}(\mathcal{S}^+|\delta) \text{ and } \mathcal{W}^+ = \mathcal{S}^{\Lambda^*_{\mathcal{W},\vec{\mathcal{U}}^*}}(\mathcal{W}^+|\delta).$$

We also have $\sigma^+ : \mathcal{R}^+ \to \mathcal{S}^+$ such that $\pi_F = \sigma^+ \circ \pi_E$ and $\sigma^+ \restriction \mathcal{R} = \sigma$. More precisely, $\sigma^+(x) = \pi_F(f)(\sigma(a))$ where $f \in \mathcal{Q}^+$, $a \in (\mathcal{R})^{<\omega}$ and $x = \pi_E(f)(a)$.

Let now $\vec{\mathcal{K}}$ be a stack on $\mathcal{R}$ such that $\Phi(\vec{\mathcal{K}}) \neq \Lambda_{\mathcal{R},\vec{\mathcal{U}}^*}(\vec{\mathcal{K}})$. Suppose that $b = \Phi(\vec{\mathcal{K}})$ and $c = \Lambda_{\mathcal{R},\vec{\mathcal{U}}^*}(\vec{\mathcal{K}})$. Notice that because both Φ and Λ are strongly fullness preserving, we must have that both $\pi_b^{\vec{\mathcal{K}}}$ and $\pi_c^{\vec{\mathcal{K}}}$ exist. Let now $\mathcal{R}_b^+$ and $\mathcal{W}_c^+$ be the last models of $\vec{\mathcal{K}}$ when it is applied to $\mathcal{R}^+$ and $\mathcal{W}^+$ respectively. Comparing $\mathcal{R}_b^+$ and $\mathcal{W}_c^+$ we get a common model $\mathcal{M}$. Let $i' : \mathcal{R}_b^+ \to \mathcal{M}$, $j' : \mathcal{W}_c^+ \to \mathcal{M}$ be iteration maps and $i = i' \circ \pi_b^{\vec{\mathcal{K}}}$, $j = j' \circ \pi_c^{\vec{\mathcal{K}}}$.

Let $C \subseteq (\delta^+)^M$ be an ω-club consisting of points β such that $\beta \in rng(i) \cap rng(j)$. Then we have that

(1) $\mathrm{crt}(i') \geq \delta^{\mathcal{R}}$ and $\mathrm{crt}(j') \geq \delta^{\mathcal{R}}$.

(2) $\delta^{\mathcal{R}} = \sup(Hull^{\mathcal{R}^+}(\mathcal{R}(\beta), i^{-1}[C]) \cap \delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{W}^+}(\mathcal{R}(\beta), j^{-1}[C]) \cap \delta^{\mathcal{R}})$.

(3) $\pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{M}}(\pi_b^{\vec{\mathcal{K}}}(\mathcal{R}(\beta)), C) \cap \pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}))$.

(4) $\pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \sup(Hull^{\mathcal{M}}(\pi_c^{\vec{\mathcal{K}}}(\mathcal{R}(\beta)), C) \cap \pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}))$.

It follows from (1), (2) , (3), and (4) that $rng(\pi_b^{\vec{\mathcal{K}}}) \cap rng(\pi_c^{\vec{K}})$ is cofinal in $\pi_b^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}}) = \pi_c^{\vec{\mathcal{K}}}(\delta^{\mathcal{R}})$ and hence (4) implies that $b = c$.

The case $\mathcal{R}(\beta+1) \neq \mathcal{R}$ (implying that $\mathcal{R}(\beta+1) \lhd \mathcal{R}$) is very similar but a bit more technical. Notice that because of our minimality assumption, we have that $\mathcal{R}(\beta+1)$ is not of lsa type. Let ν be least such that $\pi(\nu) \geq \beta + 1$ and set $\mathcal{Q}^* = (\mathcal{J}^{\vec{E},\Lambda_{\mathcal{Q}(\nu)}})^{V_\delta^M}$. Next let E be $(\delta_\nu^{\mathcal{Q}}, \delta_{\beta+1}^{\mathcal{R}})$-extender derived from π, F be $(\delta^{\mathcal{Q}}, \sigma(\delta_{\beta+1}^{\mathcal{R}}))$-extender derived from $\pi^{\vec{\mathcal{T}}}$ and H be $(\delta_\nu^{\mathcal{Q}}, \delta_{\beta+1}^{\mathcal{W}})$-extender derived from $\pi^{\vec{\mathcal{U}}^*}$, and set

$$\mathcal{R}^* = Ult(\mathcal{Q}^*, E),\ \mathcal{S}^* = Ult(\mathcal{Q}^*, F) \text{ and } \mathcal{W}^* = Ult(\mathcal{Q}, H).$$

Then let $\mathcal{R}^{**} = (\mathcal{J}^{\vec{E},\Phi_{\mathcal{R}(\beta)}})^{\mathcal{R}^*}$, $\mathcal{S}^{**} = (\mathcal{J}^{\vec{E},\Lambda_{\mathcal{S}(\sigma(\beta)),\vec{\mathcal{T}}}})^{\mathcal{S}^*}$ and $\mathcal{W}^{**} = (\mathcal{J}^{\vec{E},\Lambda_{\mathcal{W}(\beta),\vec{\mathcal{U}}^*}})^{\mathcal{W}^*}$, and finally set

$$\mathcal{R}^+ = \mathcal{S}^{\Phi_{\mathcal{R}(\beta)}}(\mathcal{R}^{**}),\ \mathcal{S}^+ = \mathcal{S}^{\Lambda_{\mathcal{S}(\sigma(\beta)),\vec{\mathcal{T}}}}(\mathcal{S}^{**}) \text{ and } \mathcal{W}^+ = \mathcal{S}^{\Lambda_{\mathcal{W}(\beta),\vec{\mathcal{U}}^*}}(\mathcal{W}^{**}).$$

We now finish by noticing that we can get an embedding $\sigma^+ : \mathcal{R}^+ \to \mathcal{S}^+$. The rest of the argument is as before. □

A variant of strong branch condensation holds for short tree strategy. The short tree strategies induced from background constructions have this form of branch condensation, but we will omit the proof of this fact because it is very similar to the proof of Theorem 4.7.2.

Definition 4.7.3 *Suppose $(\mathcal{P},\Sigma)$ is an sts hod pair. We say Σ has strong branch condensation if Σ has branch condensation and*

1. *whenever $(\vec{\mathcal{T}},\mathcal{Q},\pi,\mathcal{R},\alpha,\sigma)$ is such that*

   (a) *$(\vec{\mathcal{T}},\mathcal{Q})\in I^b(\mathcal{P},\Sigma)$ and $\mathcal{R}$ is a hod premouse,*

   (b) *$\pi:\mathcal{P}^b\to\mathcal{R}^b$, $\sigma:\mathcal{R}^b\to\mathcal{Q}^b$ and $\pi^{\vec{\mathcal{T}},b}=\sigma\circ\pi$,*

   (c) *$\alpha<\lambda^{\mathcal{R}^b}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}},\mathcal{R}(\alpha+1))\in B(\mathcal{P},\Sigma)$*

   then letting $\Lambda=\sigma$-pullback of $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$, whenever $\vec{\mathcal{W}}$ is such that $(\vec{\mathcal{W}},\mathcal{R}(\alpha+1))\in B(\mathcal{P},\Sigma)$, if there is no low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ then $\Lambda_{\mathcal{R}(\alpha+1)}=\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$.

2. *whenever $(\vec{\mathcal{T}},\mathcal{Q},\pi,\mathcal{R},\alpha,\beta,\xi,k,\sigma)$ is such that*

   (a) *$(\vec{\mathcal{T}},\mathcal{Q})\in I^b(\mathcal{P},\Sigma)$ and $\mathcal{R}$ is a hod premouse of limit type,*

   (b) *$\beta\le\lambda^{\mathcal{P}}$ and $\xi\le\lambda^{\mathcal{Q}}$ are limit ordinals such that*
   $$(\mathcal{Q}(\xi))^b=Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[(\mathcal{P}(\beta))^b]\cup\delta_\xi^{\mathcal{Q}}),$$

   (c) *$k:\mathcal{P}(\beta)^b\to(\mathcal{Q}(\xi))^b$ is $k_0\circ k_1$, where $k_1=\pi^{\vec{\mathcal{T}}}\restriction\mathcal{P}(\beta)^b$ and k_0 is the inverse of the collapse of $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[(\mathcal{P}(\beta))^b]\cup\delta_\xi^{\mathcal{Q}})$,*

   (d) *$\pi:(\mathcal{P}(\beta))^b\to\mathcal{R}^b$, $\sigma:\mathcal{R}^b\to(\mathcal{Q}(\xi))^b$ and $k=\sigma\circ\pi$,*

   (e) *$\alpha+1\le\lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}},\mathcal{R}(\alpha+1))\in B(\mathcal{P},\Sigma)\cup I(\mathcal{P},\Sigma)$*

   then letting $\Lambda=\sigma$-pullback of $\Sigma_{(\mathcal{Q}(\xi))^b,\vec{\mathcal{T}}}$, whenever $\vec{\mathcal{W}}$ is such that $(\vec{\mathcal{W}},\mathcal{R}(\alpha+1))\in B(\mathcal{P},\Sigma)\cup I(\mathcal{P},\Sigma)$, if there is no low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ then $\Lambda_{\mathcal{R}(\alpha+1)}=\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$.

Theorem 4.7.4 *Assume AD^+. Suppose for some α such that $\theta_\alpha<\Theta$, $\Gamma=\{A\subseteq\mathbb{R}:w(A)<\theta_\alpha\}$ and $\mathbb{M}=(M,\delta,\Sigma)$ is a self-capturing background triple that Suslin, co-Suslin captures Γ via (P,Ψ). Let $(\mathcal{Q}_\beta,\Sigma_\beta:\beta<\alpha^{\mathbb{M}})$ be the models and strategies of the hod pair construction of $\mathbb{M}$. Suppose ξ is such that $(\mathcal{Q}_\xi,\Sigma_\xi)\in HP^\Gamma$, $(\mathcal{Q}_\xi,\Sigma_\xi^{stc})$ is an sts hod pair and that for some $<\delta$-generic g, there is a continuous function $\sigma\in M[g]\cap\mathbb{R}$ such that $\sigma^{-1}[Code(\Psi)]=Code(\Sigma_\xi)$. Then Σ_ξ^{stc} has strong branch condensation.*

The following is an easily provable lemma, which shows that the requirement that there is no low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ is not necessary.

Lemma 4.7.5 *Suppose $(\mathcal{P},\Sigma)$ is a hod pair or an sts hod pair and Γ is a pointclass. Suppose that Σ has strong branch condensation and is Γ-strongly fullness preserving. Then the requirement that there is no low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ in clause 1 and 2 of Definition 4.7.1 and Definition 4.7.3 is not necessary.*

Proof. Since all cases are very similar, we only concentrate on clause 1 of Definition 4.7.1. Suppose then $(\vec{\mathcal{T}},\mathcal{Q},\pi,\mathcal{R},\alpha,\sigma)$ is as in clause 1 of Definition 4.7.1. Let $\vec{\mathcal{W}}$ be as in the hypothesis of Definition 4.7.1. Towards a contradiction assume that $\Lambda_{\mathcal{R}(\alpha+1)}\neq\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$. It follows that there is a low level disagreement between $\Lambda_{\mathcal{R}(\alpha+1)}$ and $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$. Let then $(\vec{\mathcal{T}}_1,\mathcal{S})\in B(\mathcal{R}(\alpha+1),\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}})\cap B(\mathcal{R}(\alpha+1),\Lambda_{\mathcal{R}(\alpha+1)})$ be a low level disagreement between $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ and $\Lambda_{\mathcal{R}(\alpha+1)}$.

Let $\mathcal{S}^+$ be the last model of $\vec{\mathcal{T}}_1$ when it is applied to $\mathcal{R}$, and let $\vec{\mathcal{T}}_1^*$ be the stack on $\mathcal{Q}$ constructed via the copying construction using π. Let $\mathcal{S}_1$ be the last model of $\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1^*$ and let $k:\mathcal{S}^+\to\mathcal{S}_1$ be the map constructed via the copying construction. Let ξ be such that $\mathcal{S}^+(\xi)=\mathcal{S}$. Then it is not hard to see that

$$(\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1,\mathcal{S}_1,\pi^{\vec{\mathcal{T}}_1}\circ\pi,\mathcal{S}^+,\xi,k)$$

is as in the hypothesis of clause 1 of Definition 4.7.1. Let Ψ be the k-pullback of $\Sigma_{\mathcal{S}_1,\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1}$. Notice that $\Psi=\Lambda_{\mathcal{S}^+,\vec{\mathcal{T}}_1}$. It then follows that there is no low level disagreement between $\Psi_{\mathcal{S}^+(\xi)}$ and $\Sigma_{\mathcal{S}^+(\xi),\vec{\mathcal{W}}^\frown\vec{\mathcal{T}}_1}$. Strong branch condensation of Σ then implies that $\Psi_{\mathcal{S}^+(\xi)}=\Sigma_{\mathcal{S}^+(\xi),\vec{\mathcal{W}}^\frown\vec{\mathcal{T}}_1}$, contradicting the fact that $\Lambda_{\mathcal{S}^+(\xi),\vec{\mathcal{T}}_1}\neq\Sigma_{\mathcal{S}^+(\xi),\vec{\mathcal{W}}^\frown\vec{\mathcal{T}}_1}$. $\square$

We finish by proving branch condensation for Γ-hod pair constructions. Our proof assumes Γ-fullness preservation, which we will establish this later on in the text (see Theorem 8.3.1).

Theorem 4.7.6 *Suppose Γ is a pointclass such that for some α,*

$$L_\alpha(\Gamma,\mathbb{R})\vDash\text{“}\mathsf{ZF}-\textit{Replacement}\text{” and }\Gamma=L_\alpha(\Gamma,\mathbb{R})\cap\wp(\mathbb{R}).$$

Suppose $\mathbb{M}=(M,\delta,\Sigma)$ is a self-capturing background triple that Suslin, co-Suslin captures Γ. Let $(\mathcal{Q}_\beta,\Sigma_\beta:\beta<\alpha^{\mathbb{M}})$ be the models and strategies of the Γ-hod pair construction of M. Suppose $\beta<\alpha^{\mathbb{M}}$ such that $(\mathcal{Q}_\beta,\Sigma_\beta)\in HP^\Gamma$ and Σ_β is strongly Γ-fullness preserving. Then Σ_β has strong branch condensation.

Proof. The proof is very similar to the proof of Theorem 4.7.2. Because of this, we will only outline it. Notice that if Σ_β does not have strong branch condensation then the witness can be found in $M[g]$ where g is $<\delta$-generic over M.

Let $\mathcal{Q} = \mathcal{Q}_\beta$ and $\Lambda = \Sigma_\beta$. We start working in $M[g]$ and just show clause (1) in the definition of strong branch condensation. What we need to show is that whenever $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$ is such that

1. $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$ and $\mathcal{R}$ is a hod premouse,
2. $\pi : \mathcal{Q} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{S}$ and $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,
3. $\beta + 1 \leq \lambda^{\mathcal{R}}$ is such that for some $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$,

then letting $\Phi = \Lambda^\sigma_{\mathcal{S}, \vec{\mathcal{T}}}$, whenever $\vec{\mathcal{U}}^*$ is such that $(\vec{\mathcal{U}}^*, \mathcal{R}(\beta+1)) \in B(\mathcal{Q}, \Lambda) \cup I(\mathcal{Q}, \Lambda)$, if there is no low level disagreement between $\Phi_{\mathcal{R}(\beta+1)}$ and $\Lambda_{\mathcal{R}(\beta+1), \vec{\mathcal{W}}}$ then $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1), \vec{\mathcal{W}}}$.

Fix then such a sequence $(\vec{\mathcal{T}}, \mathcal{S}, \pi, \mathcal{R}, \beta, \sigma)$. Let $(\vec{\mathcal{U}}^*, \mathcal{W}) \in I(\mathcal{Q}, \Lambda)$ be such that $\mathcal{R}(\beta+1) = \mathcal{W}(\beta+1)$. Let $\Phi = \Lambda^\sigma_{\mathcal{S}, \vec{\mathcal{T}}}$. It follows from strong fullness preservation of Λ that $\Phi_{\mathcal{R}^b}$ is fullness preserving.

We assume that there is no low level disagreement between $\Phi_{\mathcal{R}(\beta+1)}$ and $\Lambda_{\mathcal{R}(\beta+1), \vec{\mathcal{U}}^*}$ and want to show that $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1), \vec{\mathcal{U}}^*}$. Towards a contradiction assume that $\Phi_{\mathcal{R}(\beta+1)} \neq \Lambda_{\mathcal{R}(\beta+1), \vec{\mathcal{U}}^*}$.

It follows from Lemma 4.6.3 that either $\mathcal{R}(\beta+1)$ is of successor type or of lsa type. We then have two cases. Suppose first that $\mathcal{R} = \mathcal{W} = \mathcal{R}(\beta+1)$. Letting

$$\Lambda^* = \begin{cases} \Lambda_{\mathcal{R}(\beta)} & : \mathcal{Q} \text{ is of successor type} \\ \Lambda^{stc} & : \text{otherwise} \end{cases}$$

and

$$\Phi^* = \begin{cases} \Phi_{\mathcal{R}(\beta)} & : \mathcal{Q} \text{ is of successor type} \\ \Phi^{stc} & : \text{otherwise} \end{cases}$$

set

$$\mathcal{Q}^* = (\mathcal{J}^{\vec{E}, \Lambda^*})^{V^M_\delta} \text{ and } \mathcal{Q}^+ = \mathcal{S}^{\Lambda^*}(\mathcal{Q}^*)^{20}$$

[20] $\mathcal{S}$ here denotes the stack. See Section 5.1 of [10].

We have two cases. If $\mathcal{Q}^* \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal" then we can finish as in Theorem 4.7.2. Otherwise let $\mathcal{M}$ be the least level of $\mathcal{Q}^*$ such that $\mathcal{M} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal" and $\mathcal{J}_1[\mathcal{M}] \vDash$ "$\delta^{\mathcal{Q}}$ is not a Woodin cardinal". Notice then $\mathcal{M}$ is a $\mathcal{Q}$-structure for $\delta^{\mathcal{Q}}$ implying that Λ^* is determined by moving it correctly. This then implies that $\Phi_{\mathcal{R}(\beta+1)} = \Lambda_{\mathcal{R}(\beta+1),\vec{\mathcal{U}}^*}$.

The case $\mathcal{R}(\beta+1) \neq \mathcal{R}$ (implying that $\mathcal{R}(\beta+1) \lhd \mathcal{R}$) is very similar but a bit more technical. Notice that because of our minimality assumption, we have that $\mathcal{R}(\beta+1)$ is not of lsa type. Let ν be least such that $\pi(\nu) \geq \beta+1$. It again follows that $\nu + 1 < \lambda^{\mathcal{Q}}$.

Let then $\eta < \delta$ be the least M-cardinal above $o(\mathcal{Q})$ such that $Lp^{\Gamma,\Sigma_{\mathcal{Q}(\nu+2)}}(V_\eta^M) \vDash$ "η is a Woodin cardinal". We now repeat the proof of Theorem 4.7.2 by using V_η^M instead of V_δ^M. $\square$

4.8 Positional and commuting

In this section, our goal is to show that strong branch condensation implies commuting. Recall [10, Definition 2.35]: if M is a transitive model of a fragment of ZFC and Σ is an iteration strategy for M then we say Σ is *positional* if whenever Q is a Σ-iterate of M via $\vec{\mathcal{W}}$ and $(\vec{\mathcal{T}}, R), (\vec{\mathcal{U}}, R) \in I(Q, \Sigma_{Q,\vec{\mathcal{W}}})$, $\Sigma_{R,\vec{\mathcal{W}}^\frown\vec{\mathcal{T}}} = \Sigma_{R,\vec{\mathcal{W}}^\frown\vec{\mathcal{U}}}$. Recall that commuting means that in the above scenario, $\pi^{\vec{\mathcal{T}}} = \pi^{\vec{\mathcal{W}}}$. If $Q = M$, then we say that Σ is *weakly positional* (and *weakly commuting* respectively). Using the usual proof of the Dodd-Jensen lemma, we get that (weakly) positional implies (weakly) commuting.

Proposition 4.8.1 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair, Γ is a pointclass and Σ has strong branch condensation and is strongly Γ-fullness preserving. Then Σ is positional. Moreover, if Σ is an iteration strategy then it is also commuting.*

Proof. We just prove weak positionality and hence weak commuting. The proof of the general case is similar.

Suppose $(\vec{\mathcal{T}}, \mathcal{Q}), (\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$. We want to see that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q},\vec{\mathcal{U}}}$. Towards a contradiction, suppose not. Suppose first that either $\mathcal{P}$ is not of lsa type or if it is then $\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}} \neq \Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{U}}}$. Let then $(\vec{\mathcal{W}}, \mathcal{R}) \in B(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}}) \cap B(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{U}}})$ be a minimal lower level disagreement. Let $\mathcal{R}^+$ be the last model when we apply $\vec{\mathcal{W}}$ to $\mathcal{Q}$ and let α be such that $\mathcal{R} = \mathcal{R}^+(\alpha+1)$. We can then apply strong branch condensation to $(\vec{\mathcal{T}}^\frown\vec{\mathcal{W}}, \mathcal{R}^+, \pi^{\vec{\mathcal{T}}^\frown\vec{\mathcal{W}}}, \mathcal{R}^+, \alpha, id)$ and $(\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}, \mathcal{R}^+(\alpha))$. It follows that $\Sigma_{\mathcal{R},\vec{\mathcal{T}}^\frown\vec{\mathcal{W}}} = \Sigma_{\mathcal{R},\vec{\mathcal{U}}^\frown\vec{\mathcal{W}}}$.

Finally suppose $\mathcal{P}$ is of lsa type, Σ is an iteration strategy and $\Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma^{stc}_{\mathcal{Q},\vec{\mathcal{U}}}$. Now we can simply apply strong branch condensation (in fact, just branch condensation) to $(\vec{\mathcal{T}}, \mathcal{Q}, \pi^{\vec{\mathcal{T}}}, \mathcal{Q}, \lambda^{\mathcal{Q}}, id)$ and $(\vec{\mathcal{U}}, \mathcal{Q})$ and get that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q},\vec{\mathcal{U}}}$. □

Given a hod pair $(\mathcal{P}, \Sigma)$ and $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ such that Σ has strong branch condensation and is strongly Γ-fullness preserving for some Γ, we let $\Sigma_{\mathcal{Q}}$ be the strategy of $\mathcal{Q}$ induced by Σ. It follows from Lemma 4.8.1 that $\Sigma_{\mathcal{Q}}$ is independent of the particular iteration producing $\mathcal{Q}$.

We need commuting not only for iteration strategies but also for short tree strategies.

Definition 4.8.2 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair. We say Σ is strongly commuting if whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I^b(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in I^b(\mathcal{P}, \Sigma)$ are such that for some $\alpha \leq \lambda^{\mathcal{Q}}$, $\mathcal{R}^b = (\mathcal{Q}(\alpha))^b$ and $\mathcal{R}^b = Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{R}^b})$, then letting $k : \mathcal{P}^b \to \mathcal{R}^b$ be the inverse of the collapse of $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{R}^b})$,[21] $k = \pi^{\vec{\mathcal{U}},b}$.*

We say Σ is commuting if keeping the above notation, the conclusion holds with $\alpha = \lambda^{\mathcal{Q}}$ (in this case, $k = \pi^{\vec{\mathcal{T}},b}$).

If $(\mathcal{P}, \Sigma)$ is a hod pair. We say Σ is strongly commuting if Σ is commuting.[22]

To show strong commuting for short tree strategy, we will use AD^+ reflection. But first we need a lemma.

Lemma 4.8.3 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, Γ is a pointclass and Σ has strong branch condensation and is strongly Γ-fullness preserving. Then Σ is strongly commuting.*

Proof. Fix $(\vec{\mathcal{T}}, \mathcal{Q}), (\vec{\mathcal{U}}, \mathcal{R}) \in I^b(\mathcal{P}, \Sigma)$ and α as in Definition 4.8.2. Let k be the inverse of the collapse of $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{R}^b})$.

Using the fact that $\pi^{\vec{\mathcal{T}},b}$ and $\pi^{\vec{\mathcal{U}},b}$ exist, we can find a $\mathcal{Q}_1$ and $\mathcal{R}_1$ such that

1. $\mathcal{Q}_1$ is a cutpoint of $\vec{\mathcal{T}}$ and $\mathcal{R}_1$ is a cutpoint of $\vec{\mathcal{U}}$,
2. $\vec{\mathcal{T}}_{\geq \mathcal{Q}_1}$ is a normal tree on $\mathcal{Q}_1$ above $\mathcal{Q}_1^b$ and $\vec{\mathcal{U}}_{\geq \mathcal{R}_1}$ is a normal tree on $\mathcal{R}_1$ above $\mathcal{R}_1^b$,
3. $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{Q}_1}}$ exists and $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}_1}}$ exists, and
4. $\mathcal{Q}_1^b = \mathcal{Q}^b$ and $\mathcal{R}_1^b = \mathcal{R}^b$.

[21] From here on, we mean $k = k_0 \circ \pi^{\vec{\mathcal{T}},b}$ where k_0 is the transitive collapse of $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{R}^b})$.

[22] This terminology is so that subsequent statements of lemmas and definitions are uniform.

Let then k^+ be the inverse of the collapse of $Hull^{\mathcal{Q}_1}(\pi^{\vec{\mathcal{T}}_{\leq \mathcal{Q}_1}}[\mathcal{P}] \cup \delta^{\mathcal{R}_1^b})$. We will show that $k^+ \restriction \mathcal{P}^b = \pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}_1}, b}$. The claim then follows because $k^+ \restriction \mathcal{P}^b = k$ and $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{R}_1}, b} = \pi^{\vec{\mathcal{U}}, b}$.

Without loss of generality we can assume that $\mathcal{Q}_1 = \mathcal{Q}$ and $\mathcal{R}_1 = \mathcal{R}$. We now compare $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ with $(\mathcal{R}, \Sigma_{\mathcal{R}})$. Let $(\mathcal{T}, \mathcal{M}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{U}, \mathcal{M}) \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$ be the trees coming from the comparison of the two hod pairs. Because Σ is commuting we have that

(1) $\pi^{\vec{\mathcal{T}}^\frown \mathcal{T}} = \pi^{\vec{\mathcal{U}}^\frown \mathcal{U}}$.

It follow from strong branch condensation that $(\Sigma_{\mathcal{Q}})_{\mathcal{R}^b} = \Sigma_{\mathcal{R}^b}$ which in turns implies that

(2) $\pi^{\mathcal{T}} \restriction (\mathcal{R}^b | \delta^{\mathcal{R}^b}) = \pi^{\mathcal{U}} \restriction (\mathcal{R}^b | \delta^{\mathcal{R}^b})$.

Notice that (1) implies that

(3) $k \restriction (\mathcal{P}^b | \delta^{\mathcal{P}^b}) = \pi^{\vec{\mathcal{U}}, b} \restriction (\mathcal{P}^b | \delta^{\mathcal{P}^b})$.

To finish the proof, we have to verify (3) for subsets of $\delta^{\mathcal{P}^b}$. Let then $A \in \wp(\delta^{\mathcal{P}^b}) \cap \mathcal{P}^b$. We then have that, using (1),

(6) $k(A) = \pi^{\vec{\mathcal{T}}, b}(A) \cap \delta^{\mathcal{R}^b}$ and $\pi^{\vec{\mathcal{U}}, b}(A) = \{\xi < \delta^{\mathcal{R}^b} : \pi^{\mathcal{U}}(\xi) \in \pi^{\vec{\mathcal{T}}^\frown \mathcal{T}, b}(A)\}$.

(3) and (6) imply that $\pi^{\vec{\mathcal{U}}, b}(A) = k(A)$. □

Proposition 4.8.4 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair, Γ is a pointclass and Σ has strong branch condensation and is strongly Γ-fullness preserving. Then for some $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \mathcal{T}}$ is strongly commuting.*

Proof. If Σ is an iteration strategy then we can take $\mathcal{T} = \emptyset$ and use Proposition 4.8.1. We assume that Σ is a short tree strategy and $\mathcal{P}$ is of lsa type. Towards contradiction assume that there is no $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{Q}, \mathcal{T}}$ is strongly commuting. Let ϕ be the sentence asserting the existence of $(\mathcal{P}, \Sigma, \Gamma)$ as in the hypothesis of Proposition 4.8.4 with the property that for any $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \mathcal{T}}$ is not strongly commuting.

Let then $\Gamma^* \subseteq \wp(\mathcal{R})$ be least such that for some α, $L_\alpha(\Gamma^*, \mathbb{R}) \vDash \mathsf{ZF} - \mathit{Replacement} + \phi$ and $\Gamma^* = \wp(\mathbb{R}) \cap L_\alpha(\Gamma^*, \mathbb{R})$. Fix least α witnessing the above statement. Let now

Φ be a good pointclass such that $\Gamma^* \subseteq \underset{\sim}{\Delta}_\Phi$. Fix a triple $(\mathcal{S}, \Lambda, \Gamma^{**}) \in L_\alpha(\Gamma^*, \mathbb{R})$ such that $L_\alpha(\Gamma^*, \mathbb{R}) \vDash \phi[\mathcal{S}, \Lambda, \Gamma^{**}]$. Applying Theorem 4.1.6 to Φ, fix F as in that theorem. Fix $x \in \mathbb{R}$ such that if $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ then $(\mathcal{N}_x^*, \Sigma_x)$ Suslin, co-Suslin captures Γ^{**} and $Code(\Lambda)$.

Applying Theorem 4.7.6 and Corollary 4.6.10 to $(\mathcal{S}, \Lambda)$, we get that the Γ^{**}-hod pair construction of $\mathcal{N}_x^*$ reaches a normal iterate $\mathcal{R}$ of $\mathcal{S}$ such that if Ψ is the strategy of $\mathcal{R}$ inherited from the background construction then $\Lambda_{\mathcal{R}} = \Psi^{stc}$ and Ψ has strong branch condensation and is strongly Γ^{**}-fullness preserving. Applying Lemma 4.8.3 we get that $\Lambda_{\mathcal{R}}$ is strongly commuting. $\square$

The next lemma will be used in the proof of Theorem 6.1.5.

Lemma 4.8.5 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair or is a hod pair and $\mathcal{P}$ is non-meek. Suppose further that Γ is a pointclass and Σ has strong branch condensation and is strongly Γ-fullness preserving. Suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I^b(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in B(\mathcal{P}, \Sigma) \cup I^b(\mathcal{P}, \Sigma)$ are such that*

1. *if $\mathcal{R}^+$ is such that $(\vec{\mathcal{U}}, \mathcal{R}^+) \in I^b(\mathcal{P}, \Sigma)$ then $(\mathcal{R}^+)^b = \mathcal{R}^b$ and*
2. *there is some $\alpha \leq \lambda^{\mathcal{Q}} - 1$ and a normal tree $\mathcal{W}$ on $\mathcal{R}$ such that $(\mathcal{W}, \mathcal{Q}(\alpha)) \in I(\mathcal{R}, \Sigma_{\mathcal{R}})$.*

Then $(\mathcal{Q}(\alpha))^b = Hull^{\mathcal{Q}^b}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{(\mathcal{Q}(\alpha))^b})$.

Proof. We first present the proof under an assumptions that makes the matter somewhat simpler.

(1) Suppose that $(\vec{\mathcal{T}}, \mathcal{Q}), (\vec{\mathcal{U}}, \mathcal{R}^+) \in I(\mathcal{P}, \Sigma)$.

Apply now $\mathcal{W}$ to $\mathcal{R}^+$ and let its last model be $\mathcal{S}$. The idea now is to compare $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ with $(\mathcal{S}, \Sigma_{\mathcal{S}})$. We have that $\mathcal{Q}(\alpha) \trianglelefteq \mathcal{S}$. Suppose first the rest of the comparison between $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{S}, \Sigma_{\mathcal{S}})$ uses no extenders with critical point $\delta^{\mathcal{S}^b}$. The claim then follows from the simple facts that $\mathcal{Q}^b = Hull^{\mathcal{Q}^b}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{Q}^b})$ and $\alpha \geq \sup \pi^{\vec{\mathcal{T}},b}[\lambda^{\mathcal{Q}^b}]$.

Suppose then that the rest of the comparison between $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{S}, \Sigma_{\mathcal{S}})$ uses an extender with critical point $\delta^{\mathcal{S}^b}$. It follows from strong branch condensation and Γ-fullness preservation that $\mathcal{Q}(\alpha) \triangleleft \mathcal{S}$ and letting $E \in \vec{E}^{\mathcal{S}}$ be the extender with critical point $\delta^{\mathcal{S}^b}$ used in the aforementioned comparison, E is the least extender on the sequence of $\mathcal{S}$ with critical point $\delta^{\mathcal{S}^b}$ such that $\mathcal{Q}(\alpha) \triangleleft \mathcal{S}_1$ where $\mathcal{S}_1 = Ult(\mathcal{S}, E)$. The rest of the comparison is a comparison between $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ with $(\mathcal{S}_1, \Sigma_{\mathcal{S}_1})$. It follows from strong branch condensation of Σ (see Lemma 4.7.5) that

(2) $\Sigma_{\mathcal{Q}(\alpha)} = \Sigma_{\mathcal{S}_1(\alpha)}$.

Let now $(\mathcal{T}_1, \mathcal{K}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{T}_2, \mathcal{K}) \in I(\mathcal{S}_1, \Sigma_{\mathcal{S}_1})$ be the normal trees that achieve comparison between $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ with $(\mathcal{S}_1, \Sigma_{\mathcal{S}_1})$. It follows from the fact that Σ is commuting that

(3) $\pi^{\mathcal{T}_1,b} \circ \pi^{\vec{\mathcal{T}},b} = \pi^{\mathcal{T}_2,b} \circ (\pi^{\mathcal{S}}_E \restriction \mathcal{S}^b) \circ \pi^{\mathcal{W},b} \circ \pi^{\vec{\mathcal{U}},b}$

Because of (2) we have that

(4) $\pi^{\mathcal{T}_1} \restriction \mathcal{Q}(\alpha) = \pi^{\mathcal{T}_2} \restriction \mathcal{Q}(\alpha)$.

Using standard facts about representations of ultrapowers, we also get that

(5) $(\mathcal{Q}(\alpha))^b = Hull^{(\mathcal{S}_1)^b}(\pi_E[(\mathcal{Q}(\alpha))^b] \cup \delta^{(\mathcal{Q}(\alpha))^b})$.

Using the same standard facts, we also get that

(6) $(\mathcal{S}_1)^b = Hull^{(\mathcal{S}_1)^b}(\pi_E \circ \pi^{\mathcal{W},b} \circ \pi^{\vec{\mathcal{U}},b}[\mathcal{P}^b] \cup \delta^{(\mathcal{S}_1)^b})$ and $\mathcal{K}^b = Hull^{\mathcal{K}^b}(\pi^{\mathcal{T}_1,b} \circ \pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{\mathcal{K}^b})$.

It follows from (3)-(6) that $(\mathcal{Q}(\alpha))^b = Hull^{\mathcal{Q}^b}(\pi^{\vec{\mathcal{T}},b}[\mathcal{P}^b] \cup \delta^{(\mathcal{Q}_\alpha)^b})$, finishing the proof under our assumption that (1) holds.

Suppose next (1) fails. In this case there are a cutpoint $\mathcal{Q}_1$ of $\vec{\mathcal{T}}$ and a cutpoint $\mathcal{R}_1$ of $\vec{\mathcal{U}}$ such that

1. $(\vec{\mathcal{T}}, \mathcal{Q}_1) \in I(\mathcal{P}, \Sigma)$ and $(\mathcal{R}_1, \vec{\mathcal{U}}) \in I(\mathcal{P}, \Sigma)$,

2. $\vec{\mathcal{T}}_{\geq \mathcal{Q}_1}$ is a normal tree $\mathcal{T}$ on $\mathcal{Q}_1$ that is above $(\mathcal{Q}_1)^b$ and has a drop, and

3. $\vec{\mathcal{U}}_{\geq \mathcal{R}_1}$ is a normal tree $\mathcal{U}$ on $\mathcal{R}_1$ that is above $(\mathcal{R}_1)^b$ and has a drop.

In this case, we let $\mathcal{S} = Ult(\mathcal{R}_1, F)$ where F is $(\delta^{\mathcal{R}^b}, \delta^{(\mathcal{Q}(\alpha))^b})$-extender derived from $\pi^{\mathcal{W},b}$. The next step is to compare $(\mathcal{Q}_1, \Sigma_{\mathcal{Q}_1})$ with $(\mathcal{S}, \Sigma_{\mathcal{S}})$ The rest of the proof is word by word the same as the one from (1). Indeed, notice that $\pi^{\mathcal{T},b} = id$ and $\pi^{\mathcal{W},b} = \pi_F \restriction (\mathcal{R}_1)^b$. We leave the details to the readers. $\square$

4.9 Solidity and condensation

The main contribution of this section are Theorem 4.9.6 and Theorem 4.9.7 that can be used to show that fully backgrounded hod pair constructions converge. We start by the following version of Lemma 4.9.4 for phalanxes that is used in the proof of solidity and universality. We omit the actual proofs of Theorem 4.9.6 and Theorem 4.9.7 as, in the light of Lemma 4.9.5, the proofs of solidity and universality are trivial generalizations of the usual proofs of these facts (see Chapter 5 of [28]).

Definition 4.9.1 (Certified phalanxes) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is non-meek and $\mathcal{R}$ is a hod premouse. We say $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\pi, \mathcal{P}, \Sigma)$-certified phalanx if $\zeta > o(\mathcal{P}^b)$ and there is an embedding $\pi : \mathcal{R} \to \mathcal{P}$ such that $\zeta \leq \mathrm{crit}(\pi)$.*

Continuing with the set up of Definition 4.9.1, we let $\pi^+ : (\mathcal{P}, \mathcal{R}, \zeta) \to (\mathcal{P}, \mathcal{P}, \zeta)$ be given by (id, π), and also, we let Σ^{π^+} be the π^+-pullback of Σ.

Lemma 4.9.2 (No strategy disagreement) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is non-meek, Σ has strong branch condensation and Σ is strongly fullness preserving. Suppose $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\mathcal{P}, \Sigma)$ certified phalanx as witnessed by $\pi : \mathcal{R} \to \mathcal{P}$. Let $\Lambda = \Sigma^{\pi^+}$. Then no strategy disagreement appears in the comparison of $\mathcal{P}$ and $(\mathcal{P}, \mathcal{R}, \zeta)$ where Σ is used on $\mathcal{P}$ side and Λ is used on $(\mathcal{P}, \mathcal{R}, \zeta)$.*

Proof. Towards a contradiction suppose not. It follows from the proof of Lemma 4.6.3 that we can find a minimal low level disagreement $(\vec{\mathcal{T}}, \mathcal{Q})$ between Σ and Λ. Let then $E = E^{\vec{\mathcal{T}}}_{\mathcal{Q}}$, the un-dropping extender of $\vec{\mathcal{T}}$ restricted to $\mathcal{Q}$. We have that $\mathcal{Q} \trianglelefteq_{hod} Ult(\mathcal{P}, E)$.

Our intention now is to find a Σ-iterate $\mathcal{S}$ of $\mathcal{P}$ and an embedding $\sigma : Ult(\mathcal{P}, E) \to \mathcal{S}$ such that $\Lambda_{\mathcal{Q},\vec{\mathcal{T}}} = \Sigma^{\sigma}_{\mathcal{S},\vec{\mathcal{W}}}$ where $\vec{\mathcal{W}}$ is a stack on $\mathcal{P}$ with last model $\mathcal{S}$. Let first $\vec{\mathcal{W}}^* = \pi^+\vec{\mathcal{T}}$ and let $\mathcal{S} = Ult(\mathcal{P}, E^{\vec{\mathcal{W}}^*})$, where again $E^{\vec{\mathcal{W}}^*}$ is the un-dropping extender of $\vec{\mathcal{W}}$. Clearly $\vec{\mathcal{W}} = \vec{\mathcal{W}}^{*\frown}\{E^{\vec{\mathcal{W}}^*}\}$ works. The claim now follows from strong branch condensation of Σ applied to $(\vec{\mathcal{W}}, \mathcal{S}, \pi_E, Ult(\mathcal{P}, E), \alpha, \sigma)$, where α is such that $\mathcal{Q} = Ult(\mathcal{P}, E)(\alpha)$ and $\sigma : Ult(\mathcal{P}, E) \to Ult(\mathcal{P}, E^{\vec{\mathcal{W}}})$ is the embedding given by the copying construction. □

Definition 4.9.3 (Certified pairs) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{R}$ is a hod premouse such that both $\mathcal{P}$ and $\mathcal{R}$ are of limit type. Suppose that there is π such that $\pi : \mathcal{P}^b \to \mathcal{R}^b$. We say the pair $(\pi, \mathcal{R})$ is $(\mathcal{P}, \Sigma)$-certified by $(\sigma, \vec{\mathcal{T}}, \mathcal{Q}, \alpha)$ if*

1. $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha \leq \lambda^{\mathcal{Q}}$ *and* $\sigma : \mathcal{R} \to \mathcal{Q}(\alpha)$,
2. $(\mathcal{Q}(\alpha))^b = Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[\mathcal{P}^b] \cup \delta^{(\mathcal{Q}(\alpha))^b})$, *and*
3. *letting* $k : \mathcal{P}^b \to (\mathcal{Q}(\alpha))^b$ *be the inverse of the collapse of* $Hull^{\mathcal{Q}}(\pi^{\vec{\mathcal{T}}}[\mathcal{P}^b] \cup \delta_\alpha^{(\mathcal{Q}(\alpha))^b})$, $k = (\sigma \restriction \mathcal{R}^b) \circ \pi$.

We say $(\mathcal{R}, \Lambda)$ *is a* $(\mathcal{P}, \Sigma)$*-certified hod pair if for every* $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{R}, \Lambda)$ *there is some* π, $(\sigma, \vec{\mathcal{T}}, \mathcal{Q}, \alpha)$ *such that* $(\pi, \mathcal{S})$ *is* $(\mathcal{P}, \Sigma)$*-certified by* $(\sigma, \vec{\mathcal{T}}, \mathcal{Q}, \alpha)$ *and* $\Lambda_{\mathcal{S}^b} = \Sigma^\sigma_{\mathcal{Q}^b, \vec{\mathcal{T}}}$.

Lemma 4.9.4 *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\mathcal{P}$ *is non-lsa type non-meek hod premouse,* Γ *is a pointclass and* Σ *has strong branch condensation and is strongly* Γ*-fullness preserving. Suppose* $(\vec{\mathcal{T}}, \mathcal{R}) \in I^b(\mathcal{P}, \Sigma)$[23] *is such that for some* Λ, $(\mathcal{R}, \Lambda)$ *is* $(\mathcal{P}, \Sigma)$*-certified and there is a* $\pi : \mathcal{P} \to \mathcal{R}$ *such that* $(\pi \restriction \mathcal{P}^b, \mathcal{R})$ *is* $(\mathcal{P}, \Sigma)$*-certified by* $(\sigma, \vec{\mathcal{U}}, \mathcal{Q}, \alpha)$. *Then* $\pi^{\vec{\mathcal{T}}}$ *exists and* $\pi^{\vec{\mathcal{T}}} \leq \pi$.

Proof. Fix a $(\mathcal{P}, \Sigma)$-certificate $(\sigma, \vec{\mathcal{U}}, \mathcal{Q}, \alpha)$ for $(\mathcal{R}, \Lambda)$. Thus, $\mathcal{Q}(\alpha) = \mathcal{Q}$ and $\Lambda = \Sigma^\sigma_{\mathcal{Q}, \vec{\mathcal{U}}}$. Since Σ has strong branch condensation, it follows that $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}} = \Lambda (= \Sigma^\sigma_{\mathcal{Q}, \vec{\mathcal{U}}})$. Let $\Phi = \Sigma^{\sigma \circ \pi}_{\mathcal{Q}, \vec{\mathcal{U}}}$. It follows from strong branch condensation that $\Phi = \Sigma$. We can now apply the usual Dodd-Jensen argument to conclude that $\pi^{\vec{\mathcal{T}}}$ exists and that $\pi^{\vec{\mathcal{T}}} \leq \pi$. □

Lemma 4.9.5 (Dodd-Jensen for certified phalanxes) *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* Σ *has strong branch condensation and is strongly fullness preserving. Suppose that* $(\mathcal{P}, \mathcal{R}, \zeta)$ *is a* $(\mathcal{P}, \Sigma)$*-certified phalanx as witnessed by* $\pi : \mathcal{R} \to \mathcal{P}$. *Suppose that* $(\mathcal{T}, \mathcal{Q}) \in I((\mathcal{P}, \mathcal{R}, \zeta), \Sigma^{\pi^+})$ *and* $(\mathcal{U}, \mathcal{S}) \in I(\mathcal{P}, \Sigma)$ *are such that the last branch of* $\mathcal{T}$ *is on* $\mathcal{P}$ *and either*

1. $\mathcal{Q} \trianglelefteq_{hod} \mathcal{S}$ *and* $\pi^{\mathcal{T}}$ *exists or*
2. $\mathcal{S} \trianglelefteq_{hod} \mathcal{Q}$ *and* $\pi^{\mathcal{U}}$ *exists.*

Then $\mathcal{Q} = \mathcal{S}$ *and* $\pi^{\mathcal{T}} = \pi^{\mathcal{U}}$.

Proof.

Let $\mathcal{T}^* = \pi^+\mathcal{T}$. Let $\mathcal{Q}^*$ be the last model of $\mathcal{T}^*$ and let $\sigma : \mathcal{Q} \to \mathcal{Q}^*$ come from the copying construction. Suppose first that $\mathcal{Q} \trianglelefteq_{hod} \mathcal{S}$ and $\pi^{\mathcal{T}}$ exists. Applying (the

[23] Thus, $\pi^{\vec{\mathcal{T}}, b}$ exists, see Definition 2.6.5.

proof of) Lemma 4.9.4 to σ noting that $\mathcal{Q}^b = \mathcal{S}^b$, we get that $\pi^{\mathcal{U}}$ exists, $\mathcal{S} = \mathcal{Q}$ and $\pi^{\mathcal{U}} \leq \pi^{\mathcal{T}}$.

Suppose now $\mathcal{S} \trianglelefteq_{hod} \mathcal{Q}$ and $\pi^{\mathcal{U}}$ exists. Let Φ be $\pi^{\mathcal{U}} \circ \pi^+$-pullback of $\Sigma_{\mathcal{S}}$. We then have that $\Phi = \Sigma^{\pi^+}$. Applying Lemma 4.9.4 to $(\sigma \restriction \mathcal{S}) \circ \pi^{\mathcal{U}}$ and $(\mathcal{T}^*, \sigma(\mathcal{S}))$, we get $\sigma(\mathcal{S}) = \mathcal{Q}^*$ and $\pi^{\mathcal{T}^*}$ exists (again, we have here that $(\mathcal{Q}^*)^b = \sigma(\mathcal{S})^b$). It follows that $\Sigma^{\sigma}_{\mathcal{Q}^*, \mathcal{T}^*} = \Sigma_{\mathcal{S}}$. Therefore, the usual Dodd-Jensen argument can be used to get that $\mathcal{S} = \mathcal{Q}$ and $\pi^{\mathcal{T}} \leq \pi^{\mathcal{U}}$. Putting the two arguments together we see that $\pi^{\mathcal{U}} = \pi^{\mathcal{T}}$. $\square$

It is clear that it follows from Lemma 4.9.5 and from Lemma 4.9.2 that the usual proofs of condensation, universality and solidity go through for hod mice. We state the results without proofs (see Chapter 5 of [28] for the usual proofs of these results.)

Theorem 4.9.6 (Solidity and universality) *Suppose $k < \omega$ and $(\mathcal{P}, \Sigma)$ is a hod pair such that*

1. *$\mathcal{P}$ is k-sound non-meek hod premouse,*
2. *$\mathcal{P}$ is not of lsa type and $\rho(\mathcal{P}) > o(\mathcal{P}^b)$, and*
3. *Σ is strongly fullness preserving and has strong branch condensation.*

Let r be the $k+1$st standard parameter of $(\mathcal{P}, u_k(\mathcal{P}))$; then r is $k+1$-solid and $k+1$-universal over $(\mathcal{P}, u_k(\mathcal{P}))$.

Theorem 4.9.7 (Condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that*

1. *$\mathcal{P}$ is non-meek hod premouse,*
2. *$\mathcal{P}$ is not of lsa type and $\rho(\mathcal{P}) > o(\mathcal{P}^b)$, and*
3. *Σ is strongly fullness preserving and has strong branch condensation.*

Suppose $(\mathcal{P}, \mathcal{R}, \zeta)$ is a $(\mathcal{P}, \Sigma)$ certified phalanx as witnessed by $\pi : \mathcal{R} \to \mathcal{P}$ such that $\zeta = \mathrm{crit}(\pi) = \rho^{\mathcal{R}}_{\omega}$. Then either

1. *$\mathcal{R} \trianglelefteq_{hod} \mathcal{P}$ or*
2. *there is an extender E on the sequence of $\mathcal{P}$ such that $lh(E) = \rho^{\mathcal{R}}_{\omega}$ and $\mathcal{R} \trianglelefteq_{hod} Ult(\mathcal{P}, E)$.*

4.10 Diamond comparison

Our goal here is to provide another comparison argument, *diamond comparison*, that doesn't rely on branch condensation as heavily as our other argument (see Corollary 4.6.10). The new comparison argument follows the same line of thought as the proof of a similar comparison argument from [10] (see Theorem 2.47 of [10]).

We have two applications in mind for such a comparison argument. First we will use it to show that in some instances tails of strategies with hull condensation get branch condensation and strong branch condensation. This will appear as Theorem 5.6.8.

Next, as in [10], the diamond comparison argument can be used to show that $\mathsf{AD}^+ + \mathsf{LSA}$ is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals. This will appear as Theorem 10.3.1. In [10], a similar argument gave the consistency of $AD_{\mathbb{R}}$ + "Θ is regular" relative to a Woodin cardinal that is a limit of Woodin cardinals.

Following the proof of Theorem 2.47 of [10], we first define a *bad block* and a *bad sequence* and show that there cannot be such a bad sequence of length ω_1. We then show that the failure of comparison produces such bad sequences of length ω_1.

4.10.1 Bad sequences

For the purposes of this subsection, we make a definition of a bad block and a bad sequence. In later subsections, we will redefine these names for different objects.

Definition 4.10.1 (Bad block) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs of limit type. Then*

$$B = (((\mathcal{P}_i, \mathcal{Q}_i, \Sigma_i, \Lambda_i) : i < 5), (\vec{\mathcal{T}}_i, \vec{\mathcal{U}}_i : i < 4), (c, d))$$

is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$ if the following holds:

1. *$(\mathcal{P}_0, \Sigma_0) = (\mathcal{P}, \Sigma)$ and $(\mathcal{Q}_0, \Lambda_0) = (\mathcal{Q}, \Lambda)$.*
2. *$\vec{\mathcal{T}}_0$ is a stack according to Σ_0 on $\mathcal{P}$.*
3. *$\vec{\mathcal{U}}_0$ is a stack according to Λ_0 on $\mathcal{Q}$.*
4. *Let $\vec{\mathcal{T}}_0 = (\mathcal{M}_\beta, \vec{\mathcal{T}}_\beta, \mathcal{R}_\beta, E_\beta : \beta \leq \nu)$ and $\vec{\mathcal{U}}_0 = (\mathcal{N}_\beta, \vec{\mathcal{U}}_\beta, \mathcal{S}_\beta, F_\beta : \beta \leq \nu)$. Then $\vec{\mathcal{T}}_\nu$ and $\vec{\mathcal{U}}_\nu$ are undefined, $\mathcal{P}_1 = \mathcal{M}_\nu$ and $\mathcal{Q}_1 = \mathcal{N}_\nu$.*

5. *There is some* $\beta+1 < min(\lambda^{\mathcal{P}_1}, \lambda^{\mathcal{Q}_1})$ *such that* $\mathcal{P}_1(\beta+1) = \mathcal{Q}_1(\beta+1)$, $\mathcal{P}_1(\beta+1)$ *is of successor type,* $\Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0} \neq \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0}$ *and* $\Sigma_{\mathcal{P}_1(\beta),\vec{\mathcal{T}}_0} = \Sigma_{\mathcal{Q}_1(\beta),\vec{\mathcal{U}}_0}$.

6. $\vec{\mathcal{T}}_1$ *and* $\vec{\mathcal{U}}_1$ *are stacks on* $(\mathcal{P}_1(\beta+1), \Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0})$ *and* $(\mathcal{Q}_1(\beta+1), \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0})$ *respectively with last models* $\mathcal{R}$ *and* $\mathcal{S}$ *such that both* $(\vec{\mathcal{T}}_1, \mathcal{R}, \vec{\mathcal{U}}_1, \mathcal{S})$ *are comparison stacks for* $(\mathcal{P}_1(\beta+1), \Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0})$ *and* $(\mathcal{Q}_1(\beta+1), \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0})$[24].

7. *Keeping the above notation,* $\vec{\mathcal{T}}_1$ *and* $\vec{\mathcal{U}}_1$ *have a last normal component of successor length whose predecessor is a limit ordinal*[25] *and* $\vec{\mathcal{T}}_1^- = \vec{\mathcal{U}}_1^-$.

8. *Again keeping the above notation,* $c = \Sigma_{\mathcal{P}_1(\beta+1),\vec{\mathcal{T}}_0}(\vec{\mathcal{T}}_1^-)$, $d = \Lambda_{\mathcal{Q}_1(\beta+1),\vec{\mathcal{U}}_0}(\vec{\mathcal{U}}_1^-)$, $\mathcal{P}_2 = \mathcal{M}_c^{\vec{\mathcal{T}}_1^-}$ *and* $\mathcal{Q}_2 = \mathcal{M}_d^{\vec{\mathcal{T}}_1^-}$ *where* $\vec{\mathcal{T}}_1^-$ *is applied to* $\mathcal{P}_1$ *and* $\mathcal{Q}_1$ *respectively.*

9. $\Sigma_1 = \Sigma_{\mathcal{P}_1,\vec{\mathcal{T}}_0}$, $\Sigma_2 = \Sigma_{\mathcal{P}_2,\vec{\mathcal{T}}_0^\frown(\vec{\mathcal{T}}_1^-)^\frown\{\mathcal{P}_2\}}$, $\Lambda_1 = \Sigma_{\mathcal{Q}_1,\vec{\mathcal{U}}_0}$, *and* $\Lambda_2 = \Sigma_{\mathcal{Q}_2,\vec{\mathcal{U}}_0^\frown(\vec{\mathcal{U}}_1^-)^\frown\{\mathcal{Q}_2\}}$,

10. $\vec{\mathcal{T}}_2$ *is a stack according to* Σ_2 *on* $\mathcal{P}_2$ *with last model* $\mathcal{P}_3$ *and* $\Sigma_3 = (\Sigma_2)_{\mathcal{P}_3,\vec{\mathcal{T}}_2}$.

11. $\vec{\mathcal{U}}_2$ *is a stack according to* Λ_2 *on* $\mathcal{Q}_2$ *with last model* $\mathcal{Q}_3$ *and* $\Lambda_3 = (\Lambda_2)_{\mathcal{Q}_3,\vec{\mathcal{U}}_2}$.

12. $\vec{\mathcal{T}}_3$ *is a normal tree according to* Σ_3 *on* $\mathcal{P}_3$ *with last model* $\mathcal{P}_4$ *and* $\Sigma_4 = (\Sigma_3)_{\mathcal{P}_4,\vec{\mathcal{T}}_3}$.

13. $\vec{\mathcal{U}}_3$ *is a normal tree according to* Λ_3 *on* $\mathcal{Q}_3$ *with last model* $\mathcal{Q}_4$ *and* $\Lambda_4 = (\Lambda_3)_{\mathcal{Q}_4,\vec{\mathcal{U}}_3}$.

14. $\mathcal{P}_3^b = \mathcal{Q}_3^b$ *and* $(\Sigma_3)_{\mathcal{P}_3^b} = (\Lambda_3)_{\mathcal{Q}_3^b}$.

15. $\vec{\mathcal{T}}_3$ *and* $\vec{\mathcal{U}}_3$ *are the trees produced via extender comparison between* $\mathcal{P}_3$ *and* $\mathcal{Q}_3$.

We set $\vec{\mathcal{T}}^B = \vec{\mathcal{T}}_0^\frown\vec{\mathcal{T}}_1^\frown\vec{\mathcal{T}}_2^\frown\vec{\mathcal{T}}_3$ *and* $\vec{\mathcal{U}}^B = \vec{\mathcal{U}}_0^\frown\vec{\mathcal{U}}_1^\frown\vec{\mathcal{U}}_2^\frown\vec{\mathcal{U}}_3$. *We say* $\vec{\mathcal{T}}^B$ *is the stack on the top of* B *and* $\vec{\mathcal{U}}^B$ *is the stack in the bottom of* B.

Next we show that there cannot be a bad sequence of length ω_1.

Lemma 4.10.2 (No bad sequences) *Suppose* $(\mathcal{P}, \Sigma)$ *and* $(\mathcal{Q}, \Lambda)$ *are two hod pairs of limit type such that* $\mathcal{P}$ *and* $\mathcal{Q}$ *are countable, and both* Σ *and* Λ *are* $(\omega_1, \omega_1, \omega_1)$*-strategies. There is then no bad sequence, i.e., a sequence* $(B_\beta : \beta < \omega_1)$ *satisfying the following holds:*

[24] In particular, the two strategies agree on the last models of $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{U}}_1$. Because of Theorem 4.6.10 we can take $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{U}}_1$ to be normal trees. We will always use the diamond comparison argument in situations where Theorem 4.6.10 applies to low level strategies.

[25] Recall that in Definition 4.6.4, we required that comparison stacks have a last model.

1. *For all* $\beta < \omega_1$, $B_\beta = (((\mathcal{P}_{\beta,i}, \mathcal{Q}_{\beta,i}, \Sigma_{\beta,i}, \Lambda_{\beta,i}) : i < 5), (\vec{\mathcal{T}}_{\beta,i}, \vec{\mathcal{U}}_{\beta,i} : i < 4), (c_\beta, d_\beta))$.
2. *For all* $\beta < \omega_1$, B_β *is a bad block on* $((\mathcal{P}_{\beta,0}, \Sigma_{\beta,0}), (\mathcal{Q}_{\beta,0}, \Lambda_{\beta,0}))$.
3. *For all* $\beta < \omega_1$, $\mathcal{P}_{\beta+1,0} = \mathcal{P}_{\beta,4}$ *and* $\mathcal{Q}_{\beta+1,0} = \mathcal{Q}_{\beta,4}$.
4. *For* $\beta < \alpha < \omega_1$, *let* $\pi_{\beta,\alpha} : \mathcal{P}_{\beta,0} \to \mathcal{P}_{\alpha,0}$ *be the composition of the embeddings on the "top" and* $\sigma_{\beta,\alpha} : \mathcal{Q}_{\beta,0} \to \mathcal{Q}_{\alpha,0}$ *be the composition of the embeddings on the "bottom". Then for all limit* $\lambda < \omega_1$, $\mathcal{P}_{\lambda,0}$ *is the direct limit of* $(\mathcal{P}_\beta : \beta < \lambda)$ *under the maps* $\pi_{\beta,\alpha}$. *Similarly, for all limit* $\lambda < \omega_1$, $\mathcal{Q}_{\lambda,0}$ *is the direct limit of* $(\mathcal{Q}_\beta : \beta < \lambda)$ *under the maps* $\sigma_{\beta,\alpha}$.
5. *For a limit ordinal* $\lambda < \omega_1$, $\mathcal{P}^b_{\lambda,0} = \mathcal{Q}^b_{\lambda,0}$.
6. *For all* $\beta < \omega_1$, $\Sigma_{\beta,0} = \Sigma_{\mathcal{P}_{\beta,0}, \oplus_{\gamma<\beta} \vec{\mathcal{T}}^{B_\gamma}}$ *and* $\Lambda_{\beta,0} = \Sigma_{\mathcal{Q}_{\beta,0}, \oplus_{\gamma<\beta} \vec{\mathcal{U}}^{B_\gamma}}$.

Proof. Towards a contradiction, suppose $\vec{B} = (B_\beta : \beta < \omega_1)$ is a bad sequence. Let $\mathcal{P}_{\omega_1}$ be the direct limit of $(\mathcal{P}_{\beta,0} : \beta < \omega_1)$ under the embeddings $\pi_{\beta,\alpha}$ and $\mathcal{Q}_{\omega_1}$ be the direct limit of $(\mathcal{Q}_{\beta,0} : \beta < \omega_1)$ under the embeddings $\sigma_{\beta,\alpha}$. Let X be a countable submodel of H_{ω_3} such that letting $\tau : M \to H_{\omega_3}$ be the uncollapse map, $\vec{B} \in rng(\sigma)$. Let $\kappa = \omega_1^{\mathcal{M}}$ and notice that for every $\beta < \kappa$,

$$B^-_\beta =_{def} (((\mathcal{P}_{\beta,i}, \mathcal{Q}_{\beta,i}) : i < 5), (\vec{\mathcal{T}}_{\beta,i}, \vec{\mathcal{U}}_{\beta,i} : i < 4), (c_\beta, d_\beta)) \in M$$

and B^-_β is countable in M. It then follows that $\tau^{-1}(\mathcal{P}_{\omega_1}) = \mathcal{P}_{\kappa,0}$ and $\tau^{-1}(\mathcal{Q}) = \mathcal{Q}_{\kappa,0}$. Let

$$\pi_\beta : \mathcal{P}_{\beta,0} \to \mathcal{P}_{\omega_1} \text{ and } \sigma_\beta : \mathcal{Q}_{\beta,0} \to \mathcal{Q}_{\omega_1}$$

be the direct limit embeddings.

Standard arguments show that for all $x \in \mathcal{P}_{\kappa,0} \cap \mathcal{Q}_{\kappa,0}$,

$$\pi_\kappa(x) = \tau(x) = \sigma_\kappa(x).$$

Notice that we have that $\lambda^{\mathcal{P}_{\kappa,0}} = \lambda^{\mathcal{Q}_{\kappa,0}}$. Letting $\lambda = \lambda^{\mathcal{P}_{\kappa,0}}$, notice that $\delta^{\mathcal{P}_{\kappa,0}}_{\lambda-1} = \delta^{\mathcal{Q}_{\kappa,0}}_{\lambda-1}$. Let then $\delta = \delta^{\mathcal{P}_{\kappa,0}}_{\lambda-1}$. Let $\phi = \pi^{\vec{\mathcal{T}}_{\kappa,0}}$ and $\psi = \pi^{\vec{\mathcal{U}}_{\kappa,0}}$. It then follows that

(1) $\wp(\delta)^{\mathcal{P}_{\kappa,0}} = \wp(\delta)^{\mathcal{Q}_{\kappa,0}}$.

Let β be such that $\vec{\mathcal{T}}^-_{\kappa,1} = \vec{\mathcal{U}}^-_{\kappa,1}$ is based on $\mathcal{P}_{\kappa,1}(\beta+1) = \mathcal{Q}_{\kappa,1}(\beta+1)$. Notice that

(2) $\delta^{\mathcal{P}_{\kappa,1}(\beta+1)} = \sup\{\phi(f)(a) : f \in \mathcal{P}_{\kappa,0} \wedge f : \delta \to \delta \wedge a \in (\mathcal{P}_{\kappa,1}(\beta))^{<\omega}\}$
(3) $\delta^{\mathcal{Q}_{\kappa,1}(\beta+1)} = \sup\{\psi(f)(a) : f \in \mathcal{Q}_{\kappa,0} \wedge f : \delta \to \delta \wedge a \in (\mathcal{Q}_{\kappa,1}(\beta))^{<\omega}\}$

Let now $p = \pi_{c_\kappa}^{\vec{\mathcal{T}}_{\kappa,1}^-}$, $q = \pi_{d_\kappa}^{\vec{\mathcal{T}}_{\kappa,1}^-}$, $j : \mathcal{P}_{\kappa,2} \to \mathcal{P}_{\omega_1}$ and $i : \mathcal{Q}_{\kappa,2} \to \mathcal{Q}_{\omega_1}$ be the iteration embeddings along the top and bottom of $\vec{B}$. Notice that because

$$(\Sigma_{\kappa,2})_{\mathcal{P}_{\kappa,2}(p(\beta)+1)} = (\Lambda_{\kappa,2})_{\mathcal{Q}_{\kappa,2}(p(\beta)+1)},$$

we have that

(4) $j \restriction \mathcal{P}_{\kappa,2}(p(\beta)+1) = i \restriction \mathcal{Q}_{\kappa,2}(q(\beta)+1)$.

Let then

$$s = \{\gamma < \delta_{\beta+1}^{\mathcal{P}_{\kappa,1}} : \exists f \in (\delta^\delta)^{\mathcal{P}_{\kappa,0}} \exists a \in (\mathcal{P}_{\kappa,1}(\beta))^{<\omega}(\gamma = \phi(f)(a))\}$$
$$t = \{\gamma < \delta_{\beta+1}^{\mathcal{Q}_{\kappa,1}} : \exists f \in (\delta^\delta)^{\mathcal{Q}_{\kappa,0}} \exists a \in (\mathcal{Q}_{\kappa,1}(\beta))^{<\omega}(\gamma = \psi(f)(a))\}.$$

(1) then implies that

(5) $j \circ p[s] = i \circ q[t]$.

(4) then implies that

(6) $p[s] = q[t]$.

It follows from (2) and (3) that

(7) s and t are cofinal in $\delta^{\mathcal{P}_{\kappa,0}(\beta+1)}$.

It then follows from (6) and (7) that $c_\kappa = d_\kappa$, contradiction. □

4.10.2 The comparison argument

In this subsection we prove the following comparison theorem under the hypothesis that the *lower level comparison* holds. Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs of limit type such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving.

Definition 4.10.3 (Lower Level Comparison) *We say low level comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if*

1. *for every $(\vec{\mathcal{T}}, \mathcal{P}_1) \in B(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{Q}_1) \in B(\mathcal{Q}, \Lambda)$, comparison holds for $(\mathcal{P}_1, \Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}})$ and $(\mathcal{Q}_1, \Lambda_{\mathcal{Q}_1, \vec{\mathcal{U}}})$, and*

2. *whenever $(\vec{\mathcal{T}}, \mathcal{P}_1) \in I(\mathcal{P}, \Sigma)$, $(\vec{\mathcal{U}}, \mathcal{Q}_1) \in I(\mathcal{Q}, \Lambda)$ and β are such that $\beta + 1 \leq \min(\lambda^{\mathcal{P}_1}, \lambda^{\mathcal{Q}_1})$, $\mathcal{P}_1(\beta+1) = \mathcal{Q}_1(\beta+1)$, $\mathcal{P}_1(\beta+1)$ is meek and $\Sigma_{\mathcal{P}_1(\beta), \vec{\mathcal{T}}} = \Lambda_{\mathcal{Q}_1(\beta), \vec{\mathcal{U}}}$, there is a normal tree $\mathcal{S}$ of limit length according to both $\Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}}$ and $\Lambda_{\mathcal{Q}_1, \vec{\mathcal{U}}}$ that is based on $\mathcal{P}_1(\beta + 1)$ and is such that letting $b = \Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}}(\mathcal{S})$ and $c = \Lambda_{\mathcal{Q}_1, \vec{\mathcal{U}}}(\mathcal{S})$,*

   (a) $\pi^{\mathcal{S}}_b$ and $\pi^{\mathcal{S}}_c$ exist,

   (b) $\mathcal{M}^{\mathcal{S}}_b(\pi^{\mathcal{S}}_b(\beta + 1)) = \mathcal{M}^{\mathcal{S}}_c(\pi^{\mathcal{S}}_c(\beta + 1))$, and

   (c) $\Sigma_{\mathcal{M}^{\mathcal{S}}_b(\pi^{\mathcal{S}}_b(\beta+1)), \vec{\mathcal{T}}^\frown \mathcal{S}^\frown \{\mathcal{M}^{\mathcal{S}}_b\}} = \Lambda_{\mathcal{M}^{\mathcal{S}}_c(\pi^{\mathcal{S}}_c(\beta+1)), \vec{\mathcal{U}}^\frown \mathcal{S}^\frown \{\mathcal{M}^{\mathcal{S}}_c\}}$.

The following is then the comparison theorem we will prove in this subsection.

Theorem 4.10.4 (Diamond comparison) *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving $(\omega_1, \omega_1, \omega_1)$-strategies, $\mathcal{P}$ and $\mathcal{Q}$ are countable and are of limit type, and lower level comparison holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. Then there are $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$ such that either*

1. *$\mathcal{P}$ and $\mathcal{Q}$ are of lsa type and $\Sigma^{stc}_{\mathcal{R}, \vec{\mathcal{T}}} = \Lambda^{stc}_{\mathcal{R}, \vec{\mathcal{U}}}$ or*

2. *$\mathcal{P}$ and $\mathcal{Q}$ are not of lsa type and $\Sigma_{\mathcal{R}, \vec{\mathcal{T}}} = \Lambda_{\mathcal{R}, \vec{\mathcal{U}}}$.*

We prove the theorem by showing that the failure of its conclusion produces a bad sequence of length ω_1. Towards showing this, we prove two useful lemmas.

We say that *weak comparison* holds between $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there is $(\vec{\mathcal{T}}, \vec{\mathcal{U}}, \mathcal{R}, \mathcal{S})$ such that

1. $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$,

2. $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{Q}, \Lambda)$,

3. $\mathcal{R}^b = \mathcal{S}^b$ and $\Sigma_{\mathcal{R}^b, \vec{\mathcal{T}}} = \Lambda_{\mathcal{S}^b, \vec{\mathcal{U}}}$.

Our first lemma says that lower level comparison implies that weak comparison holds.

Lemma 4.10.5 *Suppose $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P},\Sigma) = \Gamma(\mathcal{Q},\Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving, $\mathcal{P}$ and $\mathcal{Q}$ are of limit type, and that lower level comparison holds between $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$. Then weak comparison holds between $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$.*

Proof. We inductively construct $(\mathcal{P}_i, \vec{\mathcal{T}}_i : i < \omega)$ and $(\mathcal{Q}_i, \vec{\mathcal{U}}_i : i < \omega)$ such that the following conditions hold.

1. $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{Q}_0 = \mathcal{Q}$.
2. Suppose $i = 2n$. Then the following holds.
   (a) $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{P}_i^b$ according to $\Sigma_{\mathcal{P}_i^b, \oplus_{k<i}\vec{\mathcal{T}}_k}$ with last model $\mathcal{P}_{i+1}$ (when we apply $\vec{\mathcal{T}}_i$ to $\mathcal{P}_i$).
   (b) $\vec{\mathcal{U}}_i$ is a stack on $\mathcal{Q}_i$ according to $\Lambda_{\mathcal{Q}_i, \oplus_{k<i}\vec{\mathcal{U}}_i}$ with last model $\mathcal{Q}_{i+1}$.
   (c) Letting $\gamma = \sup \pi^{\vec{\mathcal{T}}_i}[\lambda^{\mathcal{P}_i^b}]$, $\mathcal{P}_{i+1}(\gamma) \trianglelefteq_{hod} \mathcal{Q}_{i+1}^b$ and $\Lambda_{\mathcal{P}_{i+1}^b(\gamma), \oplus_{k\leq i}\vec{\mathcal{U}}_k} = \Sigma_{\mathcal{P}_{i+1}^b(\gamma), \oplus_{k\leq i}\vec{\mathcal{T}}_k}$.
3. Suppose $i = 2n+1$. Then the following holds.
   (a) $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{P}_i$ according to $\Sigma_{\mathcal{P}_i, \oplus_{k<i}\vec{\mathcal{T}}_k}$ with last model $\mathcal{P}_{i+1}$.
   (b) $\vec{\mathcal{U}}_i$ is a stack on $\mathcal{Q}_i^b$ according to $\Lambda_{\mathcal{Q}_i^b, \oplus_{k<i}\vec{\mathcal{U}}_i}$ with last model $\mathcal{Q}_{i+1}$ (when we apply $\vec{\mathcal{U}}_i$ to $\mathcal{P}_i$).
   (c) Letting $\gamma = \sup \pi^{\vec{\mathcal{U}}_i}[\lambda^{\mathcal{Q}_i^b}]$ $\mathcal{Q}_{i+1}^b(\gamma) \trianglelefteq_{hod} \mathcal{P}_{i+1}^b$ and $\Lambda_{\mathcal{Q}_{i+1}^b(\gamma), \oplus_{k\leq i}\vec{\mathcal{U}}_k} = \Sigma_{\mathcal{Q}_{i+1}^b(\gamma), \oplus_{k\leq i}\vec{\mathcal{T}}_k}$.

We show how to carry out the inductive step. Suppose we have constructed $(\mathcal{P}_i, \mathcal{Q}_i : i \leq 2n)$ and $(\vec{\mathcal{T}}_i, \vec{\mathcal{U}}_i : i < 2n)$. We now consider two cases.

Case 1. $\mathrm{cf}^{\mathcal{P}_{2n}}(\delta^{\mathcal{P}_{2n}^b})$ is not a measurable cardinal in $\mathcal{P}_{2n}$.

Notice that in this case, we have that $\mathcal{P}_1 = \mathcal{Q}_1$ and $\Sigma_{\mathcal{P}_1, \vec{\mathcal{T}}_0} = \Lambda_{\mathcal{Q}_1, \vec{\mathcal{U}}_0}$. Thus, weak comparison holds for $(\mathcal{P},\Sigma)$ and $(\mathcal{Q},\Lambda)$.

Let $(\alpha_i : i < \omega)$ be such that $\sup(\alpha_k : k < \omega) = \delta^{\mathcal{P}_{2n}^b}$. By induction we construct a sequence $(\vec{\mathcal{T}}_k^*, \mathcal{W}_k, \vec{\mathcal{S}}_k, \mathcal{R}_k, \vec{\mathcal{S}}_k^*, \mathcal{R}_k^*, \beta_k : k < \omega)$ such that the following hold.

1. $(\vec{\mathcal{S}}_0^*, \mathcal{R}_0^*) \in I(\mathcal{Q}_{2n}, \Lambda_{\mathcal{Q}_{2n}, \oplus_{m<2n}\vec{\mathcal{U}}_m})$ and
$$\Gamma(\mathcal{P}_{2n}(\alpha_0), \Sigma_{\mathcal{P}_{2n}(\alpha_0), \oplus_{m<2n}\vec{\mathcal{T}}_m}) = \Gamma(\mathcal{R}_0^*(\beta_0), \Lambda_{\mathcal{R}_0^*(\beta_0), (\oplus_{m<2n}\vec{\mathcal{U}}_m)^\frown \vec{\mathcal{S}}_0^*}).$$

Moreover, $(\vec{\mathcal{T}}_0^*, \mathcal{W}_0) \in I(\mathcal{P}_{2n}, \Sigma_{\mathcal{P}_{2n},\oplus_{m<2n}\vec{\mathcal{T}}_m})$, $(\vec{\mathcal{S}}_0, \mathcal{R}_0) \in I(\mathcal{R}_0^*, \Lambda_{\mathcal{R}_0^*,(\oplus_{m<2n}\vec{\mathcal{U}}_m)^\frown\vec{\mathcal{S}}_0^*})$ and for some ξ, $(\vec{\mathcal{T}}_0^*, \mathcal{W}_0(\xi), \vec{\mathcal{S}}_0, \mathcal{R}_0(\xi))$ are comparison stacks[26] for

$$(\mathcal{P}_{2n}(\alpha_0), \Sigma_{\mathcal{P}_{2n}(\alpha_0),\oplus_{m<2n}\vec{\mathcal{T}}_m}) \text{ and } (\mathcal{R}_0^*(\beta_0), \Lambda_{\mathcal{R}_0^*(\beta_0),(\oplus_{m<2n}\vec{\mathcal{U}}_m)^\frown\vec{\mathcal{S}}_0^*}).$$

2. For $k+1<\omega$, $(\vec{\mathcal{S}}_{k+1}^*, \mathcal{R}_{k+1}^*) \in I(\mathcal{R}_k, \Lambda_{\mathcal{R}_k,(\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown(\oplus_{m\leq k}(\vec{\mathcal{S}}_m^{*\frown}\vec{\mathcal{S}}_m))})$ and

$$\Gamma(\mathcal{W}_k(\alpha_{k+1}^*), \Sigma_{\mathcal{W}_k(\alpha_{k+1}^*),(\oplus_{m<i}\vec{\mathcal{T}}_m)^\frown\oplus_{m\leq k}\vec{\mathcal{T}}_m^*}) = \\ \Gamma(\mathcal{R}_{k+1}^*(\beta_{k+1}), \Lambda_{\mathcal{R}_{k+1}^*(\beta_{k+1}),(\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown(\oplus_{m\leq k}(\vec{\mathcal{S}}_m^{*\frown}\vec{\mathcal{S}}_m))}).$$

where $\alpha_{k+1}^* = \pi^{\oplus_{i<k}\vec{\mathcal{T}}_k^*}(\alpha_{k+1})$. Moreover,

$$(\vec{\mathcal{T}}_{k+1}^*, \mathcal{W}_{k+1}) \in I(\mathcal{W}_k, \Sigma_{\mathcal{W}_k,(\oplus_{m<i}\vec{\mathcal{T}}_m)^\frown\oplus_{m\leq k}\vec{\mathcal{T}}_m^*}), \\ (\vec{\mathcal{S}}_{k+1}, \mathcal{R}_{k+1}) \in I(\mathcal{R}_{k+1}^*, \Lambda_{\mathcal{R}_{k+1}^*,(\oplus_{m<i}\vec{\mathcal{U}}_m)^\frown(\oplus_{m\leq k}(\vec{\mathcal{S}}_m^{*\frown}\vec{\mathcal{S}}_m))^\frown\vec{\mathcal{S}}_{k+1}^*})$$

and for some ξ, $(\vec{\mathcal{T}}_{k+1}^*, \mathcal{W}_{k+1}(\xi), \vec{\mathcal{S}}_{k+1}, \mathcal{R}_{k+1}(\xi))$ are comparison stacks for

$$(\mathcal{W}_k(\alpha_k^*), \Sigma_{\mathcal{W}_k(\alpha_{k+1}^*),(\oplus_{m<2n}\vec{\mathcal{T}}_m)^\frown\oplus_{m\leq k}\vec{\mathcal{T}}_m^*}) \text{ and} \\ (\mathcal{R}_{k+1}^*(\beta_{k+1}), \Lambda_{\mathcal{R}_{k+1}^*(\beta_{k+1}),(\oplus_{m<2n}\vec{\mathcal{U}}_m)^\frown(\oplus_{m\leq k}(\vec{\mathcal{S}}_m^{*\frown}\vec{\mathcal{S}}_m))^\frown\vec{\mathcal{S}}_{k+1}^*})$$

We then let $\vec{\mathcal{T}}_{2n} = \oplus_{k<\omega}\vec{\mathcal{T}}_k^*$ and $\vec{\mathcal{U}}_{2n} = \oplus_{m<\omega}\vec{\mathcal{S}}_k^{*\frown}\vec{\mathcal{S}}$. Also, we let $\mathcal{P}_{2n}$ be the last model of $\vec{\mathcal{T}}_{2n+1}$ and $\mathcal{Q}_{2n+1}$ be the last model of $\vec{\mathcal{U}}_{2n}$.

Case 2. $\mathrm{cf}^{\mathcal{P}_{2n}}(\delta^{\mathcal{P}_{2n}^b})$ is a measurable cardinal in $\mathcal{P}$.

The difference between this case and the previous case is that here we cannot start by fixing $(\alpha_i : i<\omega)$ as above. Here is the outline of the construction of $(\vec{\mathcal{T}}_{2n}, \vec{\mathcal{U}}_{2n}, \mathcal{P}_{2n+1}, \mathcal{Q}_{2n+1})$.

Because $\Gamma(\mathcal{P}_{2n}, \Sigma_{\mathcal{P}_{2n},\oplus_{i<2n}\vec{\mathcal{T}}_i}) = \Gamma(\mathcal{Q}_{2n}, \Lambda_{\mathcal{Q}_{2n},\oplus_{i<2n}\vec{\mathcal{U}}_i})$, we can find

$$(\vec{\mathcal{S}}_0, \mathcal{R}_0) \in I(\mathcal{Q}_{2n}, \Lambda_{\mathcal{Q}_{2n},\oplus_{i<2n}\vec{\mathcal{U}}_i})$$

and $\beta < \lambda^{\mathcal{R}_0}$ such that letting $E \in \vec{E}^{\mathcal{P}_{2n}}$ be the extender of Mitchel order 0 with $\mathrm{crit}(E) = \mathrm{cf}^{\mathcal{P}_{2n}}(\delta^{\mathcal{P}_{2n}})$,

[26] In particular, $\mathcal{W}_0(\xi) = \mathcal{R}_0(\xi)$.

$\Gamma(Ult(\mathcal{P}_{2n}, E)(\lambda^{\mathcal{P}_{2n}}), \Sigma_{\mathcal{P}_{2n},(\oplus_{i<2n}\vec{\mathcal{T}}_i)^\frown\{Ult(\mathcal{P}_{2n},E)\}}) = \Gamma(\mathcal{R}_0(\beta), \Lambda_{\mathcal{Q}_{2n},(\oplus_{i<2n}\vec{\mathcal{U}}_i)^\frown\{\vec{\mathcal{S}}_0\}})$
Appealing to low level comparison, we can find

$$(\vec{\mathcal{T}}^*_{2n}, \mathcal{P}_{2n+1}) \in I(Ult(\mathcal{P}_{2n}, E), \Sigma_{\mathcal{P}_{2n},(\oplus_{i<2n}\vec{\mathcal{T}}_i)^\frown\{Ult(\mathcal{P}_{2n},E)\}}) \text{ and}$$
$$(\vec{\mathcal{S}}_1, \mathcal{R}_1) \in I(\mathcal{R}_0, \Lambda_{\mathcal{R}_0,(\oplus_{i<2n}\vec{\mathcal{U}}_i)^\frown\vec{\mathcal{S}}_0})$$

such that

1. $\vec{\mathcal{T}}^*_{2n}$ is based on $Ult(\mathcal{P}_{2n}, E)(\lambda^{\mathcal{P}_{2n}})$,
2. $\vec{\mathcal{S}}_1$ is based on $\mathcal{R}_0(\beta)$,
3. $\pi^{\vec{\mathcal{T}}^*_{2n}}(\lambda^{\mathcal{P}_{2n}}) = \pi^{\vec{\mathcal{S}}_1}(\beta) =_{def} \xi$, and
4. $\Sigma_{\mathcal{P}_{2n+1}(\xi),(\oplus_{i<2n}\vec{\mathcal{T}}_i)^\frown\{Ult(\mathcal{P}_{2n},E)\}^\frown\vec{\mathcal{T}}_{2n+1}} = \Lambda_{\mathcal{R}_1(\xi),(\oplus_{i<2n}\vec{\mathcal{U}}_i)^\frown\vec{\mathcal{S}}_0^\frown\vec{\mathcal{S}}_1}$

Let then $\vec{\mathcal{T}}_{2n} = \{Ult(\mathcal{P}_{2n}, E)\}^\frown\vec{\mathcal{T}}^*_{2n}$, $\vec{\mathcal{U}}_{2n} = \vec{\mathcal{S}}_0^\frown\vec{\mathcal{S}}_1$ and $\mathcal{Q}_{2n+1} = \mathcal{R}_1$.

The two cases above finish the construction of $(\vec{\mathcal{T}}_{2n}, \vec{\mathcal{U}}_{2n}, \mathcal{P}_{2n+1}, \mathcal{Q}_{2n+1})$. The construction of $(\vec{\mathcal{T}}_{2n+1}, \vec{\mathcal{U}}_{2n+1}, \mathcal{P}_{2n+2}, \mathcal{Q}_{2n+2})$ is very similar and we leave it to the reader.

Notice now that if $\vec{\mathcal{T}} = \oplus_{i<\omega}\vec{\mathcal{T}}_i$, $\vec{\mathcal{U}} = \oplus_{i<\omega}\vec{\mathcal{U}}_i$, $\mathcal{R}$ is the last model of $\vec{\mathcal{T}}$ and $\mathcal{S}$ is the last model of $\vec{\mathcal{U}}$ then $(\vec{\mathcal{T}}, \mathcal{R})$ and $(\vec{\mathcal{U}}, \mathcal{S})$ witness that weak comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. □

Lemma 4.10.6 *Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) =_{def} \Gamma$, both Σ and Λ are Γ-fullness preserving, both $\mathcal{P}$ and $\mathcal{Q}$ are of limit type and low level comparison holds. Suppose further that $\mathcal{P}^b = \mathcal{Q}^b$ and for all $\beta < \lambda^{\mathcal{P}} - 1$, $\Sigma_{\mathcal{P}(\beta+1)} = \Lambda_{\mathcal{Q}(\beta+1)}$. Let $(\mathcal{T}, \mathcal{R}, \mathcal{U}, \mathcal{S})$ be the trees of the extender comparison of $\mathcal{P}$ and $\mathcal{Q}$*[27]. *Suppose that either*

1. *$\mathcal{R} \neq \mathcal{S}$ or*
2. *$\mathcal{R} = \mathcal{S}$ and $\Sigma_{\mathcal{R},\mathcal{T}} \neq \Lambda_{\mathcal{S},\mathcal{U}}$.*

Then there is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$.

Proof. It follows from Lemma 4.6.3 that we can find minimal low level disagreement $(\vec{\mathcal{T}}^*, \vec{\mathcal{U}}^*, \mathcal{W})$ between $(\mathcal{R}, \Sigma_{\mathcal{R},\mathcal{T}})$ and $(\mathcal{S}, \Lambda_{\mathcal{S},\mathcal{U}})$. We then let $\mathcal{P}_1$ and $\mathcal{Q}_1$ be the last models of $\vec{\mathcal{T}}^*$ and $\vec{\mathcal{U}}^*$ when we regard them as stacks on $\mathcal{R}$ and $\mathcal{S}$ respectively.

Let $\mathcal{T}_1$ be a normal tree as in clause 2 of Definition 4.10.3 . Let $b = \Sigma(\mathcal{T}^\frown\vec{\mathcal{T}}^{*\frown}\mathcal{T}_1)$, $c = \Lambda(\mathcal{U}^\frown\vec{\mathcal{U}}^{*\frown}\mathcal{T}_1)$, $\mathcal{P}_2 = \mathcal{M}_b^{\mathcal{T}_1}$ and $\mathcal{Q}_2 = \mathcal{M}_c^{\mathcal{T}_1}$ (here we apply the stacks to $\mathcal{P}_1$ and $\mathcal{Q}_1$ respectively). We thus have that $\pi_b^{\mathcal{T}_1}$ and $\pi_c^{\mathcal{T}_1}$ exist, $\pi_b^{\mathcal{T}_1}(\mathcal{W}) = \pi_c^{\mathcal{T}_1}(\mathcal{W})$ and

[27] Thus, $\mathcal{T}$ is on $\mathcal{P}$ with last model $\mathcal{R}$ and $\mathcal{U}$ is on $\mathcal{Q}$ with last model $\mathcal{S}$.

$$\Sigma_{\pi_b^{\mathcal{T}_1}(\mathcal{W}),\mathcal{T}^\frown\vec{\mathcal{T}}^{*\frown}\mathcal{T}_1^\frown\{\mathcal{P}_2\}} = \Lambda_{\pi_c^{\mathcal{T}_1}(\mathcal{W}),\mathcal{U}^\frown\vec{\mathcal{U}}^{*\frown}\mathcal{T}_1^\frown\{\mathcal{Q}_2\}}$$

Next (appealing to Lemma 4.10.5) let $(\vec{\mathcal{T}}_2, \mathcal{P}_3)$ and $(\vec{\mathcal{U}}_2, \mathcal{Q}_3)$ witness that the weak comparison holds for

$$(\mathcal{P}_2, \Sigma_{\mathcal{P}_2,\mathcal{T}^\frown\vec{\mathcal{T}}^{*\frown}\mathcal{T}_1^\frown\{\mathcal{P}_2\}}), \text{ and}(\mathcal{Q}_2, \Lambda_{\mathcal{Q}_2,\mathcal{U}^\frown\vec{\mathcal{U}}_1^{*\frown}\mathcal{T}_1^\frown\{\mathcal{Q}_2\}}).$$

Finally, let $(\mathcal{T}_3, \mathcal{P}_4)$ and $(\mathcal{U}_3, \mathcal{Q}_4)$ be the result of extender comparison between $\mathcal{P}_3$ and $\mathcal{Q}_3$.

Next let $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{Q}_0 = \mathcal{Q}$, $\Sigma_0 = \Sigma$, $\Lambda_0 = \Lambda$, $\vec{\mathcal{T}}_0 = \mathcal{T}^\frown\vec{\mathcal{T}}^*$, and $\vec{\mathcal{U}}_0 = \mathcal{U}^\frown\vec{\mathcal{U}}^*$. Also, for $i \in \{1, 2, 3, 4\}$ let $\Sigma_i = \Sigma_{\mathcal{P}_i,\oplus_{k<i}\vec{\mathcal{T}}_k}$ and $\Lambda_i = \Lambda_{\mathcal{Q}_i,\oplus_{k<i}\vec{\mathcal{U}}_k}$, . It is then easy to see that

$$(((\mathcal{P}_i, \mathcal{Q}_i, \Sigma_i, \Lambda_i) : i < 5), (\vec{\mathcal{T}}_i, \vec{\mathcal{U}}_i : i < 4), (b, c))$$

is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$. □

We now start proving Theorem 4.10.4. Suppose that the conclusion of Theorem 4.10.4 fails. This means that

(1) whenever $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Sigma)$ and $(\vec{\mathcal{U}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$,

1. if $\mathcal{P}$ and $\mathcal{Q}$ are of lsa type then $\Sigma^{stc}_{\mathcal{R},\vec{\mathcal{T}}} \neq \Lambda^{stc}_{\mathcal{R},\vec{\mathcal{U}}}$ or
2. if $\mathcal{P}$ and $\mathcal{Q}$ are not of lsa type then $\Sigma_{\mathcal{R},\vec{\mathcal{T}}} \neq \Lambda_{\mathcal{R},\vec{\mathcal{U}}}$.

It follows from Lemma 4.10.5 that, without loss of generality, we can assume that $\mathcal{P}^b = \mathcal{Q}^b$ and for all $\beta+1 < \lambda^{\mathcal{P}^b}$, $\Sigma_{\mathcal{P}(\beta+1)} = \Lambda_{\mathcal{Q}(\beta+1)}$. We now by induction construct a bad sequence $(B_\alpha : \alpha < \omega_1)$ on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$.

It follows from Lemma 4.10.6 that there is a bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$. Let B_0 be any bad block on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$. Suppose next that we have constructed $(B_\beta : \beta < \lambda)$ for λ a limit. Let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\lambda$ be the direct limit of respectively $(\mathcal{P}_\beta : \beta < \lambda)$ and $(\mathcal{Q}_\beta : \beta < \lambda)$ under the corresponding iteration embeddings. Then letting $\Sigma_{\lambda,0}$ and $\Lambda_{\lambda,0}$ be the corresponding tails of Σ and Λ, we have that $(\mathcal{P}_\lambda, \Sigma_\lambda)$ and $(\mathcal{Q}_\lambda, \Lambda_\lambda)$ satisfy the hypothesis of Lemma 4.10.6. Let then B_λ be a bad block on $((\mathcal{P}_\lambda, \Sigma_\lambda), (\mathcal{Q}_\lambda, \Lambda_\lambda))$.

Next suppose that we have constructed $(B_\beta : \beta < \lambda + 1)$. Let $\mathcal{P}_{\lambda+1} = \mathcal{P}_{\lambda,4}$, $\mathcal{Q}_{\lambda+1} = \mathcal{Q}_{\lambda,4}$ and let $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ be the stacks respectively on the top of $(B_\beta : \beta < \lambda+1)$ and in the bottom of $(B_\beta : \beta < \lambda+1)$. We then again can find, using Lemma 4.10.6, a bad block $B_{\lambda+1}$ on $((\mathcal{P}_{\lambda+1}, \Sigma_{\mathcal{P}_{\lambda+1},\vec{\mathcal{T}}}), (\mathcal{Q}_{\lambda+1}, \Lambda_{\mathcal{Q}_{\lambda+1},\vec{\mathcal{U}}}))$. It then follows that the resulting sequence $(B_\beta : \beta < \omega_1)$ is a bad sequence on $((\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda))$. This is a contradiction to Lemma 4.10.2.

Chapter 5

Hod mice revisited

In this section we generalize the result of [10, Chapter 3] to our current context. As in [10], these results lead towards showing that given a hod pair $(\mathcal{P}, \Sigma)$, $\Gamma(\mathcal{P}, \Sigma)$ is an OD-full pointclass (see Definition 3.16 of [10]).

Recall the effect of Proposition 4.8.1; if $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and if $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, then the strategy of $\mathcal{Q}$ induced by Σ is independent of the particular iteration producing $\mathcal{Q}$. In Section 4.8, this strategy was denoted by $\Sigma_{\mathcal{Q}}$. In this chapter, whenever the strategy of a hod mouse has a strong branch condensation, we will make use of the aforementioned notation without giving any further explanation.

5.1 The uniqueness of the internal strategy

The first theorem, Theorem 5.1.2, is just a direct generalization of [10, Theorem 3.3]. It says that the internal strategies are unique. First we prove a useful lemma.

Lemma 5.1.1 *Suppose $\mathcal{P}$ is a hod premouse and $\mathcal{Q} \in Y^{\mathcal{P}}$. Suppose $\vec{\mathcal{U}} \in \mathcal{P}$ is a stack on $\mathcal{Q}$ and suppose $\mathcal{R}$ is its last model. Then for all $\nu \leq \lambda^{\mathcal{R}}$ such that $\mathcal{R} \vDash$ "$\delta_\nu^{\mathcal{R}}$ is a Woodin cardinal", $\mathrm{cf}^{\mathcal{P}}(\delta_\nu^{\mathcal{R}}) > \omega$.*

Proof. Towards a contradiction, assume not. Notice that it cannot be the case that $\delta_\nu^{\mathcal{R}}$ has a pre-image in $\mathcal{P}$. Therefore, by minimizing $\mathcal{Q}$, we can assume that $\mathcal{Q}$ is of limit type. We give the proof assuming that $\mathcal{Q}$ is of limit type. Let $(\mathcal{N}_\alpha, \vec{\mathcal{U}}_\alpha, \mathcal{Q}_\alpha, E_\alpha : \alpha \leq \eta)$ be the components of $\vec{\mathcal{U}}$. Without loss of generality we can assume that for every cutpoint $\mathcal{S}$ of $\vec{\mathcal{U}}$, $\vec{\mathcal{U}}_{\leq \mathcal{S}}$ is not a counterexample to our claim.

Let $\mathcal{S}$ be the least model in $\vec{\mathcal{U}}$ such that $\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}}$ exists and $\delta^{\mathcal{R}}_{\nu} \in rng(\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}})$. It follows that there is $\mathcal{M}$ in $\vec{\mathcal{U}}$ such that for some extender F in $\vec{\mathcal{U}}$, F is applied to $\mathcal{M}$ and $\mathcal{S} = Ult(\mathcal{M}, F)$. Let μ be such that $\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}}(\mu) = \nu$. Since $\pi^{\vec{\mathcal{U}}}_{\mathcal{S},\mathcal{R}}$ is cofinal at $\delta^{\mathcal{S}}_{\mu}$, we have that $\mathrm{cf}^{\mathcal{P}}(\delta^{\mathcal{S}}_{\mu}) = \omega$.

Let $E = E^{\vec{\mathcal{U}}_{\leq\mathcal{M}}}_{\mathcal{M}(\mu)}$ (see Definition 2.7.2) and let $\mathcal{M}^+ = Ult(\mathcal{Q}, E)$. We have that

(1) $Ult(\mathcal{M}^+, F) \vDash$ "$\delta^{\mathcal{S}}_{\mu}$ is a Woodin cardinal and hence is a regular cardinal", and
(2) there is a sequence $(h_i : i < \omega) \in \mathcal{Q}$ such that for some $(a_i : i < \omega) \in (\nu_F^{<\omega})^{\omega}$,

$$\sup_{i<\omega} \pi^{\mathcal{M}^+}_F(\pi_E(h_i))(a_i) = \delta^{\mathcal{S}}_{\mu}.$$

Notice that

(3) $G =_{def} (\pi^{\mathcal{M}^+}_F(\pi_E(h_i)) : i < \omega) \in Ult(\mathcal{M}^+, F)$.

Hence,

(4) $Ult(\mathcal{M}^+, F) \vDash \delta^{\mathcal{S}}_{\mu} = \sup_{a\in\nu_F^{<\omega}, i<\omega} G(i)(a)$.

(4) implies that $Ult(\mathcal{M}^+, F) \vDash \mathrm{cf}(\delta^{\mathcal{S}}_{\mu}) \leq \nu_F$. Clearly this contradicts (1) and the fact that $\delta^{\mathcal{S}}_{\mu} > \nu_F$. □

Theorem 5.1.2 (Uniqueness of internal strategies) *Suppose $\mathcal{P}$ is a hod premouse and $\mathcal{W} \in Y^{\mathcal{P}}$. Then $\mathcal{P} \vDash$ "$\mathcal{W}$ has a unique iteration strategy".*

Proof. Working in $\mathcal{P}$, suppose $\Lambda \neq \Sigma^{\mathcal{P}}_{\mathcal{W}}$ is another iteration strategy for $\mathcal{W}$. Let $\Sigma = \Sigma^{\mathcal{P}}_{\mathcal{W}}$. Since $\mathcal{W}$ is not of lsa type, it follows from Lemma 4.6.3 that we can find $(\vec{\mathcal{T}}, \mathcal{Q})$ that constitutes a minimal low-level disagreement between $(\mathcal{W}, \Sigma)$ and $(\mathcal{W}, \Lambda)$. Let $b = \Sigma(\vec{\mathcal{T}})$ and $c = \Lambda(\vec{\mathcal{T}})$. Let $\vec{\mathcal{S}} \in \mathcal{P}$ be a stack on $\mathcal{Q}$ according to both $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ and $\Lambda_{\mathcal{Q},\vec{\mathcal{T}}}$ and such that $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}(\vec{\mathcal{S}}) \neq \Lambda_{\mathcal{Q},\vec{\mathcal{T}}}(\vec{\mathcal{S}})$. Let $\mathcal{R}$ be a strong cutpoint of $\vec{\mathcal{S}}$ such that $\vec{\mathcal{S}}_{\geq\mathcal{R}}$ is a normal tree on $\mathcal{R}$ that is above $\delta^{\mathcal{R}}_{\lambda^{\mathcal{R}}-1}$. We now have that

(1) $\Sigma_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}} = \Lambda_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}}$ and
(2) $\Sigma_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}}(\vec{\mathcal{S}}_{\geq\mathcal{R}}) \neq \Lambda_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}}(\vec{\mathcal{S}}_{\geq\mathcal{R}})$.

It then follows that

(3) $\mathrm{cf}^{\mathcal{P}}(\delta(\vec{\mathcal{S}}_{\geq\mathcal{R}})) = \omega$.

Let now $e = \Sigma_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}}(\vec{\mathcal{S}}_{\geq\mathcal{R}})$ and $d = \Lambda_{\mathcal{R}(\lambda^{\mathcal{R}}-1),\vec{\mathcal{T}}^\frown\vec{\mathcal{S}}_{\leq\mathcal{R}}}(\vec{\mathcal{S}}_{\geq\mathcal{R}})$.

Notice that it is a consequence of (1) that it cannot be the case that both $\mathcal{Q}(e, \vec{\mathcal{S}}_{\geq\mathcal{R}})$ and $\mathcal{Q}(d, \vec{\mathcal{S}}_{\geq\mathcal{R}})$ exist as otherwise, since they are hybrid mice with respect to the same strategy, we would get that $\mathcal{Q}(e, \vec{\mathcal{S}}_{\geq\mathcal{R}}) = \mathcal{Q}(d, \vec{\mathcal{S}}_{\geq\mathcal{R}})$ which implies that $d = e$.

Without loss of generality, we assume that $\mathcal{Q}(e, \vec{\mathcal{S}}_{\geq\mathcal{R}})$ does not exist and $\pi_e^{\vec{\mathcal{S}}_{\geq\mathcal{R}}}(\delta^{\mathcal{R}}) = \delta(\vec{\mathcal{S}}_{\geq\mathcal{R}})$. It then follows from (3) that $\mathrm{cf}^{\mathcal{P}}(\delta^{\mathcal{R}}) = \omega$, contradicting Lemma 5.1.1. $\square$

The proof of Theorem 5.1.2 can be used in the context of lsa hod premice as well. We will state this result after proving the fullness preservation of the internal strategies. Essentially the internal short tree strategy is the unique short tree strategy which is internally fullness preserving. For now, we state the following corollary of the proof of Theorem 5.1.2.

Corollary 5.1.3 *Suppose $\mathcal{P}$ is an lsa type hod premouse and Λ is its internal short tree strategy. Suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Lambda)$ and $\beta + 1 < \lambda^{\mathcal{Q}}$. Then $\mathcal{P} \vDash$ "$\Lambda_{\mathcal{Q}(\beta+1),\vec{\mathcal{T}}}$ is the unique strategy of $\mathcal{Q}(\beta + 1)$".*

5.2 Generic interpretability

We now move to generic interpretability. We start by recalling and generalizing the definition of a pre-hod pair (see [10, Definition 3.7]).

Definition 5.2.1 (Prehod pair) *$(\mathcal{P}, \Sigma)$ is a prehod pair if*

1. *$\mathcal{P}$ is a countable hod premouse,*

2. *$\lambda^{\mathcal{P}}$ is a successor but $\mathcal{P}$ is not of lsa type,*

3. *Σ is an $(\omega_1, \omega_1, \omega_1)$-strategy for $\mathcal{P}$ acting on stacks based on $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$ such that $(\mathcal{P}(\lambda^{\mathcal{P}} - 1), \Sigma)$ is a hod pair and that whenever $i : \mathcal{P} \to \mathcal{Q}$ comes from an iteration according to Σ, $\Sigma^{\mathcal{Q}}_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)} = \Sigma \restriction \mathcal{Q}$,*

4. *for any $\mathcal{P}$-cardinal $\eta \in (\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \delta^{\mathcal{P}}_{\lambda})$, considering $\mathcal{P}|\eta$ as a Σ-mouse over $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$, there is an ω_1-strategy Λ for $\mathcal{P}|\eta$.*

Notice that there must be a unique strategy Λ as in 5 of Definition 5.2.1.[1] Also, recall the definition of Generic Interpretability, [10, Definition 3.8]. In our current context it takes the following form.

Definition 5.2.2 (Generic Interpretability) *Suppose $(\mathcal{P}, \Sigma)$ is a pre-hod pair, a hod pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal or an sts hod pair. We say generic interpretability holds for $(\mathcal{P}, \Sigma)$ if there is a function F such that*

1. *F is definable over $\mathcal{P}$ with no parameters,*
2. *$dom(F)$ consists of pairs $(\mathcal{Q}, \kappa)$ such that $\mathcal{Q} \in Y^{\mathcal{P}}$, $\mathcal{Q} \trianglelefteq \mathcal{P}|\delta^{\mathcal{P}}$ and $\kappa \in (\delta^{\mathcal{Q}}, \delta^{\mathcal{P}})$ is a $\mathcal{P}$-cardinal,*
3. *for $(\mathcal{Q}, \kappa) \in dom(F)$, $F(\mathcal{Q}, \kappa) = (\dot{T}, \dot{S})$ such that ,*
   (a) *$\dot{T}, \dot{S} \in \mathcal{P}^{Coll(\omega, o(\mathcal{Q}))}$,*
   (b) *$\mathcal{P} \models$ "$\Vdash_{Coll(\omega, o(\mathcal{Q}))} \dot{T}$ and $\dot{S}$ are κ-complementing",*
   (c) *for any $\nu \in (o(\mathcal{Q}), \kappa)$ and any $\mathcal{P}$-generic $g \subseteq Coll(\omega, o(\mathcal{Q}))$,*

   $\mathcal{P}[g] \models$ "$p[\dot{T}_g]$ is an $(\omega_1, \omega_1, \omega_1)$-iteration strategy for $\mathcal{Q}$ which extends $\Sigma^{\mathcal{P}}_{\mathcal{Q}}$"

   and

$$(p[\dot{T}_g])^{\mathcal{P}[g]} = \Sigma_{\mathcal{Q}} \restriction HC^{\mathcal{P}[g]}.$$

The proof that the generic interpretability holds is just like the proof of [10, Theorem 3.10] using Theorem 4.6.8 and Theorem 5.1.2 instead of [10, Lemma 2.15] and [10, Theorem 3.3]. First the proof of [10, Lemma 3.9] can be used with no changes to establish the following useful lemma.

Lemma 5.2.3 *Suppose $(\mathcal{P}, \Sigma)$ is a prehod pair and $\alpha + 1 = \lambda^{\mathcal{P}}$. Let $\kappa < \delta^{\mathcal{P}}$ be a $\mathcal{P}$-cardinal such that $\mathcal{P}$ has no extenders on its sequence with critical point $\delta^{\mathcal{P}}_{\alpha}$ and index greater than κ. Let Λ^* be the iteration strategy of $\mathcal{P}|\kappa$ as in 5 of Definition 5.2.1. Let Λ be the fragment of Λ^* that acts on non-dropping stacks. Let $g \subseteq Coll(\omega, \kappa)$ be $\mathcal{P}$-generic. Then $\mathcal{P}[g]$ locally Suslin, co-Suslin captures $Code(\Lambda^*)$ and its complement at any cardinal of $\mathcal{P}$ greater than κ.*[2]

[1] Λ is the $\mathcal{Q}$-structure guided strategy.

[2] Recall that this means that for every $\mathcal{P}$-cardinal $\nu > \kappa$, there are ν-complementing trees $U, V \in \mathcal{P}[g]$ such that for any $< \nu$-generic h, $Code(\Lambda^*) \cap P[g][h] = (p[U])^{\mathcal{P}[g][h]}$.

Fix now a prehod pair $(\mathcal{P}, \Sigma)$ and let $\mathcal{Q} \in Y^{\mathcal{P}}$. Let $\kappa < \delta^{\mathcal{P}}$ be a $\mathcal{P}$-cardinal such that $\kappa > o(\mathcal{Q})$ and $\mathcal{P}$ has no extenders with critical point $\delta^{\mathcal{Q}}$ and index greater than κ. Let $(\mathcal{R}_\beta, \Phi_\beta : \delta^{\mathcal{P}})$ be the models and strategies of the hod pair construction of $\mathcal{P}|\delta^{\mathcal{P}}$ in which extenders used have critical point $> \kappa$ (notice that we can view $(\mathcal{P}, \delta^{\mathcal{P}}, \Sigma)$ as a self-capturing background triple). Here we abuse the notation and write Φ_β both for the strategy of $\mathcal{R}_\beta$ that is internal to $\mathcal{P}$ and also for the external strategy. It follows from Theorem 4.6.8 , Lemma 5.1.2 and Lemma 5.2.3 that for some β, $(\mathcal{R}_\beta, \Phi_\beta)$ is a tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We then set

$$\mathcal{N}^{\mathcal{P}}_{\kappa,\mathcal{Q}} = \mathcal{R}_\beta \text{ and } \Lambda_{\kappa,\mathcal{Q}} = \Phi_\beta.$$

In what follows, we will omit superscript $\mathcal{P}$, but ask the reader to keep in mind that certain notions depend on $\mathcal{P}$. Also let $\pi_{\kappa,\mathcal{Q}} : \mathcal{Q} \to \mathcal{N}_{\kappa,\mathcal{Q}}$ be the iteration embedding according to $\Sigma_{\mathcal{Q}}$ and let $\mathcal{T}_{\kappa,\mathcal{Q}}$ be the tree on $\mathcal{Q}$ with last model $\mathcal{R}_\beta$. The following is a consequence of Lemma 5.2.3, hull condensation of Σ and the proof of Theorem 4.6.8.

Corollary 5.2.4 *Whenever $\eta \in (\kappa, \delta^{\mathcal{P}})$ is such that $\eta > o(\mathcal{N}_{\kappa,\mathcal{Q}})$ and $n < \omega$, there are names $(\dot{T}, \dot{S}) \in \mathcal{P}^{Coll(\omega,\eta)}$, such that*

1. *$\dot{T}, \dot{S} \in \mathcal{P}^{Coll(\omega,\eta)}$,*

2. *$\mathcal{P} \vDash$ "$\Vdash_{Coll(\omega,\mu^{\mathcal{P}}_{\beta,\xi,\gamma})} \dot{T}$ and $\dot{S}$ are $(\delta^{\mathcal{P}})^{+n}$-complementing",*

3. *for any $\lambda < (\eta, ((\delta^{\mathcal{P}})^{+n})^{\mathcal{P}})$ and any $\mathcal{P}$-generic $g \subseteq Coll(\omega, \lambda)$,*

$$\mathcal{P}[g] \vDash \text{“}p[\dot{T}_g] \textit{ is an } (\omega_1, \omega_1, \omega_1)\textit{-iteration strategy for } \mathcal{N}_{\kappa,\mathcal{Q}}\text{”}$$

and letting Φ be the $\pi^{\mathcal{P}}_{\kappa,\mathcal{Q}}$-pullback of the strategy given by $(p[\dot{T}_g])^{\mathcal{P}[g]}$ then

$$\Phi = \Sigma_{\mathcal{Q}} \restriction HC^{\mathcal{P}[g]}.$$

Our generic interpretability result can now be proved using the tree production lemma ([6, Theorem 3.3.15]) and Corollary 5.2.4. We leave the details to the reader.

Theorem 5.2.5 (The generic interpretability) *Suppose $(\mathcal{P}, \Sigma)$ is a prehod pair, is a hod pair such that $\lambda^{\mathcal{P}}$ is limit or is an sts hod pair. Assume that for every $\mathcal{Q} \in Y^{\mathcal{P}}$, $\Sigma_{\mathcal{Q}}$ has branch condensation. Then generic interpretability holds for $(\mathcal{P}, \Sigma)$.*

Next, we present our result on internal fullness preservation. The proof follows the same line of thought as the proof of [10, Theorem 3.12]. Below $\mathcal{S}^*(\mathcal{R})$ is the $*$-transform of $\mathcal{S}$ into a hybrid mouse over $\mathcal{R}$, it is defined when $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ (cf. [18]).

Definition 5.2.6 *Suppose $\mathcal{P}$ is a hod premouse and $\mathcal{Q} \in Y^{\mathcal{P}}$. We say $\Lambda = \Sigma^{\mathcal{P}}_{\mathcal{Q}}$ is internally fullness preserving if the following holds for $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$ such that $(|\vec{\mathcal{T}}|^+)^{\mathcal{P}}$ exists.*

1. *For all limit type $\mathcal{S} \in Y^{\mathcal{R}}$, if $\mathcal{M} \in \mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}}+1, (|\vec{\mathcal{T}}|^+)^{\mathcal{P}})$-iterable $\oplus_{\mathcal{K}\in Y^{\mathcal{S}^b}}\Lambda_{\mathcal{K},\vec{\mathcal{T}}}$-mouse over $\mathcal{S}|\delta^{\mathcal{S}^b}$ then $\mathcal{M} \trianglelefteq \mathcal{S}$.*

2. *Suppose $\mathcal{W} \triangleleft_{hod} \mathcal{S}$ is of lsa type and is such that $\mathcal{W} = \mathcal{M}^+(\mathcal{W}|\delta^{\mathcal{W}})$. Suppose $\mathcal{M} \in \mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}}+1, (|\vec{\mathcal{T}}|^+)^{\mathcal{P}})$-iterable $\Lambda_{\mathcal{W},\vec{\mathcal{T}}}$-sts mouse over $\mathcal{W}$. Then $\mathcal{M} \trianglelefteq \mathcal{S}$.*

3. *Suppose η is a cardinal cutpoint of $\mathcal{R}$ and suppose there are $\mathcal{R}_1, \mathcal{R}_2 \in Y^{\mathcal{R}}$ such that $\mathcal{R}_2$ is the $\mathcal{R}$-successor of $\mathcal{R}_1$ (see Definition 3.9.2), $\mathcal{R}_1$ is a cutpoint of $\mathcal{R}$ and $\eta \in (\delta^{\mathcal{R}_1}, o(\mathcal{R}_2))$. Suppose $\mathcal{M} \in \mathcal{P}$ is a sound $\max(\delta^{\mathcal{P}}+1, (|\vec{\mathcal{T}}|^+)^{\mathcal{P}})$-iterable $\Lambda_{\mathcal{R}_1,\vec{\mathcal{T}}}$-mouse over $\mathcal{R}_2|\eta$. Then $\mathcal{M} \trianglelefteq \mathcal{R}^*_2(\mathcal{R}_2|\eta)$.*

Theorem 5.2.7 (Internal fullness preservation) *Suppose $\mathcal{P}$ is a hod premouse and $\mathcal{Q} \in Y^{\mathcal{P}}$. Then $\Sigma^{\mathcal{P}}_{\mathcal{Q}}$ is internally fullness preserving.*

5.3 The derived models of hod mice

In this section, we state, without a proof, a version of [10, Theorem 3.19]. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has strong branch condensation and is fullness preserving. Suppose $\alpha \leq \lambda^{\mathcal{P}}$ is a limit ordinal such that $\mathrm{cf}^{\mathcal{P}}(\alpha)$ isn't a measurable cardinal in $\mathcal{P}$. We then let $D^*(\mathcal{P}, \Sigma, \alpha)$ be the set of all $A \subseteq \mathbb{R}$ such that for some $\beta < \alpha$ and $g \subseteq Coll(\omega, \delta^{\mathcal{P}}_{\beta})$ generic over $\mathcal{P}(\alpha)$ there are $\delta^{\mathcal{P}}_{\alpha}$-complementing trees $T, U \in \mathcal{P}(\alpha)[g]$ such that

$$x \in A \text{ if and only if there is } (\vec{\mathcal{S}}, \mathcal{R}) \in I(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)}) \text{ such that } \pi^{\vec{\mathcal{S}}} \text{ is above } \delta^{\mathcal{P}}_{\beta} \text{ and for some } \gamma < \lambda^{\mathcal{R}}, \delta^{\mathcal{R}}_{\gamma+1} \text{ is a Woodin cardinal in } \mathcal{R}, x \text{ is generic for the extender algebra of } \mathcal{R}[g] \text{ at } \delta^{\mathcal{R}}_{\gamma+1} \text{ and } \mathcal{R}[g, x] \vDash x \in p[\pi^{\vec{\mathcal{S}}}(T)].$$

Equivalently, A is Suslin, co-Suslin captured by $(\mathcal{P}(\alpha)[g], \Sigma_{\mathcal{P}(\alpha)})$. It follows from Corollary 4.6.10 and Theorem 4.8.1 that for $x \in \mathbb{R}$, the right hand side of the above equivalence is independent of the choice of $(\vec{\mathcal{S}}, \mathcal{R})$.

We let $D(\mathcal{P}, \Sigma, \alpha)$ be the derived model of $\mathcal{P}(\alpha)$ as computed by Σ, i.e., for $A \subseteq \mathbb{R}$, $A \in D(\mathcal{P}, \Sigma, \alpha)$ if there is $(\vec{\mathcal{S}}, \mathcal{Q}) \in I(\mathcal{P}(\alpha), \Sigma)$ such that $A \in D^*(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \pi^{\vec{\mathcal{S}}}(\alpha))$.

Next recall [10, Definition 3.18]. Essentially a pointclass is completely mouse-full if the next model of determinacy has the same mice relative to common iteration strategies. We introduce this notion more carefully.

Given a set of reals $A \subseteq \mathbb{R}$, we let $W_A = \{B \subseteq \mathbb{R} : B \leq_w A\}$. Next following Definition 3.13 of [10], we say $A \subseteq \mathbb{R}$ is a new set if

1. $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$,
2. $\wp(\mathbb{R}) \cap L(W_A, \mathbb{R}) = W_A$,
3. $\Theta^{L(W_A, \mathbb{R})}$ is a Suslin cardinal of $L(A, \mathbb{R})$.

The following is [10, Definition 3.17].

Definition 5.3.1 *Given a pointclass* Γ*, we say* Γ *is completely mouse full if either* $\Gamma = \wp(\mathbb{R})$ *or there is a new set* A *such that*

1. $\Gamma = W_A$,
2. *if* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $Code(\Sigma) \in \Gamma$ *and* $L(A, \mathbb{R}) \vDash$ *"*Σ *has strong branch condensation and is fullness preserving" then for every* $a \in HC$*,*

$$Lp^{\Gamma, \Sigma}(a) = (Lp^{\Sigma}(a))^{L(A, \mathbb{R})}.$$

Given two pointclasses Γ_1 and Γ_2, we write $\Gamma_1 \trianglelefteq_{mouse} \Gamma_2$ if $\Gamma_1 \subseteq \Gamma_2$ and Γ_2 has the same mice as Γ_1 relative to common iteration strategies. More precisely, if $(\mathcal{P}, \Sigma) \in \Gamma_1$ is a hod pair such that $L(\Gamma_2, \mathbb{R}) \vDash$ "Σ has strong branch condensation and is fullness preserving" then for any $a \in HC$,

$$Lp^{\Gamma_1, \Sigma}(a) = Lp^{\Gamma_2, \Sigma}(a).$$

Finally, following [10, Definition 3.18],

Definition 5.3.2 Γ *is mouse full if either it is completely mouse full or is a union of completely mouse full pointclasses* $(\Gamma_\alpha : \alpha < \Omega)$ *such that for all* α*,* $\Gamma_\alpha \trianglelefteq_{mouse} \Gamma_{\alpha+1}$ *and for all limit* α*,* $\Gamma_\alpha = \bigcup_{\beta<\alpha} \Gamma_\beta$*.*

We can now state our generalization of [10, Theorem 3.19].

Theorem 5.3.3 *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair and* Γ *is a pointclass. Suppose further that* $\mathcal{P}$ *is of limit type and* Σ *has strong branch condensation and is* Γ*-fullness preserving. Then*

1. *$\Gamma(\mathcal{P},\Sigma) = \bigcup_{\mathcal{Q}\in pI(\mathcal{P},\Sigma),\beta<\lambda^{\mathcal{Q}}} D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta)$.*

2. *For any $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$, if $\beta+\omega < \lambda^{\mathcal{P}}$ then $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta)$ is completely mouse full and if $\beta+\omega = \lambda^{\mathcal{P}}$ then $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta)$ is mouse full.*

3. *For any $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$, if $\beta < \lambda^{\mathcal{P}}$ then letting $\Gamma^* = D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta+\omega)$, if ξ is such that $\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta)})} = \theta^{\Gamma}_{\xi}$ then for every n,*

$$\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta+n)})} = \theta^{\Gamma}_{\xi+n} \text{ and } \Omega^{\Gamma} = \xi+\omega.$$

4. *$\Gamma(\mathcal{P},\Sigma)$ is a mouse full pointclass.*

We can also prove a version of Theorem 5.3.3, via exactly the same proof, for sts hod pairs.

Theorem 5.3.4 *Suppose $(\mathcal{P},\Sigma)$ is an sts hod pair and Γ is a pointclass. Suppose further that Σ has strong branch condensation and is Γ-fullness preserving. Then*

1. *$\Gamma^b(\mathcal{P},\Sigma) = \bigcup_{\mathcal{Q}\in pI^b(\mathcal{P},\Sigma),\beta<\lambda^{\mathcal{Q}}} D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta)$.*

2. *For any $\mathcal{Q} \in pI^b(\mathcal{P},\Sigma)$, $D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta)$ is completely mouse full.*

3. *For any $\mathcal{Q} \in pI^b(\mathcal{P},\Sigma)$, if $\beta < \lambda^{\mathcal{P}}$ then letting $\Gamma^* = D(\mathcal{Q},\Sigma_{\mathcal{Q}},\beta+\omega)$, if ξ is such that $\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta)})} = \theta^{\Gamma}_{\xi}$ then for every n,*

$$\theta^{\Gamma}_{Code(\Sigma_{\mathcal{Q}(\beta+n)})} = \theta^{\Gamma}_{\xi+n} \text{ and } \Omega^{\Gamma} = \xi+\omega.$$

4. *$\Gamma^b(\mathcal{P},\Sigma)$ is a mouse full pointclass.*

We finish with a theorem generalizing [10, Theorem 3.20]. It shows that $\Gamma(\mathcal{P},\Sigma)$ satisfies mouse capturing for any $\Sigma_{\mathcal{Q}}$ where $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$. Recall from [10] (the first page of the introduction of [10]) that MC stands for mouse capturing, i.e., for the statement that for $x, y \in \mathbb{R}$, $x \in OD_y$ if and only if there is an ω_1-iterable y-mouse $\mathcal{M}$ such that $x \in \mathcal{M}$. Given a hod pair $(\mathcal{P},\Sigma)$ such that Σ has strong branch condensation and is fullness preserving, we say MC holds for Σ if for $x, y \in \mathbb{R}$, $x \in OD_{y,\Sigma}$ if and only if there is an ω_1-iterable Σ-mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$. Given a mouse full pointaclass Γ and a hod pair $(\mathcal{P},\Sigma) \in \Gamma$ such that Σ is Γ-fullness preserving and has strong branch condensation, we write

$$\Gamma \vDash \text{“}\mathsf{MC} \text{ for } \Sigma\text{”}$$

if one of the following holds:

1. Γ is completely mouse full and whenever A is a new set such that $\Gamma = W_A$ then $L(A, \mathbb{R}) \vDash$ "MC for Σ".

2. Γ is not completely mouse full and if $(\Gamma_\alpha : \alpha < \Omega)$ are the completely mouse full pointclasses witnessing that Γ is mouse full then for some $\alpha < \Omega$, $L(\Gamma_\alpha, \mathbb{R}) \vDash$ "MC for Σ".

Theorem 5.3.5 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and Σ has strong branch condensation and is fullness preserving. Suppose further that there is a good pointclass Γ such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma}$. Then for every $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$,*

$$\Gamma(\mathcal{P}, \Sigma) \vDash \text{"}\mathsf{MC} \textit{ for } \Sigma_{\mathcal{Q}}\text{"}.$$

5.4 Anomalous hod premice

In this paper, we use anomalous hod premice the same way we used them in [10], to generate pointclasses that are mouse full but not completely mouse full.

Definition 5.4.1 (Anomalous hod premouse of type I) *$\mathcal{P}$ is an anomalous hod premouse of type I if there is a hod premouse $\mathcal{Q} \trianglelefteq \mathcal{P}$ such that $\mathcal{Q}$ is of successor type, $\mathcal{P} \vDash$ "$\delta^{\mathcal{Q}}$ is Woodin", $\mathcal{P}$ can be organized as $\mathcal{J}^{\vec{E}, f}(\mathcal{Q})$ where f codes a fragment of a strategy for $\mathcal{Q}$ and either $\rho(\mathcal{P}) < \delta^{\mathcal{Q}}$ or $\mathcal{J}_1[\mathcal{P}] \vDash$ "$\delta^{\mathcal{Q}}$ is not a Woodin cardinal".*

Definition 5.4.2 (Anomalous hod premouse of type II) *$\mathcal{P}$ is an anomalous hod premouse of type II if for some limit ordinal λ and some δ there is a sequence $(\mathcal{P}_\alpha : \alpha < \lambda)$ such that*

1. *$\mathcal{P}_\alpha$ is a hod premouse such that $\lambda^{\mathcal{P}_\alpha} = \alpha$,*

2. *for $\alpha < \beta < \lambda$, $\mathcal{P}_\alpha \trianglelefteq_{hod} \mathcal{P}_\beta$ and $\mathcal{P}_\alpha = \mathcal{P}_\beta(\alpha)$,*

3. *$\mathcal{P}|\delta = \bigcup_{\alpha<\lambda} \mathcal{P}_\alpha$,*

4. *$\mathcal{P}$ is a $\oplus_{\alpha<\lambda}\Sigma^{\mathcal{P}}_{\mathcal{P}(\alpha)}$-premouse over $\mathcal{P}|\delta$,*

5. *$\rho(\mathcal{P}) < \delta^{\mathcal{P}}$ but for every $\xi \in (\delta, o(\mathcal{P}))$, $\rho(\mathcal{P}|\xi) \geq \delta$.*

Definition 5.4.3 (Anomalous hod premouse of type III) *$\mathcal{P}$ is an anomalous hod premouse of type III if it is of limit type, it is not an anomalous hod premouse of type II and $\rho(\mathcal{P}) < \delta^{\mathcal{P}^b}$.*

We say $\mathcal{P}$ is an anomalous hod premouse if it is an anomalous hod premouse of type i where $i \in \{I, II, III\}$. If $\mathcal{P}$ is an anomalous hod premouse then we let $\delta^{\mathcal{P}}$ and $\lambda^{\mathcal{P}}$ be as in the above definitions. We then let $\Sigma^{\mathcal{P}}$ be the strategy that is on the sequence of $\mathcal{P}$.

Definition 5.4.4 (Anomalous hod pair) *$(\mathcal{P}, \Sigma)$ is an anomalous hod pair if $\mathcal{P}$ is an anomalous hod premouse, Σ is an iteration strategy with hull condensation and whenever $\mathcal{Q}$ is a Σ iterate of $\mathcal{P}$, $\Sigma^{\mathcal{Q}} \subseteq \Sigma \restriction \mathcal{Q}$.*

The following lemma is due to Mitchell and Steel. It appears as Claim 5 in the proof of Theorem 6.2 of [8]. In the current work, the lemma is used to show that certain hod pair constructions converge, which leads to showing that generation of pointclasses holds (see Theorem 10.1.1). It was used in [10] in a similar fashion (see [10, Lemma 3.25]).

Lemma 5.4.5 *Suppose $(\mathcal{P}, \Sigma)$ is a an anomalous hod pair, $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ and n is least such that if $\mathcal{P}$ is anomalous of type I or II then $\rho_n(\mathcal{P}) < \delta^{\mathcal{P}}$ and otherwise $\rho_n(\mathcal{P}) < \delta^{\mathcal{P}^b}$. Then $\rho_n(\mathcal{Q}) < \delta^{\mathcal{Q}}$.*

The next theorem is the adaptation of [10, Theorem 3.27] to our current setting. It generalizes our results from previous sections to anomalous hod pairs.

Theorem 5.4.6 *Suppose $(\mathcal{P}, \Sigma)$ is an anomalous hod pair of type II or III. Suppose that there is a pointclass Γ such that for any $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$ there is a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has (strong) branch condensation and is Γ-fullness fullness preserving, and there is $\pi : \mathcal{Q} \to \mathcal{R}$ such that $\Lambda^{\pi} = \Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$. Then*

1. *For every $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has (strong) branch condensation, is positional and is commuting.*
2. *Σ is strongly $\Gamma(\mathcal{P}, \Sigma)$-fullness preserving and $\Gamma(\mathcal{P}, \Sigma)$ is a mouse full pointclass.*

We omit the proof of Theorem 5.4.6 as it is only notationally more complicated than the proof of [10, Theorem 3.10]. We remind the reader that the proof of [10, Theorem 3.27] depended on generic interpretability result, which appeared as [10, Theorem 3.10]. In our current context we need to use Theorem 5.2.5. The general idea is that we can translate the properties of Σ into the derived model of $\mathcal{P}$ as computed via Σ. This fact then just gets preserved under pull-back embeddings.

It is also possible to prove a version of Theorem 5.4.6 for sts hod pairs. To prove it, we again need to use Theorem 5.2.5. We state it without a proof.

Theorem 5.4.7 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair and Γ is a pointclass. Suppose that for any $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$ there is a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has strong branch condensation and is (strongly) Γ-fullness fullness preserving, and there is $\pi : \mathcal{Q} \to \mathcal{R}$ such that $\Lambda^\pi = \Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$. Then*

1. *For every $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has (strong) branch condensation, is positional and is commuting.*

2. *Σ is strongly $\Gamma^b(\mathcal{P}, \Sigma)$-fullness preserving and $\Gamma^b(\mathcal{P}, \Sigma)$ is a mouse full pointclass.*

The following is an easy corollary of Theorem 5.4.6.

Corollary 5.4.8 (Branch condensation pulls back) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit and Σ has (strong) branch condensation. Suppose $\pi : \mathcal{Q} \to \mathcal{P}$ is elementary. Then for every $\beta < \lambda^{\mathcal{Q}}$, $(\Sigma^\pi)_{\mathcal{Q}(\beta)}$ has (strong) branch condensation.*

5.5 Strong branch condensation and correctness of $\mathcal{Q}$-structures

Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair. There is one potential problem with our definition of short tree strategy indexing scheme (see Definition 3.8.2). Suppose $\mathcal{M}$ is an unambiguous Σ-sts premouse and $\mathcal{T}$ is an ambiguous tree on $\mathcal{P}$. Suppose there is an $\mathcal{M}$-shortness witness (β, γ, b) for $\mathcal{T}$ and let $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$. It is not immediately clear that $\mathcal{Q}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T}), \mathcal{T}}$-sts premouse. More precisely, it is not clear that $\Sigma^{\mathcal{Q}} \subseteq \Sigma_{\mathcal{M}^+(\mathcal{T}), \mathcal{T}} \restriction \mathcal{Q}$. In this section, we show that if Σ has strong branch condensation then $\mathcal{Q}$ is indeed $\Sigma_{\mathcal{M}^+(\mathcal{T}), \mathcal{T}}$-sts premouse. The following lemma is the crux of our argument.

Lemma 5.5.1 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair and Γ is a pointclass. Suppose further that Σ has a strong branch condensation and is strongly Γ-fullness preserving.*

Suppose $t = (\mathcal{P}_0, \mathcal{U}_0, \mathcal{P}_1, \mathcal{U}_1, \mathcal{P}_2, \vec{\mathcal{U}})$ is a stack of length 3 on $\mathcal{P}$ such that $\mathcal{U}_0$ is according to Σ and $(\mathcal{P}_1, \mathcal{U}_1, \mathcal{P}_2, \vec{\mathcal{U}})$ is $(\mathcal{P}, \Sigma)$-authentic. Then $(\mathcal{P}_1, \mathcal{U}_1, \mathcal{P}_2, \vec{\mathcal{U}})$ is according to $\Sigma_{\mathcal{P}_1, \mathcal{U}_0}$.

Proof. The proof is a routine application of strong branch condensation. We first prove that $\mathcal{U}_1$ is according to $\Sigma_{\mathcal{P}_1}$.

Fix a cutpoint $\mathcal{S}$ of $\mathcal{U}_1$ such that $\pi^{(\mathcal{U}_1)_{\leq\mathcal{S}},b}$ exists and $(\mathcal{U}_1)_{\leq\mathcal{S}}$ is according to $\Sigma_{\mathcal{P}_1,\mathcal{U}_0}$. Let $\mathcal{K}$ be the longest initial segment of $(\mathcal{U}_1)_{\geq\mathcal{S}}$ that is above $\delta^{\mathcal{S}^b}$. We claim that

Claim 1. $\mathcal{K}$ is according to $\Sigma_{\mathcal{S}}$.

Proof. Suppose first that $\mathcal{K}$ doesn't have fatal drops. Fix a limit ordinal $\gamma < lh(\mathcal{K})$ such that $\mathcal{K} \restriction \gamma$ is according to $\Sigma_{\mathcal{S}}$ and let b be the branch of $\mathcal{K} \restriction \gamma$ in $\mathcal{K}$. We want to see that b is according to $\Sigma_{\mathcal{S}}$.

We have that $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$ exists and is $(\mathcal{P}, \Sigma)$-authentic. Let then $\mathcal{T}$ be a tree on $\mathcal{P}$ according to Σ that authenticates $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$. Let $\mathcal{W} = \pi^{\mathcal{T},b}(\mathcal{P}^b)$. Also let $\mathcal{U}$ be the $\mathcal{T}$-authentication tree on $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$ and (α, ξ) be the $\mathcal{T}$-authentication ordinals. Thus, $\xi \leq o(\mathcal{W}(\alpha))$ and $\mathcal{W}||\xi$ is the last model of $\mathcal{U}$. Let $k : \mathcal{P}^b \to (\mathcal{W}(\alpha))^b$ be the uncollapse map of $Hull^{\mathcal{W}}(\pi^{\mathcal{T},b}[\mathcal{P}^b] \cup \delta^{(\mathcal{W}(\alpha))^b})$. It follows from clause 3 of Definition 3.7.1 and Lemma 4.8.3 that $k = \pi^{\mathcal{U},b} \circ \pi^{\mathcal{U}_0^\frown(\mathcal{U}_1)_{\leq\mathcal{S}},b}$.

We now want to show that $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$ is a $\mathcal{Q}$-structure of a correct kind, a kind that $\Sigma_{\mathcal{S}}$ chooses. Let $e = \Sigma_{\mathcal{S}}(\mathcal{K} \restriction \gamma)$, $\mathcal{Q} = \mathcal{M}_e^{\mathcal{K}\restriction\gamma}|(\delta(\mathcal{K} \restriction \gamma)^+)^{\mathcal{M}_e^{\mathcal{K}\restriction\gamma}}$, and let Λ be the $\pi^{\mathcal{U}}$-pullback of $\Sigma_{\mathcal{W}||\xi}$. We now compare $\mathcal{Q}$ with $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$ using respectively $\Sigma_{\mathcal{Q}}$ and Λ. Because Σ is strongly $\Gamma^b(\mathcal{P}, \Sigma)$-fullness preserving (see Theorem 5.4.7) and $\Lambda \in \Gamma^b(\mathcal{P}, \Sigma)$ (see Definition 3.9.7), if the comparison halts then $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$ must be an initial segment of $\mathcal{Q}$ implying that $b = e$. Therefore, the comparison cannot halt.

Using the proof of Lemma 4.6.3, we can find a low level disagreement between Λ and $\Sigma_{\mathcal{Q}}$. Let then $(\vec{\mathcal{T}}_0, \mathcal{R}) \in B(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\vec{\mathcal{T}}_1, \mathcal{R}) \in B(\mathcal{Q}(b, \mathcal{K} \restriction \gamma), \Lambda)$ constitute a low level disagreement between $\Sigma_{\mathcal{Q}}$ and Λ. Let $\mathcal{R}_0^+$ and $\mathcal{R}_1^+$ be the last models of $\vec{\mathcal{T}}_0$ and $\vec{\mathcal{T}}_1$ when we regard them as stacks on $\mathcal{Q}$ and $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$. Notice that $\pi^{\vec{\mathcal{T}}_0,b}$ and $\pi^{\vec{\mathcal{T}}_1,b}$ exists. Let $\vec{\mathcal{T}}_1^* = \pi^{\mathcal{U}}\vec{\mathcal{T}}_1$ be the stack on $\mathcal{W}||\xi$ constructed via a copying construction using $\pi^{\mathcal{U}}$ and let $\mathcal{R}^*$ be its last model. There is then

$$\sigma : (\mathcal{R}_1^+)^b \to (\mathcal{R}^*)^b$$

such that

$$\pi^{\vec{\mathcal{T}}_1^*,b} \circ k = \sigma \circ \pi^{\vec{\mathcal{T}}_1,b} \circ \pi^{\mathcal{U}_0^\frown(\mathcal{U}_1)_{\leq\mathcal{S}},b},$$

and $\Lambda_{\mathcal{R},\vec{\mathcal{T}}_1}$ is σ-pullback of $\Sigma_{\sigma(\mathcal{R})}$. It follows from strong branch condensation that $\Lambda_{\mathcal{R},\vec{\mathcal{T}}_1} = \Sigma_{\mathcal{R}}$.

Suppose now that $\mathcal{K}$ has a fatal drop. The proof is very similar to the proof given above. Without loss of generality we can assume that $\mathcal{K}$ has a fatal drop at $\mathcal{S}$. Fix then η such that $\mathcal{K}$ is a normal tree on $\mathcal{O}_\eta^{\mathcal{S}}$ above η. We can then mimic the above proof using $\mathcal{O}^{\mathcal{S}}_\eta$ instead of $\mathcal{Q}(b, \mathcal{K} \restriction \gamma)$. □

To finish proving that $\mathcal{U}_1$ is according to $\Sigma_{\mathcal{P}_1}$, it is enough to establish the following claim.

Claim 2. Suppose $\mathcal{S}$ is a cutpoint of $\mathcal{U}_1$ such that $\pi^{(\mathcal{U}_1)_{\leq \mathcal{S}}, b}$ exists and $(\mathcal{U}_1)_{\leq \mathcal{S}}$ is according to $\Sigma_{\mathcal{P}_1, \mathcal{U}_0}$. Let $\mathcal{K}$ be the longest initial segment of $(\mathcal{U}_1)_{\geq \mathcal{S}}$ that is based on $\mathcal{S}^b$. Then $\mathcal{K}$ is according to $\Sigma_{\mathcal{S}^b}$.

Proof. We know that $\mathcal{S}^b$ is $(\mathcal{P}, \Sigma)$-authentic. Fix then a normal iteration tree $\mathcal{T}$ on $\mathcal{P}$ according to Σ that authenticates $\mathcal{S}^b$. Let $\mathcal{W} = \pi^{\mathcal{T}, b}(\mathcal{P}^b)$, and let $\mathcal{U}$ be the $\mathcal{T}$-authentication tree on $\mathcal{S}^b$. Let α be such that $\mathcal{W}(\alpha)$ is the last model of $\mathcal{U}$. Then $\mathcal{K}$ is according to $\pi^{\mathcal{U}}$-pullback of $\Sigma_{\mathcal{W}(\alpha)}$.

Let $k : \mathcal{P}^b \to \mathcal{W}(\alpha)$ be the inverse map of the collapse of $Hull^{\mathcal{W}}(\pi^{\mathcal{T}, b}[\mathcal{P}^b] \cup \delta^{\mathcal{W}}_\alpha)$. We then have that $k = \pi^{\mathcal{U}} \circ \pi^{\mathcal{U}_0 {}^\frown (\mathcal{U}_1)_{\leq \mathcal{S}}, b}$. It now easily follows from strong branch condensation that $\mathcal{K}$ is according to $\Sigma_{\mathcal{S}^b}$. □

Finally we want to see that $\vec{\mathcal{U}}$ is according to $\Sigma_{\mathcal{P}_2}$. The proof is very similar to the proof given above. Fix $\mathcal{T}$ that authenticates $\mathcal{P}_2^b$, and let $\mathcal{S}$ be the $\mathcal{T}$-authentication tree on $\mathcal{P}_2^b$. Let $\mathcal{W} = \pi^{\mathcal{T}, b}(\mathcal{P}^b)$. Let α be such that $\mathcal{W}(\alpha)$ is the last model of $\mathcal{S}$. Then $\vec{\mathcal{U}}$ is according to $\pi^{\mathcal{S}}$-pullback of $\Sigma_{\mathcal{W}(\alpha)}$.

Let $k : \mathcal{P}^b \to \mathcal{W}(\alpha)$ be the inverse map of the collapse of $Hull^{\mathcal{W}}(\pi^{\mathcal{T}, b}[\mathcal{P}^b] \cup \delta^{\mathcal{W}}_\alpha)$. We then have that $k = \pi^{\mathcal{S}} \circ \pi^{\mathcal{U}_0 {}^\frown \mathcal{U}_1, b}$. It now easily follows from strong branch condensation that $\vec{\mathcal{U}}$ is according to $\Sigma_{\mathcal{P}_2^b}$, as it is according to $\pi^{\mathcal{U}}$-pullback of $\Sigma_{\mathcal{W}(\alpha)}$. □

Corollary 5.5.2 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, Γ is a pointclass and Σ has a strong branch condensation and is strongly Γ-fullness preserving. Suppose further that $\mathcal{M}$ is an unambiguous Σ-sts mouse, $\mathcal{T} \in \mathcal{M}$ is a normal $\mathcal{M}$-ambiguous tree on $\mathcal{P}$ according to $\Sigma^{\mathcal{M}}$ and (β, γ, b) is an $\mathcal{M}$-shortness witness for $\mathcal{T}$. Then $b = \Sigma(\mathcal{T})$.*

Proof. It is enough to show that $\mathcal{Q}(b, \mathcal{T})$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$. Let Φ be the iteration strategy of $\mathcal{M}|\beta$ induced from the iteration strategy of $\mathcal{M}$. Thus, Φ witnesss that $\mathcal{M}|\beta$ is a Σ-sts mouse. Let $(\delta_i : i < \omega)$ be a sequence of Woodin cardinals of $\mathcal{M}|\beta$ witnessing that that clause 4 of Definition 3.8.2 holds. Also, let $\Lambda \in \mathcal{M}|\beta$ be an iteration strategy for $\mathcal{Q}(b, \mathcal{T})$ as in clause 4 of Definition 3.8.2.

Notice that it follows from minimality of β that $\rho(\mathcal{J}_1(\mathcal{M}|\beta)) < \delta_0$, implying that Φ is commuting. It then follows (using clause 4 of Definition 3.8.2) that Λ has an extension Λ^+ that acts on all trees in V. It is then enough to show that Λ^+ witnesses that $\mathcal{Q}(b, \mathcal{T})$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse.

Let $\mathcal{U}$ be a tree on $\mathcal{Q}(b, \mathcal{T})$ above $\delta(\mathcal{T})$ according to Λ^+. Let $\mathcal{R}$ be the last model of $\mathcal{U}$. We need to see that $\mathcal{R}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-premouse. Let $\mathcal{P}_1 = \mathcal{M}^+(\mathcal{T})$ and fix $(\mathcal{P}_1, \mathcal{T}_1, \mathcal{P}_2, \vec{\mathcal{T}}) \in \mathcal{R}$ a finite stack of length 2 on $\mathcal{P}_1$ that is according to $\Sigma^{\mathcal{R}}$. It follows from clause 4 of Definition 3.8.2 that for some Φ-iterate $\mathcal{N}$ of $\mathcal{M}|\beta$, $(\mathcal{P}_1, \mathcal{T}_1, \mathcal{P}_2, \vec{\mathcal{T}})$ is $(\mathcal{P}, \Sigma^{\mathcal{N}})$-authentic. It then follows that $(\mathcal{P}, \mathcal{T}, \mathcal{P}_1, \mathcal{T}_1, \mathcal{P}_2, \vec{\mathcal{T}})$ satisfies the hypothesis of Lemma 5.5.1. Hence, $(\mathcal{P}_1, \mathcal{T}_1, \mathcal{P}_2, \vec{\mathcal{T}})$ is according to $\Sigma_{\mathcal{P}_1, \mathcal{T}}$. □

Remark 5.5.3 (On hod pair constructions) *Suppose $(\mathcal{P}, \Lambda)$ is an sts hod pair. Recall Definition 4.2.1, which introduces fully backgrounded constructions relative to Λ. In particular, recall the Important Anomaly in clause 2.c of Definition 4.2.1. The main point of Corollary 5.5.2 is to show that this anomaly cannot occur. What follows is an explanation of how fully backgrounded constructions relative to Λ and in general, hod pair constructions are carried out (the Important Anomaly appears in such constructions as well, for instance, see clause 3.a of Definition 4.3.9).*

Suppose (M, δ, Σ) is a background triple and we want to show that the hod pair construction of M doesn't break down because of Important Anomaly. Let $(\mathcal{P}, \Lambda)$ be some pair that appears in the fully backgrounded hod pair construction of M. Suppose further that $\mathcal{P}$ is of lsa type. It follows from Theorem 4.5.4 that Λ^{sts} is strongly Γ-fullness preserving for some Γ. It also follows from Theorem 4.7.4 that Λ^{sts} has strong branch condensation. It then follows from Corollary 5.5.2 that Important Anomaly cannot happen in fully backgrounded constructions relative to Λ^{sts}.

5.6 From condensation to strong condensation

In this section we show that strategies with branch condensation acquire strong branch condensation on a tail. However, we don't quite get strong branch condensation for lsa type hod pairs. Nevertheless, in the case of lsa type hod pairs we get *low level strong branch condensation*. In the case of limit type hod premice that are not of lsa type, low level strong branch condensation and strong branch condensation coincide. The difference between Definition 4.7.1 and Definition 5.6.1 is just the requirement in clause 3 that $\alpha + 1 < \lambda^{\mathcal{R}}$.

Definition 5.6.1 (Low level strong branch condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is of limit type. We say Σ has low level strong branch condensation if Σ has branch condensation and whenever $(\vec{\mathcal{T}}, \mathcal{Q}, \pi, \mathcal{R}, \alpha, \sigma)$ is such that*

1. $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ *and* $\mathcal{R}$ *is a hod premouse,*

2. $\pi : \mathcal{P} \to \mathcal{R}$, $\sigma : \mathcal{R} \to \mathcal{Q}$ *and* $\pi^{\vec{\mathcal{T}}} = \sigma \circ \pi$,

3. $\alpha + 1 < \lambda^{\mathcal{R}}$ *is such that for some* $\vec{\mathcal{U}}$, $(\vec{\mathcal{U}}, \mathcal{R}(\alpha+1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$

then letting $\Lambda = \sigma$*-pullback of* $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$, *whenever* $\vec{\mathcal{W}}$ *is such that* $(\vec{\mathcal{W}}, \mathcal{R}(\alpha+1)) \in B(\mathcal{P}, \Sigma) \cup I(\mathcal{P}, \Sigma)$, *if there is no low level disagreement between* $\Lambda_{\mathcal{R}(\alpha+1)}$ *and* $\Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$ *then* $\Lambda_{\mathcal{R}(\alpha+1)} = \Sigma_{\mathcal{R}(\alpha+1),\vec{\mathcal{W}}}$.

Theorem 5.6.2 (From condensation to strong condensation) *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* Σ *has branch condensation and* $\mathcal{P}$ *is of limit type. Then there is some* $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ *such that* $(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ *has a low level strong branch condensation.*

We spend the rest of this section proving Theorem 5.6.2. The idea is just like the idea behind the diamond comparison proof. If there is no tail with (low level) strong branch condensation then we obtain a certain bad sequence of length ω_1. As is expected, such sequences cannot exist. We start by describing the blocks of our bad sequences.

Definition 5.6.3 (A bad diamond) *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair of limit type. We say* $((\mathcal{P}_i : i < 2), (\vec{\mathcal{T}}_i : i < 3), (\vec{\mathcal{U}}_i : i < 3), (\mathcal{R}_i : i < 2), (\mathcal{S}_i : i < 2), k, \xi)$ *is a bad diamond on* $(\mathcal{P}, \Sigma)$ *if it satisfies the following conditions:*

1. $\mathcal{P}_0 = \mathcal{P}$, *for* $i < 2$, $\mathcal{P}_i, \mathcal{R}_i$ *and* $\mathcal{S}_i$ *are hod premice and* $k : \mathcal{P}_0 \to \mathcal{R}_0$.

2. $(\vec{\mathcal{U}}_0, \mathcal{S}_0) \in I(\mathcal{P}, \Sigma)$, $(\vec{\mathcal{U}}_1, \mathcal{S}_1) \in I(\mathcal{S}_0, \Sigma_{\mathcal{S}_0,\vec{\mathcal{U}}_0})$, $(\vec{\mathcal{U}}_2, \mathcal{P}_1) \in I(\mathcal{S}_1, \Sigma_{\mathcal{S}_1,\vec{\mathcal{U}}_0{}^\frown\vec{\mathcal{U}}_1})$, *and* $\vec{\mathcal{U}}_1$ *is a normal tree on* $\mathcal{S}_0$.

3. $\vec{\mathcal{T}}_0 = \emptyset$, $\vec{\mathcal{T}}_1$ *is a normal tree on* $\mathcal{R}_0$ *with last model* $\mathcal{R}_1$ *and* $\vec{\mathcal{T}}_2$ *is a stack on* $\mathcal{R}_1$ *with last model* $\mathcal{P}_1$,.

4. $\xi + 1 < \lambda^{\mathcal{S}_0}$, $\mathcal{S}_0(\xi+1) = \mathcal{R}_0(\xi+1)$, $\vec{\mathcal{T}}_1^- = \vec{\mathcal{U}}_1^-$[3] *is a normal tree based on* $\mathcal{S}(\xi+1)$ *such that it has a* $\preceq^{\vec{\mathcal{T}}_1^-,s}$*-maximal cutpoint* $\mathcal{N}$ *such that* $(\vec{\mathcal{T}}_1^-)_{\geq \mathcal{N}}$ *is based on* $\mathcal{N}(\nu+1)$ *where* $\nu = \pi^{(\vec{\mathcal{T}}_1^-)_{\leq \mathcal{N}}}(\xi)$.

5. $\delta_{\xi+1}^{\mathcal{R}_0} = \sup\{k(f)(a) : f \in \mathcal{P}_0 \wedge a \in (\mathcal{R}(\xi))^{<\omega}\}$.

6. *If* b *is the branch of* $\vec{\mathcal{T}}_1^-$ *in* $\vec{\mathcal{T}}_1$ *then* $b \neq \Sigma_{\mathcal{S}_0}(\vec{\mathcal{U}}_1^-)$.

[3] Recall that this is just $\vec{\mathcal{T}}_1$ without its last model.

7. *Letting* $\gamma = \pi^{\vec{\mathcal{T}}_1}(\xi) = \pi^{\vec{\mathcal{U}}_1}(\xi)$, $\mathcal{R}_1(\gamma+1) = \mathcal{S}_1(\gamma+1)$. *If* $\vec{\mathcal{W}}$ *is the part of* $\vec{\mathcal{T}}_2$ *based on* $\mathcal{R}_1(\gamma+1)$ *then* $\vec{\mathcal{W}}$ *is according to* $\Sigma_{\mathcal{S}_1(\gamma+1)}$.

Lemma 5.6.4 *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* Σ *has branch condensation and* $\mathcal{P}$ *is of limit type. Suppose further that* Σ *doesn't have low level strong branch condensation. Then there is a bad diamond on* $(\mathcal{P}, \Sigma)$.

Proof. Let $(\vec{\mathcal{T}}, \mathcal{Q}, \pi, \mathcal{R}, \alpha, \sigma)$ be a witness to the failure of low level strong branch condensation of $(\mathcal{P}, \Sigma)$. Let $(\vec{\mathcal{U}}_0, \mathcal{S}_0) \in I(\mathcal{P}, \Sigma)$ be such that $\mathcal{R}(\alpha+1) = \mathcal{S}_0(\alpha+1)$. We let $\pi = k$, $\mathcal{R}_0 = \mathcal{R}$ and $\xi = \alpha$. Let $\Lambda = \Sigma^{\sigma}_{\mathcal{Q}, \vec{\mathcal{T}}}$. Notice that clause 5 of Definition 5.6.3 is satisfied because $\delta^{\mathcal{R}}_{\alpha+1}$ is the least $\nu > \delta^{\mathcal{R}}_{\alpha}$ such that $Lp^{\Gamma(\mathcal{P},\Sigma), \Lambda_{\mathcal{R}(\alpha)}}(\mathcal{R}|\nu) \vDash$ "ν is a Woodin cardinal" (to see this, we use Theorem 5.4.6 and Corollary 5.4.8).

Let $\mathcal{T}$ be a normal tree on $\mathcal{R}(\alpha+1)$ according to both $\Sigma_{\mathcal{S}_0, \vec{\mathcal{U}}_0}$ and Λ and such that $\Sigma_{\mathcal{S}_0, \vec{\mathcal{U}}_0}(\mathcal{T}) \neq \Lambda(\mathcal{T})$ but letting $b = \Sigma_{\mathcal{S}_0, \vec{\mathcal{U}}_0}(\mathcal{T})$, $c = \Lambda(\mathcal{T})$, $\mathcal{S}_1 = \mathcal{M}^{\mathcal{T}}_b$ and $\mathcal{R}_1 = \mathcal{M}^{\mathcal{T}}_c$ then

$$\Sigma_{\mathcal{S}_1(\pi^{\mathcal{T}}_b(\alpha+1)), \vec{\mathcal{U}}_0^\frown \mathcal{T}^\frown \{\mathcal{S}_1\}} = \Lambda_{\mathcal{R}_1(\pi^{\mathcal{T}}_c(\alpha+1)), \mathcal{T}}.$$

Such a $\mathcal{T}$ can be found using the Theorem 4.6.10. Notice that Theorem 4.6.10 is applicable because both Σ and Λ are $\Gamma(\mathcal{S}_0(\alpha+\omega), \Sigma_{\mathcal{S}_0, \vec{\mathcal{U}}_0})$-fullness preserving (here we need to use Corollary 5.4.8 to conclude that $\Lambda_{\mathcal{R}(\alpha+\omega)}$ has branch condensation). Let $\vec{\mathcal{T}}_1 = \mathcal{T}^\frown\{\mathcal{M}^{\mathcal{T}}_c\}$, $\vec{\mathcal{U}}_1 = \mathcal{T}^\frown\{\mathcal{M}^{\mathcal{T}}_b\}$, $\mathcal{R}_1 = \mathcal{M}^{\mathcal{T}}_c$ and $\mathcal{S}_1 = \mathcal{M}^{\mathcal{T}}_b$.

Next we would like to compare $(\mathcal{R}_1, \Lambda_{\mathcal{R}_1, \vec{\mathcal{T}}_1})$ and $(\mathcal{S}_1, \Sigma_{\mathcal{S}_1, \vec{\mathcal{U}}_0^\frown \mathcal{T}^\frown\{\mathcal{S}_1\}})$. To do this, we can use Corollary 5.4.8 and Theorem 4.10.4. Let then $(\vec{\mathcal{T}}_2, \mathcal{P}_1) \in I(\mathcal{R}_1, \Lambda_{\mathcal{R}_1, \vec{\mathcal{T}}_1})$ and $(\vec{\mathcal{U}}_2, \mathcal{P}_1) \in I(\mathcal{S}_1, \Sigma_{\mathcal{S}_1, \vec{\mathcal{U}}_0^\frown \mathcal{T}^\frown\{\mathcal{S}_1\}})$ be such that $\Sigma_{\mathcal{P}_1, \vec{\mathcal{U}}_0^\frown \mathcal{T}^\frown\{\mathcal{S}_1\}^\frown \vec{\mathcal{U}}_2} = \Lambda_{\mathcal{P}_1, \vec{\mathcal{T}}_1^\frown \vec{\mathcal{T}}_2}$. It is then not hard to see that

$$((\mathcal{P}_i : i < 2), (\vec{\mathcal{T}}_i : i < 3), (\vec{\mathcal{U}}_i : i < 3), (\mathcal{R}_i : i < 2), (\mathcal{S}_i : i < 2), k, \xi)$$

is a bad diamond on $(\mathcal{P}, \Sigma)$. □

Now we want to show that there cannot be an ω_1-sequence of bad diamonds on $\mathcal{P}$.

Definition 5.6.5 (A bad diamond sequence of length β) *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\lambda^{\mathcal{P}}$ *is limit. We say* $\vec{D} = \langle D_\alpha : \alpha < \beta \rangle$ *is a bad diamond sequence of length* β *if* $D_\alpha = ((\mathcal{P}^\alpha_i : i < 2), (\vec{\mathcal{T}}^\alpha_i : i < 3), (\vec{\mathcal{U}}^\alpha_i : i < 3), (\mathcal{R}^\alpha_i : i < 2), (\mathcal{S}^\alpha_i : i < 2), k^\alpha, \xi^\alpha)$ *and the following holds:*

1. D_0 *is a bad diamond on* $(\mathcal{P}, \Sigma)$ *and* $\mathcal{P}^1_0 = \mathcal{P}^0_1$.

2. *For all* $\alpha < \beta$, $\mathcal{P}^0_\alpha \in pI(\mathcal{P}, \Sigma)$, D_α *is a bad diamond on* $(\mathcal{P}^\alpha_0, \Sigma_{\mathcal{P}^\alpha_0, \oplus_{\gamma<\alpha}\vec{\mathcal{U}}_\gamma})$ *and* $\mathcal{P}^{\alpha+1}_0 = \mathcal{P}^\alpha_1$.

3. *For* $\nu < \alpha < \beta$, *let* $\pi_{\nu,\alpha} : \mathcal{P}^\nu_0 \to \mathcal{P}^\alpha_0$ *be the embedding obtained by composing the embeddings* $k^\gamma \circ \pi^{\vec{\mathcal{T}}^\gamma_1 {}^\frown \vec{\mathcal{T}}^\gamma_2}$ *for* $\nu \leq \gamma < \alpha$, *and let* $\sigma_{\nu,\alpha} : \mathcal{P}^\nu_0 \to \mathcal{P}^\alpha_0$ *be the iteration embedding obtained by composing the embeddings* $\pi^{\vec{\mathcal{U}}^\gamma_0 {}^\frown \vec{\mathcal{U}}^\gamma_1 {}^\frown \vec{\mathcal{U}}^\gamma_2}$ *for* $\nu \leq \gamma < \alpha$. *Then for limit* $\lambda < \beta$, $\mathcal{P}^\lambda_0$ *is the direct limit of* $(\mathcal{P}^\gamma_0 : \gamma < \lambda)$ *under the embeddings* $\sigma_{\nu,\alpha}$, *and* $(\mathcal{P}^\lambda_0)^b$ *is the direct limit of* $((\mathcal{P}^\gamma_0)^b : \gamma < \lambda)$ *under the embeddings* $\pi_{\nu,\alpha}$.

We say that π *embeddings are the top embeddings of* $\vec{D}$ *and* σ *embeddings are the bottom embeddings of* $\vec{D}$.

Lemma 5.6.6 (No bad diamond sequence of length ω_1) *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair such that* $\lambda^{\mathcal{P}}$ *is limit and* Σ *has a branch condensation. Then there is no bad diamond sequence of length* ω_1 *based on* $(\mathcal{P}, \Sigma)$.

Proof. Suppose not and let $\vec{D} = (D_\beta : \beta < \omega_1)$ be a bad diamond sequence of length ω_1. Let $\tau : H \to H_{\omega_2}$ be a countable submodel such that $\{\vec{D}, (\mathcal{P}, \Sigma)\} \in rng(\tau)$. Let $\kappa = \omega^H_1$. Notice that $\kappa = \text{crit}(\tau)$. Let for $\xi < \beta \leq \omega_1$, $\pi_{\xi,\beta} : \mathcal{P}^\xi_0 \to \mathcal{P}^\beta_0$ be the composition of the top embedding of $\vec{D}$ and let $\sigma_{\xi,\beta} : \mathcal{P}^\xi_0 \to \mathcal{P}^\beta_0$ be the composition of the bottom embeddings of $\vec{D}$. Let $\mathcal{P}^{\omega_1} = \tau(\mathcal{P}^\kappa_0)$. Standard arguments show that

(1) $\tau \restriction (\mathcal{P}^\kappa_0)^b = \pi_{\kappa,\omega_1} \restriction (\mathcal{P}^\kappa_0)^b = \sigma_{\kappa,\omega_1} \restriction (\mathcal{P}^\kappa_0)^b$.

Let $j : \mathcal{R}^\kappa_1 \to \mathcal{P}^{\omega_1}$ and $m : \mathcal{S}^\kappa_1 \to \mathcal{P}^{\omega_1}$ be the composition of respectively the top and the bottom embeddings of $\vec{D}$. Let $\gamma = \pi^{\vec{\mathcal{T}}^\kappa_1}(\xi^\kappa)$. Because the top and bottom embeddings of $\vec{D}$ move $\mathcal{R}^\kappa_1(\gamma+1)$ and $\mathcal{S}^\kappa_1(\gamma+1)$ correctly (this is a consequence of our choice of $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{U}}_1$), we have that

(2) $j \restriction \mathcal{R}^\kappa_1(\gamma+1) = m \restriction \mathcal{S}^\kappa_1(\gamma+1)$.

Notice also that

(3) $\delta^{\mathcal{R}^\kappa_1}_{\gamma+1} = \sup\{\pi^{\vec{\mathcal{T}}^\kappa_1} \circ k^\kappa(f)(a) : a \in (\mathcal{R}^\kappa_1(\gamma))^{<\omega} \wedge f \in \mathcal{P}^\kappa_0\}$ and $\delta^{\mathcal{S}^\kappa_1}_{\gamma+1} = \sup\{\pi^{\vec{\mathcal{U}}^\kappa_0 {}^\frown \vec{\mathcal{U}}^\kappa_1}(f)(a) : a \in (\mathcal{S}^\kappa_1(\gamma))^{<\omega} \wedge f \in \mathcal{P}^\kappa_0\}$

The first equality in (3) follows from standard iteration facts and clause 5 of Definition 5.6.3. It follows from (1) and (2) that

(4) for all $f \in (\mathcal{P}_0^\kappa)^b$ and $a \in (\mathcal{S}_1^\kappa(\gamma))^{<\omega}$, $\pi^{\vec{\mathcal{T}}_0^\kappa \frown \vec{\mathcal{T}}_1^\kappa}(f)(a) = \pi^{\vec{\mathcal{U}}_0^\kappa \frown \vec{\mathcal{U}}_1^\kappa}(f)(a)$.

It then follows from (2) and (3) that

(5) $\delta_{\gamma+1}^{\mathcal{R}_1} = \sup(rng(\pi^{\vec{\mathcal{T}}_1}) \cap rng(\pi^{\vec{\mathcal{U}}_1}))$

contradicting the fact that $\vec{\mathcal{T}}_1$ isn't according to $\Sigma_{\mathcal{S}_0^\kappa, (\oplus_{\alpha<\kappa}(\vec{\mathcal{U}}_0^\alpha \frown \vec{\mathcal{U}}_1^\alpha \frown \vec{\mathcal{U}}_3^\alpha)) \frown \vec{\mathcal{U}}_0^\kappa}$. $\square$

The next lemma finishes the proof of Theorem 5.6.2. Its proof is straightforward, and can be obtained by a consecutive application of Lemma 5.6.4.

Lemma 5.6.7 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit, Σ has branch condensation and for every $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}, \vec{\mathcal{T}}})$ doesn't have low level strong branch condensation. Then there is a bad diamond sequence on $(\mathcal{P}, \Sigma)$ of length ω_1.*

We end this section with a statement of a generalization of Theorem 3.28 of [10]. The theorem shows that we can get branch condensation on a tail by starting with a pair that has only hull condensation. Just like in [10], this result will be used when proving generation of pointclasses (Theorem 10.1.1). The proof is very much like the proof of Theorem 5.6.2, and the proof of Theorem 3.28 of [10].

Theorem 5.6.8 (Getting branch condensation) *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an anomalous hod pair of type II or III with the property that $\mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$ is measurable in $\mathcal{P}$. Suppose further that whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in B(\mathcal{P}, \Sigma)$, $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has branch condensation. Then there is $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}}$ has branch condensation.*

Chapter 6

The internal theory of lsa hod mice

A major shortcoming of our treatment of short-tree-strategy mice is that we did not add branches to all trees. Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, X is a self-well-ordered set such that $\mathcal{P} \in X$ and $\mathcal{M}$ is a Σ-sts premouse over X based on $\mathcal{P}$. Recall short tree strategy indexing scheme Definition 3.8.2. Recall that our strategy for indexing branches was to consider two kinds of iterations, unambiguous and ambiguous. We outright index the branches of unambiguous iterations. However, we only consider a subclass of ambiguous trees. If for some $\beta < o(\mathcal{M})$, $\mathcal{T} \in dom(\Sigma^{\mathcal{M}|\beta})$ is an $\mathcal{M}|\beta$-ambiguous tree then (i) $\mathcal{T}$ is a result of comparing $\mathcal{P}$ with a certain background construction of $\mathcal{M}|\beta$ and (ii) we index the branch of $\mathcal{T}$ after we find a certain certificate of shortness (recall Definition 3.8.2). It is then not clear from our definition that $\Sigma \restriction \mathcal{M} \subseteq \mathcal{M}$ has branches of all trees. The main goal of this chapter is to show that, provided $\mathcal{M}$ is sufficiently closed, $\Sigma \restriction \mathcal{M} \subseteq \mathcal{M}$. Below we make our goal more precise.

Motivational Question. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair, X is a self-well-ordered set such that $\mathcal{P} \in X$ and $\mathcal{M}$ is a Σ or Σ-sts mouse over X (see Definition 3.8.6). Is $\Sigma \restriction \mathcal{N}$ definable over $\mathcal{N}$? Is $\Sigma \restriction \mathcal{N}[g]$ definable over $\mathcal{N}[g]$ where g is $\mathcal{N}$-generic?

In Section 5.2 we gave an answer to Motivational Question in the case $\mathcal{M}$ is $\mathcal{P}$ itself (see Theorem 5.2.5). Another answer was given by [10, Lemma 3.35], where it was shown that $\Sigma \restriction \mathcal{N}[g]$ is definable over $\mathcal{N}[g]$ provided $\mathcal{P}$ is doesn't have non-meek levels. Here, we are mainly concerned with proving a version of [10, Lemma 3.35] in the case of a non-meek hod premice. Because of this we will state many of our definitions and theorems for hod pairs or sts hod pairs $(\mathcal{P}, \Sigma)$ such that $\mathcal{P}$ is non-meek (see Definition 2.4.8, recall that non-meek means that $\lambda^{\mathcal{P}}$ is a successor ordinal and $\delta^{\mathcal{P}}_{\lambda-1}$ is a measurable cardinal). To simplify our terminology, we will say $(\mathcal{P}, \Sigma)$ is a non-meek hod pair if $\mathcal{P}$ is a non-meek hod premouse and Σ is either an iteration strategy or a short-tree-strategy (this is only allowed in the case $\mathcal{P}$ is of lsa type).

While a positive answer to the Motivational Questions is desirable, it is naive to hope that one exists for all such $\mathcal{N}$. A positive answer depends on how closed $\mathcal{N}$ is. If for instance the branch of $\mathcal{T}$ is given via a $\mathcal{Q}$-structure that is beyond the #-operator while our $\mathcal{N}$ is only closed under the #-operator then, in most cases, identifying the correct branch of $\mathcal{T}$ inside $\mathcal{N}$ via a procedure that is uniform in $\mathcal{T}$ will be impossible. In this chapter, we give a positive answer to the Motivational Question provided our $\mathcal{N}$ is sufficiently closed. We make this notion more precise.

Suppose $(\mathcal{P}, \Sigma)$ is a non-meek hod pair and $\mathcal{N}$ is a Σ-mouse such that $\mathcal{N} \vDash$ $\mathsf{ZFC}-$*Replacement*. We say $\mathcal{N}$ is Σ-closed if $\Sigma \restriction \mathcal{N} \subseteq \mathcal{N}$. We say $\mathcal{N}$ is generically Σ-closed if $\mathcal{N}$ is Σ-closed and whenever g is $\mathcal{N}$-generic, $\Sigma \restriction \mathcal{N}[g]$ is definable over $(\mathcal{N}[g], \in)$ (in the language of Σ-premice) without parameters. It is worth remarking that the structure $(\mathcal{N}[g], \in)$ is a structure in the language of Σ-premice and in particular, there are names for $\vec{E}^{\mathcal{N}}$ and $\Sigma^{\mathcal{N}}$.

Definition 6.0.9 *We say $\mathcal{N}$ is uniformly generically Σ-closed if $\mathcal{N}$ is generically Σ-closed and there are formulas ϕ and ψ (in the language of Σ-premice) such that for any $\mathcal{N}$-generic g, any stack $\vec{\mathcal{T}} \in \mathcal{N}[g]$ on $\mathcal{P}$ and any $b \in \mathcal{N}[g]$,*

$$\vec{\mathcal{T}} \in dom(\Sigma) \leftrightarrow (\mathcal{N}[g], \in) \vDash \phi[\vec{\mathcal{T}}]$$
$$\Sigma(\vec{\mathcal{T}}) = b \leftrightarrow (\mathcal{N}[g], \in) \vDash \psi[\vec{\mathcal{T}}, b]$$

The main theorem of this chapter is Theorem 6.1.5. It gives a positive answer to our Motivational Question in the case $\mathcal{N}$ is Σ-closed and has fullness preserving iteration strategy (see Definition 6.1.1 and Definition 6.1.3). The main idea behind the proof of Theorem 6.1.5 is that the branch of an iteration tree $\mathcal{T}$ on $\mathcal{P}$ can be identified by the authentication process introduced in Definition 3.7.2.

Recall that given a transitive set X, we let $\mathcal{M}^+(X)$ be the least sound active mouse over X. Also recall that if X is any set and $A \subseteq X^2$ then $p[A]$ is the projection of A onto one of the coordinates of A. The specific coordinate onto which we project will always be clear from the context. Also, if X is a transitive set then $o(X) = Ord \cap X$.

6.1 Internally Σ-closed mice

In this section we introduce a kind of closure property of hybrid mice for which we can give a positive answer to our motivational question. The first such closure property is internal closure, which postulates that our mouse has enough of the strategy.

Definition 6.1.1 (Internally Σ-closed mouse) *Suppose $(\mathcal{P}, \Sigma)$ is a non-meek hod pair (possibly an sts hod pair) and $\mathcal{N}$ is a Σ-premouse.*

1. *We say $\mathcal{N}$ is an internally Σ-closed premouse if for every $\mathcal{N}$-cardinal κ there is $\mathcal{M} \trianglelefteq \mathcal{N}$ such that $\mathcal{M} \vDash \mathsf{ZFC}$, $\mathcal{N}||\kappa \trianglelefteq \mathcal{M}$ and for every $\eta \in [\kappa, o(\mathcal{M}))$, letting $\mathcal{S}$ be the output of the $(\mathcal{P}, \Sigma^{\mathcal{M}})$-hod pair construction of $\mathcal{M}$ (cf. Definition 3.5.1) in which extenders used have critical points $> \eta$ reaches a $\Sigma^{\mathcal{M}}$-iterate $\mathcal{Q}$ of $\mathcal{P}$ via a normal tree $\mathcal{T}$ such that $\pi^{\mathcal{T},b}$ exists, $\lambda^{\mathcal{S}} = \pi^{\mathcal{T},b}(\lambda^{\mathcal{P}})$ and in the case Σ is an iteration strategy, $\pi^{\mathcal{T}}$-exists.*

2. *If $\mathcal{M}, \mathcal{N}$ and κ are as above then we say $\mathcal{M}$ witnesses the internal Σ-closure of $\mathcal{N}$ at κ.*

3. *We say $\mathcal{N}$ is an internally Σ-closed mouse if it an internally Σ-closed premouse and has a (k, ω_1)-iteration strategy Λ witnessing that $\mathcal{N}$ is a Σ-mouse.*

Two remarks are in order. First notice that internal Σ-closure is a first order property of $\mathcal{N}$, and in clause 3 above we do not need to require that Λ-iterates of $\mathcal{N}$ are internally Σ-closed as this is just a consequence of elementarity.

Secondly, we cannot in general hope to prove that generic interpretability holds for internally Σ-closed mice. The reason is that there might be $\mathcal{Q} \in B(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{Q}}$ is beyond the iteration strategy of $\mathcal{N}$ (in the sense that $\Lambda <_w \Sigma_{\mathcal{Q}}$), and if such a $\mathcal{Q}$ is generic over $\mathcal{N}$ then it is not wise to hope that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N}$ would be definable over $\mathcal{N}[\mathcal{Q}]$. In order to prove generic interpretability result for internally Σ-closed premice we need to find a *fullness condition* that would let us take care of examples as above. In particular, we seem to need to require that any $\Sigma_{\mathcal{Q}}$ as above is strictly below the strategy of $\mathcal{N}$. The next couple of paragraphs make this intuitive notion more precise.

Suppose $\mathcal{N}$ is an internally Σ-closed mouse, κ is an $\mathcal{N}$-cardinal and $\mathcal{M}$ is as in Definition 6.1.1. We then let $\mathcal{S}^{\mathcal{M}}_\eta$ be the $\Sigma^{\mathcal{M}}$-iterate of $\mathcal{P}$ constructed via the $(\mathcal{P}, \Sigma^{\mathcal{M}})$-coherent fully backgrounded construction where critical points of extenders used are $> \eta$. We let $\mathcal{U}^{\mathcal{M}}_\eta$ be the normal tree on $\mathcal{P}$ with last model $\mathcal{S}^{\mathcal{M}}_\eta$ and

$$\pi^{\mathcal{M}}_\eta = \begin{cases} \pi^{\mathcal{U}^{\mathcal{M}}_\eta, b} & : \mathcal{P} \text{ is of lsa type} \\ \pi^{\mathcal{U}^{\mathcal{M}}_\eta} & : \text{otherwise.} \end{cases}$$

Notice that $\pi^{\mathcal{M}}_{\eta} \in \mathcal{N}$.

Keeping the notation and terminology of Definition 6.1.1, suppose Λ is an iteration strategy for $\mathcal{N}$ (witnessing that $\mathcal{N}$ is an internally Σ-closed mouse). We then let $\Gamma(\mathcal{N}, \Lambda)$ be the collection of all sets $A \subseteq \mathbb{R}$ such that for some $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{N}, \Lambda)$, there are

1. an $\mathcal{R}$-cardinal κ,
2. $\mathcal{M} \trianglelefteq \mathcal{R}$ witnessing that $\mathcal{R}$ is internally Σ-closed at κ,
3. $\eta \in [\kappa, o(\mathcal{M}))$,
4. $\alpha < \lambda^{\mathcal{S}^{\mathcal{M}}_{\eta}} - 1$ and
5. $\mathcal{S}^{\mathcal{M}}_{\eta} \vDash$ "$\delta^{\mathcal{S}^{\mathcal{M}}_{\eta}}_{\alpha}$ is a Woodin cardinal"

such that

$$A \leq_w Code(\Sigma_{\mathcal{S}^{\mathcal{M}}_{\eta}(\alpha), \mathcal{U}^{\mathcal{M}}_{\eta}})$$

Remark 6.1.2 *For convenience, we will use the notation $\Gamma(\mathcal{P}, \Sigma)$ for both sts pairs and hod pairs. In the case of sts hod pairs, it is just $\Gamma^b(\mathcal{P}, \Sigma)$.*

Definition 6.1.3 *We then say that Λ is a fullness preserving iteration strategy for $\mathcal{N}$ if for every $\mathcal{N}$-cardinal η, letting Λ^{η} be the fragment of Λ that acts on stacks above η, $\Gamma(\mathcal{N}, \Lambda^{\eta}) = \Gamma(\mathcal{P}, \Sigma)$.*

The following is a useful lemma.

Lemma 6.1.4 *Suppose $(\mathcal{P}, \Sigma)$ is a non-meek hod pair and $\mathcal{N}$ is a Σ-closed mouse with a fullness preserving iteration strategy Λ. Fix an $\mathcal{N}$-cardinal κ and $\mathcal{M} \trianglelefteq \mathcal{N}$ such that $\mathcal{M}$ witnesses the internal closure of $\mathcal{N}$ at κ. Let $\eta \in [\kappa, o(\mathcal{M}))$ and let $\alpha < \lambda^{\mathcal{S}^{\mathcal{M}}_{\eta}} - 1$. Then there is an $\mathcal{N}$-cardinal $\nu > \eta$, $\mathcal{M}_1 \trianglelefteq \mathcal{N}$ witnessing the internal Σ-closure of $\mathcal{N}$ at ν and an increasing sequence of $\mathcal{M}_1$-cardinals $(\eta_i : i < \omega)$ such that letting $\eta_\omega = \sup_{i<\omega} \eta_i$ and $\mathcal{Q} = \mathcal{S}^{\mathcal{M}}_{\eta}$,*

1. *for every $i < \omega$, $Lp^{\Gamma(\mathcal{P},\Sigma), \Sigma_{\mathcal{Q}(\alpha)}}(\mathcal{M}_1|\eta_i) \vDash$ "η_i is a Woodin cardinal",*
2. *$Lp^{\Gamma(\mathcal{P},\Sigma), \Sigma_{\mathcal{Q}(\alpha)}}_{\omega}(\mathcal{M}_1|\eta_\omega) \in \mathcal{M}_1$, and*
3. *letting $\mathcal{S} = \mathcal{M}_1|(\eta^{+\omega}_{\omega})^{\mathcal{M}_1}$ and Φ be the fragment of $\Lambda_{\mathcal{S}}$ that acts on non-dropping trees that are above ν, whenever $\mathcal{R} \in I(\mathcal{S}, \Phi)$ and $\xi > \nu$ is a cardinal of $\mathcal{R}$, then $Lp^{\Gamma(\mathcal{P},\Sigma), \Sigma_{\mathcal{Q}(\alpha)}}(\mathcal{R}|\xi) \in \mathcal{R}$.*

Proof. Fix $\nu_0 > \eta$ such that there is $\mathcal{M}_1 \trianglelefteq \mathcal{N}$ witnessing the internal Σ-closure of $\mathcal{N}$ at ν_0 and such that for some $\nu \in [\nu_0, o(\mathcal{M}_1)]$ and some $\beta < \lambda^{\mathcal{S}_\nu^{\mathcal{M}_1}} - 1$,

(1) $Code(\Sigma_{\mathcal{Q}(\alpha+2)}) <_w Code(\Sigma_{\mathcal{S}_\nu^{\mathcal{M}_1}(\beta)})$ and $\mathcal{S}_\nu^{\mathcal{M}_1} \vDash$ "$\delta_\beta^{\mathcal{S}_\nu^{\mathcal{M}_1}}$ is a Woodin cardinal".

Fix ν satisfying (1). We claim that $\mathcal{M}_1$ is as desired. Clearly $\mathcal{M}_1$ witnesses the internal Σ-closure of $\mathcal{N}$ at ν. It is then enough to show that there is a sequence $(\eta_i : i \leq \omega)$ satisfying clause 1-3 above. Let $\delta = \delta_\beta^{\mathcal{S}_\nu^{\mathcal{M}_1}}$ and $\mathcal{R} = \mathcal{S}_\nu^{\mathcal{M}_1}$. Because δ is a Woodin cardinal inside $\mathcal{R}$, it follows from standard S-construction arguments (see [10, Proposition 3.39]) that

(2) $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\beta)}}(\mathcal{M}_1|\delta) \vDash$ "δ is a Woodin cardinal" and $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\beta)}}(\mathcal{M}_1|\delta) \in \mathcal{M}_1$.

Moreover, it follows from fullness preservation of Λ that

(3) the fragment of Λ acting on non-dropping stacks based on $\mathcal{M}_1|(\delta^+)^{\mathcal{M}_1}$ that are above ν is $(\Sigma_1^2(Code(\Sigma_{\mathcal{R}(\beta)})))^{\Gamma(\mathcal{P},\Sigma)}$-fullness preserving.

Next, notice that it follows from (1) that

(4) for some $\gamma < \beta$, $\mathcal{R}(\gamma)$ is a $\Sigma_{\mathcal{Q}(\alpha)}$-iterate of $\mathcal{Q}(\alpha)$.

It follows from (2) and (4) that if $\mathcal{K}$ is the output of the $\Sigma_{\mathcal{R}(\gamma)}$-fully backgrounded construction of $\mathcal{R}|\delta$ in which all extenders used have critical point $> \delta_{\beta-1}^{\mathcal{R}}$, then

(5) $\mathcal{K} \vDash$ "the least $< \delta$-strong cardinal is a limit of Woodin cardinals".

(5) is a standard fact. It can be proven as follows. First it follows from standard genericity iteration arguments that

(6) $\mathcal{R} \vDash \mathrm{cf}(Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{K})) \leq \delta_{\gamma+2}^{\mathcal{R}}$.

It follows from (1) that $\delta_{\gamma+2}^{\mathcal{R}} < \delta_\beta^{\mathcal{R}}$. Using (6), a Skolem hull argument and fullness preservation of $\Sigma_{\mathcal{R}(\gamma+2)}$, we get that

(7) there are unboundedly many $\xi < \delta$ such that $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{K}|\xi) \vDash$ "ξ is a Woodin

cardinal".[1]

It can be shown using S-constructions (see [10, Proposition 3.39]) and (7) that

(8) there are unboundedly many $\xi < \delta$ such that $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{M}_1|\xi) \vDash$ "ξ is a Woodin cardinal".

(5) easily follows from (8). Continuing with the proof, let $(\eta_i : i < \omega)$ be the first ω many cardinals of $\mathcal{K}$ such that for each i, $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{M}|\eta_i) \vDash$ "η_i is Woodin". Let $\eta_\omega = \sup_{i<\omega}\eta_i$. We claim that $(\eta_i : i < \omega)$ is as desired. It can be shown using S-constructions that

(9) for every $i < \omega$, $Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{M}_1|\eta_i) \vDash$ "η_i is a Woodin cardinal".

It also follows from (2) and (3) that

(10) the fragment of Λ acting on non-dropping stacks based on $\mathcal{M}_1|(\delta_\omega^+)^{\mathcal{M}_1}$ that are above ν is $(\Sigma_1^2(Code(\Sigma_{\mathcal{R}(\gamma)})))^{\Gamma(\mathcal{P},\Sigma)}$-fullness preserving.

It follows from the fact that the iteration embedding $\pi : \mathcal{Q}(\alpha) \to \mathcal{R}(\gamma)$ is in $\mathcal{M}_1$ and (10) that

(11) the fragment of Λ acting on non-dropping stacks based on $\mathcal{M}_1|(\delta_\omega^+)^{\mathcal{M}_1}$ that are above ν is $(\Sigma_1^2(Code(\Sigma_{\mathcal{Q}(\alpha)})))^{\Gamma(\mathcal{P},\Sigma)}$-fullness preserving.

(11) finishes the proof of lemma.

□

We will state our generic interpretability result for internally Σ-closed mice $\mathcal{N}$ that have a fullness preserving iteration strategy.

Theorem 6.1.5 *Suppose $(\mathcal{P}, \Sigma)$ is a non-meek hod pair , Γ is a pointclass and $\mathcal{N}$ is an internally Σ-closed premouse. Suppose Σ is strongly Γ-fullness preserving and has strong branch condensation. Then the following hold.*

1. *If $(\mathcal{P}, \Sigma)$ is a hod pair then for any $\mathcal{N}$-generic g, $\mathcal{N}[g]$ is Σ-closed and $\Sigma \restriction \mathcal{N}[g]$ is uniformly in g definable over $\mathcal{N}[g]$.*

[1] To prove (7), simply iterate $\mathcal{R}(\gamma+1)$ (above $\delta_\gamma^{\mathcal{R}}$) to make $\mathcal{K}$ generic. Then the Woodin cardinal has to be mapped to $o(Lp^{\Gamma(\mathcal{P},\Sigma),\Sigma_{\mathcal{R}(\gamma)}}(\mathcal{K}))$. Then take Skolem hulls that are transitive below $\delta_\beta^{\mathcal{R}}$ and contain the embedding coming from the aforementioned genericity iteration.

2. *If $(\mathcal{P}, \Sigma)$ is an sts hod pair and $\mathcal{N}$ has fullness preserving iteration strategy then for any $\mathcal{N}$-generic g, $\mathcal{N}[g]$ is Σ-closed and $\Sigma \restriction \mathcal{N}[g]$ is uniformly in g definable over $\mathcal{N}[g]$.*

In the next few sections, we will develop the terminology we need to prove Theorem 6.1.5. We will not give the proof of clause 1 of Theorem 6.1.5. It is much easier than the proof of clause 2 of Theorem 6.1.5 and it is very much like the proof of [10, Theorem 3.10]. Thus, we only concentrate on sts hod pairs.

6.2 Authentication procedure revisited

Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, $\mathcal{N}$ is an internally Σ-closed premouse, g is $\mathcal{N}$-generic and $\mathcal{T} \in dom(\Sigma^{stc}) \cap \mathcal{N}[g]$ is an irreducible tree on $\mathcal{P}$ above $\mathcal{P}^b$ such that $\mathcal{T}$ doesn't have fatal drops. Suppose first that $\mathcal{T} \in b(\Sigma^{stc})$. In this case, we would like to identify $\mathcal{Q}(b, \mathcal{T})$ in $\mathcal{N}[g]$ via a procedure that is uniform in $\mathcal{T}$. Here $b = \Sigma(\mathcal{T})$. Clearly if $\mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{M}^+(\mathcal{T})$ then we can easily identify $\mathcal{Q}(b, \mathcal{T})$. Suppose then $\mathcal{M}^+(\mathcal{T}) \triangleleft \mathcal{Q}(b, \mathcal{T})$. We now face two problems.

The first problem is showing that $\mathcal{Q}(b, \mathcal{T}) \in \mathcal{N}[g]$ and the second is showing that $\mathcal{Q}(b, \mathcal{T})$ can be identified by $\mathcal{N}$ in a uniform manner. Both of these require more of $\mathcal{N}$ than just internal Σ-closure. To prove both of these facts, we will need that $\mathcal{N}$ has a fullness preserving iteration strategy. Our strategy for finding $\mathcal{Q}(b, \mathcal{T})$ in $\mathcal{N}$ is that if $\mathcal{N}$ is sufficiently rich then some backgrounded construction will reach $\mathcal{Q}(b, \mathcal{T})$. To execute this plan, we first need to describe the sort of backgrounded constructions that we will consider. In what follows, we borrow ideas from Section 3.7. In particular, it will be helpful to recall Definition 3.7.3 and other definitions from that section.

Definition 6.2.1 ($(\mathcal{N}, X)$-authenticated iteration strategy) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, $X \subseteq \mathcal{P}^b$ and $\mathcal{N}$ is a Σ-sts premouse such that $X \in \mathcal{N}$. Suppose that $g \subseteq \mathbb{P}$ is $\mathcal{N}$-generic for some poset $\mathbb{P} \in \mathcal{N}$ and $\mathcal{R} \in \mathcal{N}[g]$ is an lsa type hod premouse. We define a partial short tree strategy $\Phi_{\mathcal{R}}^{\mathcal{N},X,g}$ without a model component for $\mathcal{R}$ as follows. $\Phi_{\mathcal{R}}^{\mathcal{N},X,g}$ acts on finite stacks of length 2.*

1. *$t = (\mathcal{R}_0, \mathcal{T}, \mathcal{R}_1, \vec{\mathcal{U}}) \in dom(\Phi_{\mathcal{R}}^{\mathcal{N},X,g}) \cap \mathcal{N}[g]$ if and only if t is $(\mathcal{P}, \Sigma^{\mathcal{N}}, X)$ authenticated.*

2. *Given $t = (\mathcal{R}_0, \mathcal{T}, \mathcal{R}_1, \vec{\mathcal{T}}) \in dom(\Phi_{\mathcal{R}}^{\mathcal{N},X,g}) \cap \mathcal{N}[g]$, $\Phi_{\mathcal{R}}^{\mathcal{N},X,g}(\vec{\mathcal{T}}) = b$ if and only if $t^\frown\{\mathcal{M}_b^t\}$ is $(\mathcal{P}, \Sigma^{\mathcal{N}}, X)$-authenticated, where $\mathcal{M}_b^t$ is the direct limit of models along b.*

When $X = \mathcal{P}^b$ we simply omit it from our terminology.

Continuing with the $\mathcal{R}, \mathcal{N}$ of Definition 6.2.1, we next define an $\mathcal{N}$-authenticated backgrounded construction over $\mathcal{R}$. This is essentially a fully backgrounded construction relative to $\Phi_{\mathcal{R}}^{\mathcal{N},g}$ (see Definition 4.2.1).

Definition 6.2.2 (($\mathcal{N}, X$)-authenticated backgrounded constructions) *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair, $X \subseteq \mathcal{P}^b \cap \mathcal{N}$ and $\mathcal{N}$ is a Σ-sts premouse such that $X \in \mathcal{N}$. Suppose that $g \subseteq \mathbb{P}$ is $\mathcal{N}$-generic for some poset $\mathbb{P} \in \mathcal{N}$ and $Y, \mathcal{R} \in \mathcal{N}[g]$ are such that Y is a self-well-ordered set and $\mathcal{R} \in Y$ is an lsa type hod premouse. Suppose further that κ is an $\mathcal{N}$-cardinals such that $\{\mathbb{P}, \mathcal{R}, Y\} \in \mathcal{N}|\kappa[g]$.*

We then say that $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ is the ηth initial segment of the output of the $(\mathcal{N}, X)$-authenticated fully backgrounded construction over Y based on $\mathcal{R}$ in which extenders used have critical points $> \kappa$ if $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ is the ηth initial segment of the output of the fully backgrounded construction of $\mathcal{N}$ over Y relative to $\Phi_{\mathcal{R}}^{\mathcal{N},X,g}$ in which all extenders used have critical points $> \kappa$.

Finally, we say $\mathcal{Q}$ is an $(\mathcal{N}, X)$-authenticated sts mouse over Y based on $\mathcal{R}$ if $\mathcal{Q} \in \mathcal{N}$ and for some ν, $\{\mathbb{P}, \mathcal{R}, Y, \mathcal{Q}\} \in \mathcal{N}|\nu[g]$ and $\mathcal{Q}$ appears as a model in the $(\mathcal{N}, X)$-authenticated fully backgrounded construction over Y based on $\mathcal{R}$ in which extenders used have critical points $> \nu$. When $X = \mathcal{P}^b$ we simply omit it from our terminology.

Suppose now that $(\mathcal{P}, \Sigma)$ is an sts hod pair, $X \subseteq \mathcal{P}^b$ and $\mathcal{N}$ is an internally Σ-closed mouse with a fullness preserving iteration strategy Λ such that $X \in \mathcal{N}$. We let

$$\begin{gathered} Lp^{\mathcal{N},X,sts}(Y, \mathcal{R}) = \bigcup\{\mathcal{Q} \in \mathcal{N}[g] : \text{there is an } \mathcal{N}\text{-cardinal } \kappa \text{ such that} \\ \{\mathbb{P}, \mathcal{R}, Y, \mathcal{Q}\} \in \mathcal{N}|\kappa[g] \text{ and an } \mathcal{M} \trianglelefteq \mathcal{N} \text{ witnessing that } \Lambda \text{ is fullness preserving at } \kappa \\ \text{such that } \mathcal{Q} \text{ is an } (\mathcal{M}, X)\text{-authenticated sound sts mouse over } Y \text{ based on } \mathcal{R} \text{ such} \\ \text{that } \rho(\mathcal{Q}) = o(Y)\} \end{gathered}$$

Again, if $X = \mathcal{P}^b$, then we omit it from the notation.

Notice that we do not know that $Lp^{\mathcal{N},X,sts}(Y, \mathcal{R})$ is a meaningful object, since we do not know that if $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are authenticated by $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively then they are compatible. This, however, is true when $\mathcal{R}$ is an iterate of $\mathcal{P}$ and Σ has strong branch condemnation and is strongly Γ-fullness preserving for some Γ (see, for instance, Corollary 5.5.2). This fact will also be verified in the next section.

We can then define $(Lp_\alpha^{\mathcal{N},X,sts}(Y, \mathcal{R}) : \alpha < o(\mathcal{N}))$ by induction as usual. More precisely, the sequence is defined via the following recursion.

1. $Lp_0^{\mathcal{N},X,sts}(Y,\mathcal{R}) = Lp^{\mathcal{N},Xsts}(Y,\mathcal{R})$.
2. $Lp_{\alpha+1}^{\mathcal{N},X,sts}(Y,\mathcal{R}) = Lp^{\mathcal{N},X,sts}(Lp_\alpha^{\mathcal{N},X,sts}(Y,\mathcal{R}))$.
3. $Lp_\lambda^{\mathcal{N},X,sts}(Y,\mathcal{R}) = \cup_{\alpha<\lambda} Lp_\alpha^{\mathcal{N},X,sts}(Y,\mathcal{R})$.

When $Y = \mathcal{R}$ or $X = \mathcal{P}^b$, we omit them from the above notation. We can now describe the $\mathcal{N}$-authenticated iterations of $\mathcal{P}$.

Definition 6.2.3 ($\mathcal{N}$-authenticated iteration) *Suppose $(\mathcal{P},\Sigma)$ is an sts pair, Γ is a pointclass and $\mathcal{N}$ is an internally Σ-closed mouse with a fullness preserving iteration strategy Λ. Suppose further that Σ has strong branch condensation and is strongly Γ-fullness preserving. Also suppose that $g \subseteq \mathbb{P}$ is $\mathcal{N}$-generic for some poset $\mathbb{P} \in \mathcal{N}$ and $\vec{\mathcal{T}} = (\mathcal{S}_i, \vec{\mathcal{T}}_i : i \leq m) \in \mathcal{N}[g]$ is a stack on $\mathcal{P}$. We say $\vec{\mathcal{T}}$ is $\mathcal{N}$-authenticated if the following conditions hold.*

1. *For every $i \leq m$, $\mathcal{S}_i$ is an lsa type hod premouse such that*
$$\mathcal{S}_i = Lp_\omega^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{S}_i|\delta^{\mathcal{S}_i})).$$
2. *For every $i < m$, $\pi^{\vec{\mathcal{T}}_i,b}$ exists.*
3. *For all cutpoints $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{S}},b}$ exists, letting $\mathcal{W}$ be the longest normal initial segment of $\vec{\mathcal{T}}_{\geq\mathcal{S}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}^b}$, for all limit ordinal $\gamma < lh(\mathcal{W})$ such that $\mathcal{W}\restriction\gamma$ is $\mathcal{N}$-ambiguous,*
   - (a) *if $Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{W}\restriction\gamma)) \vDash$ "$\delta(\mathcal{W}\restriction\gamma)$ is a Woodin cardinal" then $\mathcal{W}$ doesn't have a branch for $\mathcal{W}\restriction\gamma$ and $\mathcal{M}_\gamma^{\mathcal{W}} = \mathcal{S}_i$ for some $i \leq m$, and*
   - (b) *if $Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{W}\restriction\gamma)) \vDash$ "$\delta(\mathcal{W}\restriction\gamma)$ is not a Woodin cardinal" then $\mathcal{W}$ has a branch b for $\mathcal{W}\restriction\gamma$ such that $\mathcal{Q}(b,\mathcal{W}\restriction\gamma)$ exists and $\mathcal{Q}(b,\mathcal{W}\restriction\gamma) \trianglelefteq Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{W}\restriction\gamma))$.*
4. *For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{S}},b}$ exists, letting $\vec{\mathcal{U}}$ be the largest initial segment of $\vec{\mathcal{T}}$ based on $\mathcal{S}^b$, $(\mathcal{S}^b,\vec{\mathcal{U}})$ is an $\mathcal{N}$-authenticated iteration (see Definition 3.7.2).*
5. *For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{S}},b}$ exists, letting $\mathcal{U}$ be the longest normal initial segment of $\vec{\mathcal{T}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}^b}$ and is such that for some $\eta \in (\delta^{\mathcal{S}^b},\delta^{\mathcal{S}})$, $\mathcal{U}$ is based on $\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta}$ and is above η, then $(\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta},\mathcal{U})$ is an $\mathcal{N}$-authenticated iteration (see Definition 3.7.2).*

6. *For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{S}},b}$ exists and $\vec{\mathcal{T}}_{\geq\mathcal{S}}$ is a normal tree on $\mathcal{S}^b$ above $\delta^{\mathcal{S}^b}$, then $(\mathcal{S}^b, \vec{\mathcal{T}}_{\geq\mathcal{S}})$ is an $\mathcal{N}$-authenticated iteration (see Definition 3.7.2).*

7. *If for some $k < \omega$, $\vec{\mathcal{T}}_m$ is a normal tree $\mathcal{T}$ on some $\mathcal{Q} \trianglelefteq \mathcal{S}_m|((\delta^{\mathcal{S}_m})^{+k+1})^{\mathcal{S}_m}$ above $((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ such that $\rho(\mathcal{Q}) = ((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ then there is an $\mathcal{N}$-cardinal κ such that $\{\mathbb{P}, \vec{\mathcal{T}}\} \in \mathcal{N}|\kappa[g]$ and $\mathcal{M} \trianglelefteq \mathcal{N}$ such that letting $\mathcal{W} = \mathcal{S}^{\mathcal{M}}_{\kappa}$,*

   (a) $\mathcal{M}$ witnesses internal Σ-closure of $\mathcal{N}$ at κ,

   (b) for some $\beta < \lambda^{\mathcal{W}} - 1$, $\mathcal{Q}$ appears as a model in $\mathcal{M}|\delta^{\mathcal{W}}_{\beta+\omega}$-authenticated fully backgrounded construction over $\mathcal{S}_m|((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ in which extenders used have critical points $> \delta^{\mathcal{R}}_{\beta}$,

   (c) there is β as in clause 7.b such that letting $\mathcal{K} = \mathcal{M}|((\delta^{\mathcal{W}}_{\beta+\omega})^{+})^{\mathcal{W}}$, $\mathcal{K} \vDash$ "$\mathcal{Q}$ is $<Ord$-iterable above $((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ via a strategy Φ such that $\mathcal{T}$ is according to Φ and for every generic $h \subseteq Coll(\omega, <\delta^{\mathcal{W}}_{\beta+\omega})$, Φ has an extension $\Phi^+ \in D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h)$ such that $D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h) \vDash$ "Φ^+ is an ω_1-iteration strategy" and whenever $\mathcal{R} \in D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h)$ is a Φ^+-iterate of $\mathcal{Q}$ and $t \in \mathcal{R}$ is a stack on $\mathcal{M}^+(\mathcal{S}_m|\delta^{\mathcal{S}_m})$ of length 2 then t is $(\mathcal{P}, \Sigma^{\mathcal{K}})$-authenticated".

6.3 Generic interpretability in internally Σ-closed premice

In this section, we prove our main theorem, Theorem 6.1.5. As we said before, we will only prove clause 2. We start by fixing an sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ has strong branch condensation, a pointclass Γ such that Σ is strongly Γ-fullness preserving and an internally Σ-closed premouse $\mathcal{N}$ such that $\mathcal{N}$ has a fullness preserving iteration strategy Λ. We want to show that $\mathcal{N}$ is uniformly generically Σ-closed.

Fix a poset $\mathbb{P} \in \mathcal{N}$ and an $\mathcal{N}$-generic $g \subseteq \mathbb{P}$. We start by defining a short tree iteration strategy Φ for $\mathcal{P}$. Φ will be defined over $\mathcal{N}[g]$ in a uniform manner. Its domain consists of $\mathcal{N}$-authenticated iterations (see Definition 6.2.3). Given an $\mathcal{N}$-authenticated iteration $\vec{\mathcal{T}} = (\mathcal{S}_i, \vec{\mathcal{T}}_i : i \leq m) \in \mathcal{N}[g]$ of limit length, we set $\Phi(\vec{\mathcal{T}}) = x$ if and only if the following conditions hold.

1. There is a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq\mathcal{S}},b}$ exists, $\vec{\mathcal{T}}_{\geq\mathcal{S}}$ is a normal tree on $\mathcal{S}$ above $\mathcal{S}^b$, $Lp^{\mathcal{N},sts}(\vec{\mathcal{T}}_{\geq\mathcal{S}}) \vDash$ "$\delta(\vec{\mathcal{T}}_{\geq\mathcal{S}})$ is a Woodin cardinal" and $x = Lp^{\mathcal{N},sts}_{\omega}(\vec{\mathcal{T}}_{\geq\mathcal{S}})$.

2. There is no cutpoint $\mathcal{S}$ as in clause 1, $x \in \mathcal{N}$ is a branch of $\vec{\mathcal{T}}_m$ such that $\mathcal{N} \vDash$ "x is a cofinal well-founded branch of $\vec{\mathcal{T}}$" and $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}$ is $\mathcal{N}$-authenticated.

To complete the proof of Theorem 6.1.5 we need to show that

(1) whenever $\vec{\mathcal{T}} \in dom(\Phi) \cap dom(\Sigma)$, $\Phi(\vec{\mathcal{T}})$ is defined and is equal to $\Sigma(\vec{\mathcal{T}})$.

Fix then $\vec{\mathcal{T}} = (\mathcal{S}_i, \vec{\mathcal{T}}_i : i \leq m) \in \mathcal{N}[g]$ such that $\vec{\mathcal{T}} \in dom(\Phi) \cap dom(\Sigma)$. Suppose first that clause 1 in the definition of Φ holds. Fix then a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{S}$ above $\delta^{\mathcal{S}^b}$ such that $Lp^{\mathcal{N},sts}(\vec{\mathcal{T}}_{\geq \mathcal{S}}) \vDash$ "$\delta(\vec{\mathcal{T}}_{\geq \mathcal{S}})$ is a Woodin cardinal". Let $\mathcal{T} = \vec{\mathcal{T}}_{\geq \mathcal{S}}$. We need to show that

(2) $Lp_\omega^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{M}^+(\mathcal{T})}}(\mathcal{M}^+(\mathcal{T})) = Lp_\omega^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$

We prove notationally less cumbersome version of (2) and leave the full proof of (2), which is only notationally more complicated, to the reader. The following is what we will prove.

(3) $Lp^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{M}^+(\mathcal{T})}}(\mathcal{M}^+(\mathcal{T})) = Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$

Towards proving (3), let $\mathcal{W} \trianglelefteq Lp^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{M}^+(\mathcal{T})}}(\mathcal{M}^+(\mathcal{T}))$ be such that $\rho(\mathcal{W}) = \delta(\mathcal{T})$. We want to show that

Claim 1. $\mathcal{W} \trianglelefteq Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$.

Proof. Recall from Definition 3.9.7 that $\mathcal{W}$ has a strategy in $\Psi \in \Gamma^b(\mathcal{P},\Sigma)$ witnessing that $\mathcal{W}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$. Let κ be an $\mathcal{N}$-cardinal such that $\{\mathbb{P}, \vec{\mathcal{T}}\} \in \mathcal{N}|\kappa[g]$. Using fullness preservation of Λ, fix an iteration tree $\mathcal{U}$ on $\mathcal{N}$ above κ and according to Λ with last model $\mathcal{N}_1$ such that $\pi^{\mathcal{U}}$ exists and there is an $\mathcal{M} \trianglelefteq \mathcal{N}_1$ such that

1. $\mathcal{M}$ witnesses internal Σ-closure of $\mathcal{N}_1$ at κ and
2. for some $\alpha < \lambda^{(\mathcal{S}_\kappa^{\mathcal{M}})^b}$, $\Gamma(\mathcal{W},\Psi), Code(\Psi) <_w Code(\Sigma_{\mathcal{S}_\kappa^{\mathcal{M}}(\alpha)})$.

Fix a real x that witnesses that $Code(\Psi) <_w Code(\Sigma_{\mathcal{S}_\kappa^{\mathcal{M}}(\alpha)})$. Let ν, $\mathcal{M}_1$ and $(\eta_i : i \leq \omega)$ be as in Lemma 6.1.4 applied to $\mathcal{N}_1$, $\mathcal{M}$, α and κ (we take $\eta = \kappa$). Let Φ be the fragment of $\Lambda_{\mathcal{N}_1,\mathcal{U}}$ that acts on non-dropping trees that are above ν. Recall from Definition 6.1.3, Φ is $\Gamma(\mathcal{P},\Sigma)$-fullness preserving.

Let $\mathcal{U}_1$ be an iteration tree on $\mathcal{M}_1|(\eta_\omega^+)^{\mathcal{M}_1}$ based on $\mathcal{M}_1|\eta_0$ according to Φ and above ν that is constructed according to the rules of x-genericity iteration. Let $\pi = \pi^{\mathcal{U}_1}$ and let $\mathcal{M}_2$ be the last model of $\mathcal{U}_1$. Then we have that x is generic for the extender algebra of $\mathcal{M}_2$ at $\pi(\eta_0)$. It follows that

(4) $\Psi \restriction \mathcal{M}_2|\pi(\eta_\omega)[g][x] \in \mathcal{M}_2$.

Finally, let $\mathcal{S}$ be the output of the $\mathcal{M}_2$-authenticated fully backgrounded construction over $\mathcal{M}^+(\mathcal{T})$ done inside $\mathcal{M}_2|\pi(\eta_1)[g][x]$ using extenders with critical points above $\pi(\eta_0)$. Next we make the following assumption.

(5) Coiteration of $\mathcal{W}$ and the construction producing $\mathcal{S}$ halts.

We then compare $\mathcal{W}$ with $\mathcal{S}$. It follows from (4) that $\mathcal{S}$ doesn't move. We then have two cases. Suppose first that $\mathcal{W}$ side loses. It follows that $\mathcal{W} \trianglelefteq \mathcal{S}$. By elementarity of π, it follows that $\mathcal{W} \trianglelefteq \pi^{-1}(\mathcal{S})$, and further, using elementarty of $\pi^{\mathcal{U}}$, we conclude that $\mathcal{W} \trianglelefteq Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$.

Suppose next that $\mathcal{W}$ wins the coiteration with $\mathcal{S}$. Let $\mathcal{U}_2$ be a tree on $\mathcal{W}$ such that $\mathcal{M}(\mathcal{U}_2) = \mathcal{S}$. Let $e = \Psi(\mathcal{U}_2)$. It follows from (4) that $e \in \mathcal{M}_2[g][x]$, contradicting universality of $\mathcal{S}$.

It follows that it is enough to show that (5) holds. Suppose then (5) fails. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ be the models of the construction producing $\mathcal{S}$. Since (5) fails, we must have that there is an iteration tree $\mathcal{K}$ on $\mathcal{W}$ according to Ψ with last model $\mathcal{K}_1$ such that for some β and $\gamma < \eta$, $\mathcal{K}_1|\beta = \mathcal{N}_\gamma|\beta$, $\mathcal{K}_1||\beta \neq \mathcal{N}_\gamma||\beta$ and $\beta \notin dom(\vec{E}^{\mathcal{K}_1})$.

It follows that there is a stack $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{T}_0, \mathcal{Q}, \vec{\mathcal{U}}) \in \mathcal{K}_1|\beta$ of length 2 such that either the $\mathcal{K}_1$ side or the $\mathcal{N}_\gamma$ side has a branch of t indexed at β. Notice that it follows from sts indexing scheme that it is not the case that $\vec{\mathcal{U}} = \emptyset$ and $\mathcal{T}_0$ is $\mathcal{K}_1|\beta$-ambiguous. Indeed, suppose that $\vec{\mathcal{U}} = \emptyset$ and $\mathcal{T}_0$ is $\mathcal{K}_1|\beta$-ambiguous. But then the branch indexed at β, either in $\mathcal{K}_1$ or in $\mathcal{N}_\gamma$, depends only on $\mathcal{K}_1|\beta = \mathcal{N}_\gamma|\beta$. Hence, these two branches have to be the same.

We thus have that $\vec{\mathcal{U}} \neq \emptyset$. Notice that, because $Code(\Sigma_{\mathcal{Q}^b}) <_w Code(\Sigma_{\mathcal{S}_\kappa^{\mathcal{M}}(\alpha)})$, we have that $(\mathcal{Q}^b, \vec{\mathcal{U}})$ is indeed $\mathcal{M}_2$-authenticated iteration. It follows that $\mathcal{N}_\gamma$ side must have a branch of t indexed at β. Because $\mathcal{K}_1$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse, it follows that $\mathcal{K}_1$ also has a branch of t indexed at β. Moreover, because $(\mathcal{Q}^b, \vec{\mathcal{U}})$ is $\mathcal{M}_2$-authenticated iteration, it follows that the branch on the $\mathcal{N}_\gamma$ side is $\Sigma_{\mathcal{M}^+(\mathcal{T})}(t)$, which is exactly the same branch on $\mathcal{K}_1$-side. $\square$

Claim 1 implies that $Lp^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{M}^+(\mathcal{T})}}(\mathcal{M}^+(\mathcal{T})) \trianglelefteq Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$. Let now $\mathcal{W} \trianglelefteq Lp^{\mathcal{N},sts}(\mathcal{M}^+(\mathcal{T}))$ be such that $\rho(\mathcal{W}) = \delta(\mathcal{T})$. We want to show that

Claim 2. $\mathcal{W} \trianglelefteq Lp^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{M}^+(\mathcal{T})}}(\mathcal{M}^+(\mathcal{T}))$.

Proof. To prove the claim, we need to show that $\mathcal{W}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-mouse over $\mathcal{M}^+(\mathcal{T})$. Let $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta))$ be the models of $\mathcal{N}$-certified fully backgrounded construction over $\mathcal{M}^+(\mathcal{T})$. It is enough to show that for every γ, $\mathcal{N}_\gamma$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-mouse.

Fix $\gamma \leq \eta$ and let Ψ be the strategy of $\mathcal{N}_\gamma$ inherited from Λ. Let $\vec{\mathcal{U}}$ be a stack according to Ψ on $\mathcal{N}_\gamma$ whose last normal component has a limit length and let $\vec{\mathcal{U}}^+$ be the resurrection of $\vec{\mathcal{U}}$ onto $\mathcal{N}$. Let $e = \Lambda(\vec{\mathcal{U}}^+)$. We then have $\sigma : \mathcal{M}_e^{\vec{\mathcal{U}}} \rightarrow \pi_e^{\vec{\mathcal{U}}^+}(\mathcal{N}_\gamma)$. It follows from hull condensation of $\Sigma_{\mathcal{M}^+(\mathcal{T})}$ that to show that $\mathcal{M}_e^{\vec{\mathcal{U}}}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$, it is enough to show that $\pi_e^{\vec{\mathcal{U}}^+}(\mathcal{N}_\gamma)$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$. In what follows we will show that $\mathcal{N}_\gamma$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$. The same proof also would show that $\pi_e^{\vec{\mathcal{U}}^+}(\mathcal{N}_\gamma)$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$.

Suppose $\mathcal{N}_\gamma$ is not a $\Sigma_{\mathcal{M}^+(\mathcal{T})}$-sts mouse over $\mathcal{M}^+(\mathcal{T})$. This can happen in two ways. Either we indexed a wrong branch or we skipped an iteration. We now investigate both of these cases.

Suppose first that we indexed a wrong branch in $\mathcal{N}_\gamma$. It follows from hull condensation of Σ that there is $\xi \leq \gamma$ such that we indexed branch b at $\mathcal{M}_\xi$ and b is not according to Σ. Suppose first that $\mathcal{M}_\xi$ is ambiguous and let $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{T}_0, \mathcal{Q}, \vec{\mathcal{U}}) \in \mathcal{M}_\xi$ be the least stack of length 2 witnessing this. Because of our minimality assumption, we have that t is according to $\Sigma_{\mathcal{M}^+(\mathcal{T})}$. Let $\alpha < \lambda^{\mathcal{Q}^b}$ be such that $\vec{\mathcal{U}}$ is based on $\mathcal{Q}(\alpha)$ and let e be the branch indexed in $\mathcal{M}_\xi$. Because $(\mathcal{Q}(\alpha), \vec{\mathcal{U}}^\frown\{\mathcal{M}_e^{\vec{\mathcal{U}}}\})$ is an $\mathcal{N}$-authenticated iteration, it follows from Lemma 5.5.2 that $e = \Sigma_{\mathcal{M}^+(\mathcal{T})}(t)$, contradiction!

Next suppose that $\mathcal{M}_\xi$ is unambiguous. It follows that the branch indexed at β is a branch for an $\mathcal{M}_\xi$-ambiguous tree $\mathcal{T}_0$. It follows that there is a triple (ν, ϕ, d) that is a $\mathcal{M}_\xi$-shortness witness for $\mathcal{T}_0$. Let $\mathcal{Q} = \mathcal{Q}(d, \mathcal{T}_0)$. It follows from Lemma 5.5.2 that $\mathcal{Q}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{T}_0)}$-sts mouse over $\mathcal{M}^+(\mathcal{T}_0)$, and hence, $d = \Sigma_{\mathcal{M}^+(\mathcal{T})}(\mathcal{T}_0)$.

It remains to show that we never skip iterations. To show this, it is enough to show that if $\mathcal{M}_\xi$ is an ambiguous level of the construction and $t \in \mathcal{M}_\xi$ is the least finite stack of length 2 witnessing ambiguity of $\mathcal{M}_\xi$, then $\Phi^{\mathcal{N},g}_{\mathcal{M}^+(\mathcal{T})}(t)$ is defined. Let $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{T}_0, \mathcal{Q}, \vec{\mathcal{U}})$. It is enough to show that $\Sigma_{\mathcal{Q}^b} \restriction \mathcal{N}[g] \in \mathcal{N}[g]$. To see this, we claim that

(6) for every ζ such that $t \in \mathcal{N}|\zeta[g]$ there is an $\mathcal{M} \trianglelefteq \mathcal{N}$ witnessing the internal Σ-closure of $\mathcal{N}$ at ζ and such that $\mathcal{Q}^b$ is $\mathcal{M}$-authenticated.

Fix then a ζ as in (6). Using fullness preservation of Λ, find a tree $\mathcal{U}$ on $\mathcal{N}$ according to Λ with last model $\mathcal{N}_1$ such that $\mathcal{U}$ is above ζ and there is an $\mathcal{M} \trianglelefteq \mathcal{N}_1$ witnessing internal Σ-closure at ζ and such that for some $\nu < \lambda^{\mathcal{S}_\zeta^{\mathcal{M}}}$, $\Sigma_{\mathcal{Q}^b} <_w \Sigma_{\mathcal{S}_\zeta^{\mathcal{M}}(\nu)}$. It then follows that $\mathcal{Q}^b$ is $\mathcal{M}$-authenticated. By elementarity, $\mathcal{Q}^b$ is $\mathcal{N}$-authenticated. $\square$

Claim 1 and claim 2 finish the proof of (1) provided clause 1 in the definition of Φ holds. We now consider clause 2, which is done by considering all the clauses of Definition 6.2.3. The rest of the proof is very similar to the proof given above, and so we will only outline it. Let $b = \Sigma(\vec{\mathcal{T}})$. Suppose

(7) there is a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such is $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists, $\mathcal{W} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a normal tree on $\mathcal{S}$ above $\mathcal{S}^b$ without fatal drops, and $\mathcal{Q}(b, \mathcal{W})$ exists.

We then claim that $b \in \mathcal{N}[g]$ and $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ satisfies clause 3.b of Definition 6.2.3. We have two cases, either $\mathcal{Q}(b, \mathcal{W}) \trianglelefteq \mathcal{M}^+(\mathcal{W})$ or $\mathcal{Q}(b, \mathcal{W})$ is a $\Sigma_{\mathcal{M}^+(\mathcal{W})}$-sts mouse over $\mathcal{M}^+(\mathcal{W})$. The first case is trivial. In the second case we can use the proof of Claim 2 to show that $\mathcal{Q}(b, \mathcal{W}) \in \mathcal{N}[g]$ and $\mathcal{Q}(b, \mathcal{W}) \trianglelefteq Lp^{\mathcal{N}, sts}(\mathcal{M}^+(\mathcal{W}))$.

Suppose next that

(8) there is a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists and $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a stack on $\mathcal{S}^b$.

In this case, we can use fullness preservation to show that $\mathcal{S}^b$ is $\mathcal{N}$-authenticated. The proof is like the last two paragraphs of the proof of Claim 1. There we showed that $\mathcal{Q}^b$ is $\mathcal{N}$-authenticated. To show that $b \in \mathcal{N}[g]$ use Lemma 6.1.4 as it was used in the proof of Claim 1 to show that $\mathcal{W} \in \mathcal{N}[g]$. Then it follows that $(\mathcal{S}^b, \vec{\mathcal{T}}_{\geq \mathcal{S}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\})$ is an $\mathcal{N}$-authenticated iteration.

Suppose now that

(9) there is a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists and for some $\mathcal{S}$-cutpoint $\eta \in (\delta^{\mathcal{S}^b}, \delta^{\mathcal{S}})$, $\mathcal{W} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{S}}$ is a normal tree based on $\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta}$.

Let $E_\alpha \in \vec{E}^{\mathcal{S}}$ be the least extender with critical point $\delta^{\mathcal{S}^b}$ such that $\nu(E) \geq o(\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta})$. Let $\mathcal{Q} = Ult(\mathcal{S}, E)(\lambda^{\mathcal{S}^b}+2)$. Using the proof of (8) we can show that $(\mathcal{Q}, \mathcal{W}^\frown\{\mathcal{M}_b^{\mathcal{W}}\})$ is an $\mathcal{N}$-authenticated iteration.

Suppose next that

(10) there is a cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists and letting $\mathcal{W} = \vec{\mathcal{T}}_{\geq \mathcal{S}}$, $\mathcal{W}$

is a normal tree on $\mathcal{S}^b$ above $\delta^{\mathcal{S}^b}$.

Again, applying the proof of (8), we can show that $(\mathcal{S}^b, \mathcal{W}^\frown\{\mathcal{M}_b^{\mathcal{W}}\})$ is an $\mathcal{N}$-authenticated iteration.

Suppose finally that

(11) for some $k < \omega$, $\mathcal{T} =_{def} \vec{\mathcal{T}}_m$ is a normal tree on some $\mathcal{Q} \trianglelefteq \mathcal{S}_m|((\delta^{\mathcal{S}_m})^{+k+1})^{\mathcal{S}_m}$ above $((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ such that $\rho(\mathcal{Q}) = ((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$.

What we need to show now is that there is an triple $(\mathcal{M}, \kappa, \beta)$ satisfying clause 7.a and 7.c of Definition 6.2.3.

Using the proof of Claim 1, we can find $(\mathcal{M}, \kappa)$ such that, letting Ψ be a strategy witnessing that $\mathcal{Q}$ is a $\Sigma_{\mathcal{M}^+(\mathcal{S}_m|\delta^{\mathcal{S}_m})}$-sts mouse,

1. κ is an $\mathcal{N}$-cardinal such that $\vec{\mathcal{T}} \in \mathcal{N}|\kappa[g]$,
2. $\mathcal{M}$ witnesses the internal Σ-closure of $\mathcal{N}$ at κ,
3. $\mathcal{Q} \trianglelefteq Lp^{\mathcal{M},sts}(((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}, \mathcal{M}^+(\mathcal{S}_m|\delta^{\mathcal{S}_m}))$,
4. for some $\beta < \lambda^{\mathcal{S}_\kappa^{\mathcal{M}}}$, $Code(\Psi) <_w Code(\mathcal{S}_\kappa^{\mathcal{M}}(\beta))$.

We claim $(\mathcal{M}, \kappa, \beta)$ are as desired. To see this we need to show that

(12) letting $\mathcal{W} = \mathcal{S}_\kappa^{\mathcal{M}}$ and $\mathcal{K} = \mathcal{M}|((\delta^{\mathcal{W}}_{\beta+\omega})^+)^{\mathcal{W}}$, $\mathcal{K} \vDash$ "$\mathcal{Q}$ is $< Ord$-iterable above $((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ via a strategy Φ such that $\mathcal{T}$ is according to Φ and for every generic $h \subseteq Coll(\omega, < \delta^{\mathcal{W}}_{\beta+\omega})$, Φ has an extension $\Phi^+ \in D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h)$ such that $D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h) \vDash$ "Φ^+ is an ω_1-iteration strategy" and whenever $\mathcal{R} \in D(\mathcal{K}, \delta^{\mathcal{W}}_{\beta+\omega}, h)$ is a Φ^+-iterate of $\mathcal{Q}$ and $t \in \mathcal{R}$ is a stack on $\mathcal{M}^+(\mathcal{S}_m|\delta^{\mathcal{S}_m})$ of length 2 then t is $(\mathcal{P}, \Sigma^{\mathcal{K}})$-authenticated".

In what follows, we show how to obtain the strategy Φ and its extension Φ^+. The rest of the proof is like the proof of the previous cases, and so we will leave it to the reader.

For $i \leq \omega$, let $\eta_i = \delta^{\mathcal{W}}_{\beta+i+2}$. It follows from the proof of Lemma 6.1.4 that

(13) if $\mathcal{M}_1 = \mathcal{M}|(\eta_\omega^+)^{\mathcal{M}}$ and Δ is the strategy of $\mathcal{M}_1$ that acts on non-dropping trees above $\delta^{\mathcal{W}}_{\beta+2}$ then whenever $\mathcal{M}_2$ is a Δ-iterate of $\mathcal{M}_1$ and $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is an iteration embedding then $Lp^{\Gamma^b(\mathcal{P},\Sigma),\Sigma_{\mathcal{W}(\beta)}}(\mathcal{M}_2|\pi(\eta_\omega)) \subseteq \mathcal{M}_2$.

Using the proof of Claim 1 we can show that

(14) for every $i \geq 1$, $\mathcal{Q}$ can be constructed via $\mathcal{M}_1|\eta_i$-authenticated background construction over $\mathcal{S}_m|((\delta^{\mathcal{S}_m})^{+k})^{\mathcal{S}_m}$ based on $\mathcal{M}^+(\mathcal{S}_m|\delta^{\mathcal{S}_m})$ and using extenders with critical point $> \eta_{i-1}$.

(13) is needed to prove (14). In that proof, (13) is used instead of fullness preservation of Λ. It follows from (13) that $\mathcal{Q}$ has an iteration strategy $\Phi \in \mathcal{M}_1$. To show that Φ has the desired properties it is enough to show that

(15) For every $\mathcal{M}_1$-generic $h \subseteq Coll(\omega, < \eta_\omega)$, if $\sigma = \bigcup_{\xi < \eta_\omega} \mathbb{R}^{\mathcal{M}_1[h \cap Coll(\omega, <\xi)]}$ then for every $i < \omega$, $\Sigma_{\mathcal{W}(\beta+i+2)} \restriction (\mathcal{M}_1|\eta_\omega(\sigma)) \in \mathcal{M}_1(\sigma)$.

(15) follows from Theorem 5.2.5.

6.4 S-constructions

Our definition of sts mice makes heavy use of the fact that the set X is a self-well-ordered set. In particular, our definition cannot be used to define sts mice over $\mathbb{R}$. Another shortcoming of our definition is that it does not explain how to do S-constructions. In this short section, motivated by Section 3.38 of [10], we indicate how to use Theorem 6.1.5 to redefine hod mice in a way that one can define sts mice over $\mathbb{R}$ and perform S-constructions.

Recall the difficulty with defining hybrid mice over $\mathbb{R}$. In our definition, we always choose the least stack of some sort for which the branch has not been added and index a branch. Since $\mathbb{R}$ may not be self-well-ordered, we do not have the luxury of choosing the least such stack.

The problem with S-constructions is very similar. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair and N and M are two transitive models of some fragment of set theory such that $\mathcal{J}_\omega(M) \subseteq \mathcal{J}_\omega(N)$ and for some poset $\mathbb{P} \in \mathcal{J}_\omega(M)$ and some $\mathbb{P}/M$-generic G, $\mathcal{J}_\omega(N) = \mathcal{J}_\omega(M)[G]$. Suppose further that both M and N are Σ-closed and $\mathcal{P} \in N \cap M$. For us, S-constructions are constructions that translate Σ-mice over N to Σ-mice over M. For more details consult Section 3.38 of [10].[2]

The difficulty in performing S-constructions is the following. Suppose $\mathcal{N}$ is a Σ-mouse over N, and we want to translate it onto a Σ-mouse over M. Suppose our translation has produced a Σ-mouse $\mathcal{M}$ over M, and our indexing scheme demands that a branch of some stack $\vec{\mathcal{T}} \in \mathcal{N}$ be indexed in the very next step in the translation

[2] In [18], this process is called P-constructions.

procedure. The problem is that $\vec{\mathcal{T}}$ may not be a stack in $\mathcal{M}$ nor may it be the stack whose branch is indexed in $\mathcal{M}$.

To solve this problem, we changed the definition of hybrid premouse in a way that the iterations whose branches are indexed do not depend on generic extensions. In particular, instead of indexing iterations according to Σ, we considered generic genericity iterations on $\mathcal{M}_1^{\#,\Sigma}$. Such iteration make levels of the model generically generic and do not depend on generic extensions. This move solves both problems. In the first case what is important is that the indexed iterations do not depend on the well-ordering of the model, and in the second case what is important is that the indexed iterations do not depend on generic extensions. For more details consult Definition 3.37 of [10] or [20] for a similar construction.

Here our solution is similar. Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair and $\mathcal{M}$ is an Σ-sts mouse over some set X such that $\mathcal{P} \in X$. Then the iterations of $\mathcal{P}$ that are indexed in $\mathcal{M}$ are of the form $t = (\mathcal{P}, \mathcal{T}, \mathcal{Q}, \vec{\mathcal{U}})$, where t is a stack on $\mathcal{P}$ of length 2. $\mathcal{T}$ is always the result of comparing $\mathcal{P}$ with a certain backgrounded construction. Notice that this neither depends on the well-ordering of $\mathcal{M}$ nor on small generic extensions. $\vec{\mathcal{U}}$ is a stack on $\mathcal{Q}^b$ and, in Definition 3.8.2, we chose the least such stack. Thus the choice of $\vec{\mathcal{U}}$ depends on both the well-ordering of $\mathcal{M}$ and small generic extensions (small in the sense that the generic is smaller than the critical point of the first background extender used in the construction). To solve the issue, we will start considering stacks $s = (\mathcal{P}, \mathcal{T}, \mathcal{Q}, \mathcal{U})$ where $\mathcal{T}$ is as before but now $\mathcal{U}$ is a generic genericity iteration on $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$ to make a level of the model generically generic. We only consider such generic genericity iterations of $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$ that are based on the first Woodin of $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$.

The reason we choose $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$ is that we want to use clause 1 of Theorem 6.1.5. It is not hard to see that if $\delta_0 < \delta_1$ are the first two Woodin cardinals of $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$ and $g \subseteq Coll(\omega, \delta_0)$ then $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}|\delta_1[g]$ is internally $\Sigma_{\mathcal{Q}^b}$-closed. Clause 1 of Theorem 6.1.5 is a weaker result than [10, Lemma 3.35], which is what is used to reorganize hod mice in [10]. We could prove an equivalent of [10, Lemma 3.35], but doing this is much harder than proving clause 1 of Theorem 6.1.5.

To show that the resulting structure $\mathcal{M}$ is closed under Σ, we will first show that we can find branches of stacks of length 2. Given such a stack $t = (\mathcal{P}, \mathcal{T}, \mathcal{Q}, \vec{\mathcal{U}})$ let $\mathcal{W}$ be an iteration of $\mathcal{M}_2^{\#,\Sigma_{\mathcal{Q}^b}}$ such that $(\mathcal{P}, \mathcal{T}, \mathcal{Q}, \mathcal{W})$ is indexed in $\mathcal{M}$ and if $\mathcal{S}$ is the last model of $\mathcal{W}$ then $\vec{\mathcal{U}}$ is generic over $\mathcal{S}$ for $\mathbb{B}_\delta^{\mathcal{S}}$ where δ is the least Woodin of $\mathcal{S}$ and $\mathbb{B}_\delta^{\mathcal{S}}$ is the extender algebra of $\mathcal{S}$ at δ. It then follows from Theorem 6.1.5 that $\Sigma_{\mathcal{Q}^b} \restriction \mathcal{S}|\eta[\vec{\mathcal{U}}] \in \mathcal{S}$ where η is the second Woodin cardinal of $\mathcal{S}$. The rest of the proof is just repeating the proof of Theorem 6.1.5. We start by redefining what an sts indexing scheme is.

Definition 6.4.1 (Revised unambiguous sp) *Suppose $\mathcal{M}$ is an sp over some self-well-ordered set X based on hod-like lsa type lsp $\mathcal{P}$. We say $\mathcal{M}$ is revised unambiguous if $\mathcal{M}$ is closed under sharps and whenever $t = (\mathcal{P}_0, \mathcal{T}_0, \mathcal{P}_1, \mathcal{U}) \in \mathcal{M}$ is according to $\Sigma^{\mathcal{M}}$ such that either*

1. *$\mathcal{U} = \emptyset$ and $\mathcal{M} \vDash$ "$\mathcal{T}_0$ is an unambiguous tree of limit length" or*

2. *$\mathcal{U}$ is a nonempty stack of limit length*

then $t \in \mathrm{dom}(\Sigma^{\mathcal{M}})$. We say $\mathcal{M}$ is revised ambiguous if it is not revised unambiguous.

Definition 6.4.2 (ϕ-sts indexing scheme revisited) *Suppose $\psi(x)$ and $\phi(x, y)$ are two formulas in the language of sp. We say ψ is a ϕ-sts indexing scheme for ϕ if whenever X is a self-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp and $\mathcal{N}$ is an sp over X based on $\mathcal{P}$ then $\mathcal{N} \vDash \psi[c]$ if and only if*

1. *$\mathcal{N}$ is closed under sharps,*

2. *$\mathcal{N} \vDash$ "$\Sigma^{\mathcal{N}}$ is a partial faithful short tree strategy without model component",*

3. *for some finite sequence $t = (\mathcal{P}, \mathcal{T}, \mathcal{P}_1, \mathcal{U}) \in \mathcal{N}$ such that*

   (a) *t is according to $\Sigma^{\mathcal{N}}$ and $\Sigma^{\mathcal{N}}(t)$ is undefined,*

   (b) *there is (ν, ξ) such that letting $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \eta), (F_\gamma : \gamma < \eta), (\mathcal{T}_\gamma : \gamma < \eta))$ be the output of the $(\mathcal{P}, \Sigma^{\mathcal{N}})$-coherent fully backgrounded construction of $\mathcal{N}$ in which extenders used have critical points $> \nu$ (see Definition 3.5.1), $\mathcal{T}_\xi = \mathcal{T}$,*

   (c) *either $\pi^{\mathcal{T}}$ exists and $\mathcal{P}_1$ is the last model of $\mathcal{T}$ or $\pi^{\mathcal{T},b}$ exists, $\mathcal{T}$ is $\mathcal{N}$-ambiguous and $\mathcal{P}_1 = \mathcal{M}^+(\mathcal{T})$,*

   (d) *for some $\gamma < o(\mathcal{N})$ such that $\mathcal{N}|\gamma \vDash \mathsf{ZF}$, letting $\mathcal{M} \trianglelefteq \mathcal{P}_1$ be $\mathcal{M}_2^{\#,(\Sigma_{\mathcal{P}_1^b})^{\mathcal{P}_1}}$, $\mathcal{U}$ is build according to the rules of the $\mathcal{N}|\gamma$-generic genericity iteration of $\mathcal{M}|\delta$ where δ is the least Woodin cardinal of $\mathcal{M}$,*

   (e) *if $\mathcal{N}$ is revised unambiguous[3] then $\mathcal{N} \vDash$ "there is a unique cofinal well-founded branch $b \in \mathcal{N}$ of $\mathcal{T}$ such that $\phi[\mathcal{T}, b]$ holds", and*

   (f) *if $\mathcal{N}$ is revised ambiguous then t witnesses this,*

[3] This implies that $\mathcal{U} = \emptyset$ and $\mathcal{T}$ is $\mathcal{N}$-ambiguous.

$c = \mathcal{J}_\omega(t)$, and

4. *there is $t = (\mathcal{P}, \mathcal{T}, \mathcal{P}_1, \mathcal{U})$ as in clause 3 above such that if (ν, ξ, γ) are lexicographically least witnessing that properties 3.a-3.f hold for t then there is no triple $(\nu_0, \xi_0, \gamma_0) <_{lex} (\nu, \xi, \gamma)$ such that for some $s = (\mathcal{P}, \mathcal{W}, \mathcal{Q}, \mathcal{S})$, (ν_0, ξ_0, γ_0) witnesses that s has properties 3.a-3.f above.*

Definition 6.4.3 (Revised sts indexing scheme) *Suppose $(\psi_\beta : \beta < \alpha)$ have been defined. We let ψ_α be the following formula in the language of sp. Suppose X is a sellf-well-ordered set, $\mathcal{P} \in X$ is a hod-like lsa type lsp and $\mathcal{M}$ is an unambiguous sp over X based on $\mathcal{P}$. Then $\mathcal{M} \vDash \psi_\alpha[\mathcal{T}, b]$ if and only if $(\mathcal{T}, b)$ is the $\mathcal{M}$-lexicographically least pair such that $\mathcal{T}$ is an $\mathcal{M}$-terminal tree on $\mathcal{P}$ and b is a cofinal branch through $\mathcal{T}$ such that for some pair (β, γ) such that $\gamma < \alpha$ and $\beta < o(\mathcal{M})$,*

1. *$\mathcal{M}|\beta$ is revised unambiguous (see Definition 6.4.1) and $\mathcal{M}|\beta \vDash \mathsf{ZFC} +$ "there are infinitely many Woodin cardinals $> \delta(\mathcal{T})$",*

2. *$b \in \mathcal{M}|\beta$ and $\mathcal{M}|\beta \vDash$ "b is well-founded branch",*

3. *$\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b, \mathcal{T})$ exists and is an sts ψ_γ-premouse over $\mathcal{M}(\mathcal{T})$" and*

4. *letting $(\delta_i : i < \omega)$ be the first ω Woodin cardinals $> \delta(\mathcal{T})$ of $\mathcal{M}|\beta$, $\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b, \mathcal{T})$ is $<$ Ord-iterable above $\delta(\mathcal{T})$ via a strategy Σ such that letting $\lambda = \sup_{i<\omega} \delta_i$, for every generic $g \subseteq Coll(\omega, < \lambda)$, Σ has an extension $\Sigma^+ \in D(\mathcal{M}|\beta, \lambda, g)$ such that $D(\mathcal{M}, \lambda, g) \vDash$ "Σ^+ is an ω_1-iteration strategy" and whenever $\mathcal{R} \in D(\mathcal{M}|\beta, \lambda, g)$ is a Σ^+-iterate of $\mathcal{Q}(b, \mathcal{T})$ and $t \in \mathcal{R}$ is a stack on $\mathcal{M}^+(\mathcal{T})$ of length 2 then t is $(\mathcal{P}, \Sigma^{\mathcal{M}})$-authenticated".*

The lexicographically least pair (β, γ) satisfying the above conditions is called the least $(\mathcal{M}, \psi_\alpha^w)$-witness for $(\mathcal{T}, b)$. We also say that (β, γ, b) is an $\mathcal{M}$-minimal shortness witness for $\mathcal{T}$.

We leave the rest of the definitions unchanged. We say $\mathcal{P}$ is a revised hod premouse if it is indexed according to our revised indexing scheme. We say $(\mathcal{P}, \Sigma)$ is revised hod pair if $\mathcal{P}$ is revised hod premouse and Σ is an iteration strategy for $\mathcal{P}$.

Theorem 6.4.4 *Suppose $(\mathcal{P}, \Sigma)$ is a revised hod premouse such that Σ is strongly Γ-fullness preserving for some pointclass Γ and Σ has strong branch condensation. Then for any $\mathcal{Q} \in Y^{\mathcal{P}}$ and $\mathcal{P}$-generic g,*

1. *if $\mathcal{Q}$ is not of lsa type then $\Sigma_{\mathcal{Q}} \restriction \mathcal{P}[g]$ is uniformly in $\mathcal{Q}$ definable over $\mathcal{P}[g]$, and*

2. *if $\mathcal{Q}$ is of lsa type then the fragment of $\Sigma_{\mathcal{Q}}^{stc} \restriction \mathcal{P}[g]$ that acts on stacks of length 2 is uniformly in $\mathcal{Q}$ definable over $\mathcal{P}[g]$.*

We now just carry our lemmas on S-construction from Section 3.8 of [10] to our current context. Let $(\mathcal{P}, \Sigma)$ be a hod pair or an sts pair such that Σ has the strong branch condensation and is strongly Γ-fullness preserving for some pointclass Γ. Suppose $\mathcal{M}$ is a sound Σ-mouse and δ is a cutpoint cardinal of $\mathcal{M}$. Suppose further that $\mathcal{N} \in \mathcal{M}|\delta + 1$ is such that $\delta \subseteq \mathcal{N} \subseteq H_\delta^{\mathcal{M}}$, $\mathcal{N}$ models a sufficiently strong fragment of $\mathsf{ZF}-$*Replacement*, $\mathcal{N}$ is a Σ-mouse or a Σ-sts mouse and there is a partial ordering $\mathbb{P} \in L_\omega[\mathcal{N}]$ such that $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $L_\omega[\mathcal{N}]$. We would like to define S-construction of $\mathcal{M}$ over $\mathcal{N}$ relative to Σ.

Definition 6.4.5 *An S-construction of $\mathcal{M}$ over $\mathcal{N}$ relative to Σ is a sequence $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha \leq \eta)$ of Σ-mice over $\mathcal{N}$ such that*

1. *$\mathcal{S}_0 = L_\omega[\mathcal{N}]$,*
2. *if $\mathcal{M}|\delta$ is generic over $\bar{\mathcal{S}}_\alpha$ for a forcing in $L_\omega[\mathcal{N}]$ then*
   (a) *if $\mathcal{M}||(\omega \cdot \alpha)$ is active and has a last branch b then $\mathcal{S}_\alpha$ is the expansion of $\bar{\mathcal{S}}_\alpha$ by b and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$.*
   (b) *if $\mathcal{M}||(\omega \cdot \alpha)$ is active and has a last extender E then $\mathcal{S}_\alpha$ is the expansion of $\bar{\mathcal{S}}_\alpha$ by E and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,*
   (c) *if $\mathcal{M}||(\omega \times \alpha)$ is passive then $\mathcal{S}_\alpha = \bar{\mathcal{S}}_\alpha$ and $\bar{\mathcal{S}}_{\alpha+1} = rud(\mathcal{S}_\alpha)$,*
3. *if λ is limit then $\bar{\mathcal{S}}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{S}_\alpha$.*

The following is the restatement of Lemma 3.42 of [10].

Lemma 6.4.6 *Suppose $(\mathcal{P}, \Sigma)$, $\mathcal{M}, \mathcal{N}$ are as above and δ is a strong cutpoint cardinal of $\mathcal{M}$. Suppose further that $\mathcal{N} \in \mathcal{M}|\delta+1$ is such that $\delta \subseteq \mathcal{N} \subseteq H_\delta^{\mathcal{M}}$ and there is a partial ordering $\mathbb{P} \in L_\omega[\mathcal{N}]$ such that whenever $\mathcal{Q}$ is a Σ-mouse over $\mathcal{N}$ such that $H_\delta^{\mathcal{Q}} = \mathcal{N}$ then $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $\mathcal{Q}$. Then there is a Σ-mouse $\mathcal{S}$ over $\mathcal{N}$ such that $\mathcal{M}|\delta$ is generic over $\mathcal{S}$ and $\mathcal{S}[\mathcal{M}|\delta] = \mathcal{M}$.*

The following is just the restatement of Lemma 3.43 of [10].

Lemma 6.4.7 *Suppose $(\mathcal{P}, \Sigma)$, $\mathcal{M}$ and $\mathcal{N}$ are as above. Suppose further that $\mathcal{M} \vDash \mathsf{ZFC}-$Replacement is a Σ-mouse and η is a strong cutpoint non-Woodin cardinal of $\mathcal{M}$. Suppose $\gamma > \eta$ is a cardinal of $\mathcal{M}$ and $\mathcal{N} = (\mathcal{J}^{\vec{E},\Sigma})^{\mathcal{M}|\gamma}$. Suppose $\mathcal{J}_\omega(\mathcal{N}|\eta) \vDash$ "η is Woodin". Let $(\mathcal{S}_\alpha, \bar{\mathcal{S}}_\alpha : \alpha < \nu)$ be the $\mathcal{S}$-construction of $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ over $\mathcal{N}|\eta$ relative to Σ. Then for some $\alpha < \nu$, $\mathcal{S}_\alpha \vDash$ "η isn't Woodin".*

Chapter 7

Analysis of HOD

In this chapter we analyze V_Θ^{HOD} of the minimal model of the Largest Suslin Axiom. The analysis is very much like the analysis of V_Θ^{HOD} in the minimal model of $AD^+ + \theta_1 = \Theta$, which appeared in [10, Chapter 4]. Just like in [10, Chapter 4], we need to introduce the notion of suitable pair, B-iterable pair and etc. The proof of Theorem 7.2.2 is very much like the proof of [10, Theorem 4.24].

7.1 B-iterability

In this section, we import B-iterability technology to our current context. Most of what we will need was laid out in [10, Section 4.1 and Section 4.2]. Here we will only sketch the necessary arguments.

Definition 7.1.1 (Suitable pair) $(\mathcal{P}, \Sigma)$ *is a* suitable pair *if*

1. *$\mathcal{P}$ is a hod premouse, $\lambda^{\mathcal{P}}$ is a successor ordinal and $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is a Woodin cardinal",*
2. *if $\mathcal{P}$ is not of lsa type then $(\mathcal{P}(\lambda^{\mathcal{P}} - 1), \Sigma)$ is a hod pair such that Σ has strong branch condensation and is strongly fullness preserving,*
3. *if $\mathcal{P}$ is of lsa type then $(\mathcal{P}, \Sigma)$ is an sts hod pair such that Σ has strong branch condensation and is strongly fullness preserving,*
4. *if $\mathcal{P}$ is not of lsa type then $\mathcal{P}$ is a $\Sigma_{\mathcal{P}(\lambda^{\mathcal{P}}-1)}$-mouse above $\mathcal{P}(\lambda^{\mathcal{P}} - 1)$,*
5. *if $\mathcal{P}$ is not of lsa type then for any $\mathcal{P}$-cardinal $\eta > \delta^{\mathcal{P}}_{\lambda-1}$, if η is a strong cutpoint then $\mathcal{P}|(\eta^+)^{\mathcal{P}} = Lp^{\Sigma}(\mathcal{P}|\eta)$*

Notation 7.1.2 *We let*

$$\mathcal{P}^- = \begin{cases} \mathcal{P} & : \mathcal{P} \text{ is of lsa type} \\ \mathcal{P}(\lambda^{\mathcal{P}} - 1) & : \text{otherwise.} \end{cases}$$

Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are hod pairs or sts hod pairs such that Σ and Λ have strong branch condensation and are strongly fullness preserving. We then let

$$(\mathcal{P}, \Sigma) \leq_{DJ} (\mathcal{Q}, \Lambda)$$

if and only if $(\mathcal{P}, \Sigma)$ loses the coiteration with $(\mathcal{Q}, \Lambda)$. Notice that $\leq_{DJ}$ is a well-founded relation. We then let $\alpha(\mathcal{P}, \Sigma) = |(\mathcal{P}, \Sigma)|_{\leq_{DJ}}$, and we let $[\mathcal{P}, \Sigma]$ be the $=_{DJ}$ equivalence class of $(\mathcal{P}, \Sigma)$, i.e.,

$(\mathcal{Q}, \Lambda) \in [\mathcal{P}, \Sigma]$ iff $(\mathcal{Q}, \Lambda)$ is a hod pair such that Λ has branch condensation and is super fullness preserving and $\alpha(\mathcal{Q}, \Lambda) = \alpha(\mathcal{P}, \Sigma)$.

Notice that $[\mathcal{P}, \Sigma]$ is independent of $(\mathcal{P}, \Sigma)$. We let

$$\mathbb{B}(\mathcal{P}, \Sigma) = \{B \subseteq [\mathcal{P}, \Sigma] \times \mathbb{R} : B \text{ is } OD\}.$$

Note that $\mathbb{B}(\mathcal{P}, \Sigma)$ is defined for hod pairs or sts hod pairs, but not for suitable pairs that are not sts hod pairs.

The following standard lemma features prominently in our computations of HOD. The proof is very much like the proof of Lemma 4.16 of [10]. Below SMC stands for *Strong Mouse Capturing*. More precisely, SMC states that for any hod pair or sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ is strongly fullness preserving and has strong branch condensation then for any $x, y \in \mathbb{R}$, $x \in OD_{y,\Sigma}$ if and only if $x \in Lp^{\Sigma}(y)$.

Lemma 7.1.3 *Assume SMC and suppose $(\mathcal{P}, \Sigma)$ is a suitable pair and $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. Suppose κ is a $\mathcal{P}$-cardinal such that if $\mathcal{P}$ is of lsa type then $\kappa > ((\delta^{\mathcal{P}})^+)^{\mathcal{P}}$ and otherwise $\kappa > \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$. Then there is $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ such that $(\mathcal{P}, \tau)$ locally term captures $B_{(\mathcal{P},\Sigma)}$ at κ for a comeager set of $g \subseteq Coll(\omega, \kappa)$ such that g is $\mathcal{P}$-generics.*

If B is locally term captured for comeager many set generics over a suitable pair $(\mathcal{P}, \Sigma)$ then we let $\tau^{\mathcal{P},\Sigma}_{B,\kappa}$ be the invariant term in $\mathcal{P}$ locally term capturing B at κ for comeager many set generics. One way to get term capturing for all generics is to show that a suitable pair can be extended to a structure that has one more Woodin.

Definition 7.1.4 (n-Suitable pair) *$(\mathcal{P}, \Sigma)$ is an* n-suitable pair *if there is δ such that*

1. $(\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}, \Sigma)$ *is suitable,*

2. $\mathcal{P} \vDash \mathsf{ZFC}-$*Replacement* + "*there are* n *Woodin cardinals,* $\eta_0 < \eta_1 < ... < \eta_{n-1}$ *above* δ",

3. $o(\mathcal{P}) = \sup_{i<\omega}(\eta_n^{+i})^{\mathcal{P}}$,

4. *if* $\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}$ *is of lsa type then* $\mathcal{P}$ *is a* Σ*-sts premouse over* $\mathcal{M}^+(\mathcal{P}|\delta)$ *and otherwise* $\mathcal{P}$ *is a* Σ*-premouse over* $\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}$,

5. *for any* $\mathcal{P}$*-cardinal* $\eta > \delta$, *if* η *is a strong cutpoint then* $\mathcal{P}|(\eta^+)^{\mathcal{P}} = Lp^{\Sigma}(\mathcal{P}|\eta)$,

6. *Letting* $\Gamma = \Sigma_1^2(Code(\Sigma))$, *if* $\xi > \delta$ *is such that* $\mathcal{P} \vDash$ "ξ *is not Woodin*", *then* $C_\Gamma(\mathcal{P}|\xi) \subseteq \mathcal{P}$ *and* $C_\Gamma(\mathcal{P}|\xi) \vDash$ "ξ *is not Woodin*".

If $(\mathcal{P}, \Sigma)$ is n-suitable then we let $\delta^{\mathcal{P}}$ be the δ of Definition 7.1.4 and

$$\mathcal{P}^- = ((\mathcal{P}|((\delta^{\mathcal{P}})^{+\omega})^{\mathcal{P}})^-.$$

We let $\lambda^{\mathcal{P}} = \lambda^{\mathcal{P}^-} + 1$. Clearly 0-suitable pair is just a suitable pair. The following are easy consequences of Lemma 7.1.3.

Lemma 7.1.5 *Assume* SMC. *Suppose* $(\mathcal{P}, \Sigma)$ *is an* n*-suitable pair and* $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. *Suppose* κ *is a* $\mathcal{P}$*-cardinal such that if* $\mathcal{P}^-$ *is of lsa type then* $\kappa > ((\delta^{\mathcal{P}})^+)^{\mathcal{P}}$ *and otherwise* $\kappa > \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$. *Then there is* $\tau \in \mathcal{P}^{Coll(\omega,\kappa)}$ *such that* $(\mathcal{P}, \tau)$ *locally term captures* $B_{(\mathcal{P},\Sigma)}$ *at* κ *for comeager set of* $g \subseteq Coll(\omega, \kappa)$ *such that* g *is* $\mathcal{P}$*-generic.*

Corollary 7.1.6 *Assume* SMC. *Suppose* $(\mathcal{P}, \Sigma)$ *is an* n*-suitable pair and* $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. *Let* $\nu = ((\delta^{\mathcal{P}})^{+\omega})^{\mathcal{P}}$. *Suppose* κ *is a* $\mathcal{P}$*-cardinal such that if* $\mathcal{P}^-$ *is of lsa type then* $\kappa > ((\delta^{\mathcal{P}})^+)^{\mathcal{P}}$ *and otherwise* $\kappa \in (\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}, \nu)$. *Then* $(\mathcal{P}|\nu, \tau^{\mathcal{P},\Sigma}_{B,\kappa})$ *locally term captures* $B_{(\mathcal{P},\Sigma)}$ *at* κ *for comeager set of* $g \subseteq Coll(\omega, \kappa)$ *such that* g *is* $\mathcal{P}$*-generic.*

Corollary 7.1.6 is our main method of showing that various B are term captured over the hod mice that we will construct. Suppose now that $(\mathcal{P}, \Sigma)$ is a hod pair. It is now a trivial matter to import the terminology of [10, Section 4.1] to our current context. We will have that $S(\Sigma)$ consists of those $\mathcal{Q}$ such that $\mathcal{Q}^- \in pI(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-})$ is a suitable pair. Given $\mathcal{Q} \in S(\Sigma)$, we let $f_B(\mathcal{Q}) = \oplus_{\kappa<o(\mathcal{Q})}\tau^{\mathcal{Q},\Sigma_{\mathcal{Q}^-}}_{B,\kappa}$. Then the rest of the notions are defined for $F = \{f_B : B \in \mathbb{B}(\mathcal{P}, \Sigma)\}$. Therefore, in the sequel, we will freely use the terminology of [10, Section 4.1].

7.2 The computation of HOD

Throughout this section we assume $\mathsf{AD}^+ + \mathsf{SMC}$ and let $\langle\theta_\alpha : \alpha \leq \Omega\rangle$ be the Solovay sequence. Our goal is to compute $V_{\theta_\alpha}^{\mathrm{HOD}}$ for $\alpha \leq \Omega$. We will do it under some additional hypothesis described below. In the next few chapters, we will prove that our additional hypothesis essentially follows from AD^+ + "No initial segment of the Solovay sequence satisfies LSA".

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts pair such that Σ has strong branch condensation and is strongly fullness preserving. We will continue using the notation $\alpha(\mathcal{P}, \Sigma)$ and $\mathcal{P}^-$ from the previous section.

Suppose first that $\alpha + 1 = \Omega$. We then let $\mathcal{I} = \{(\mathcal{Q}, \Lambda, \vec{B} = (B_0, ..., B_n)) :$

1. $(\mathcal{Q}, \Lambda)$ is suitable, Λ is strongly fullness preserving and has strong branch condensation, and $\alpha(\mathcal{Q}^-, \Lambda) = \alpha$,
2. for every $i < n$, $B_i \in \mathbb{B}(\mathcal{Q}^-, \Lambda)$, and
3. $(\mathcal{Q}, \Lambda)$ is strongly $\vec{B}$-iterable $\}$.

$\mathcal{I}$ may be empty. But the results of Theorem 8.1.14 and Section 10.1 show that it is not. Define $\preceq$ on $\mathcal{I}$ by

$$(\mathcal{P}, \Sigma, \vec{B}) \preceq (\mathcal{Q}, \Lambda, \vec{C}) \leftrightarrow \vec{B} \subseteq \vec{C} \text{ and } (\mathcal{Q}, \Lambda, \vec{B}) \text{ is a } \vec{B}\text{-tail of } (\mathcal{P}, \Sigma, \vec{B}).$$

When $(\mathcal{R}, \Psi, \vec{B}) \preceq (\mathcal{Q}, \Lambda, \vec{C})$, there is a canonical map

$$\pi : H_{\vec{B}}^{\mathcal{R},\Psi} \rightarrow H_{\vec{B}}^{\mathcal{Q},\Lambda},$$

which is independent of $\vec{B}$-iterable branches. We let $\pi_{(\mathcal{R},\Psi,\vec{B}),(\mathcal{Q},\Lambda,\vec{B})}$ be this map. We then have that $(\mathcal{I}, \preceq)$ is a directed. Let

$$\mathcal{F} = \{H_{\vec{B}}^{\mathcal{Q},\Lambda} : (\mathcal{Q}, \Lambda, \vec{B}) \in \mathcal{I}\}.$$

and also let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ under the iteration maps $\pi_{(\mathcal{R},\Psi,\vec{B}),(\mathcal{Q},\Lambda,\vec{B})}$. Let $\delta_\infty = \delta^{\mathcal{M}_\infty}$. For $(\mathcal{Q}, \Lambda, B) \in \mathcal{I}$, we let $\pi_{(\mathcal{Q},\Lambda,B),\infty} : H_B^{\mathcal{Q},\Lambda} \rightarrow \mathcal{M}_\infty$. Standard arguments show that $\mathcal{M}_\infty$ is well-founded.

Following [10, Section 4.4], we let ϕ be the following sentence: for every $\alpha+1 < \Omega$, letting $\Gamma_\alpha = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$, there is a hod pair $(\mathcal{P}, \Sigma)$ such that

1. $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$,
2. Σ is strongly fullness preserving and has strong branch condensation,

3. for any $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$, if $\lambda^{\mathcal{Q}}$ is a successor ordinal then
   (a) there is a sequence $\langle B_i : i < \omega \rangle \subseteq \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ which guides $\Sigma_{\mathcal{Q}}$ and
   (b) for any $B \in \mathbb{B}(\mathcal{Q}^-, \Sigma_{\mathcal{Q}^-})$ there is $\mathcal{R} \in pI(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ such that $\Sigma_{\mathcal{R}}$ respects B.

Our additional hypothesis, ψ, is a conjunction of ϕ with the following statement: If $\Omega = \alpha + 1$ then there is a suitable $(\mathcal{P}, \Sigma)$ which is $\emptyset$-iterable, $\lambda^{\mathcal{P}}$ is a successor and such that

1. $(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ is either a hod pair or an sts pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$ and $\Sigma_{\mathcal{P}^-}$ is strongly fullness preserving and has strong branch condensation,
2. for any $B \in \mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-})$ there is an $\emptyset$-iterate $(\mathcal{Q}, \Phi)$ of $(\mathcal{P}, \Sigma)$ such that $(\mathcal{Q}, \Phi)$ is strongly B-iterable.
3. $\mathcal{M}_\infty$ is well-founded and $\delta_\infty = \Theta = \theta_{\alpha+1}$.

We will use the following lemma to establish ψ. It can be proved exactly the same way as [10, Lemma 4.23].

Lemma 7.2.1 *Suppose $\Gamma \subseteq \wp(\mathbb{R})$ is such that $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{SMC} + \Omega = \alpha + 1$ and $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Suppose $\Gamma^* \subseteq \wp(\mathbb{R})$ is such that $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*, \mathbb{R}) \vDash \mathsf{AD}^+$ and there is a hod a pair $(\mathcal{P}, \Sigma) \in \Gamma^*$ such that the following holds.*

1. *Σ has strong branch condensation and is strongly Γ-fullness preserving.*
2. *$\lambda^{\mathcal{P}}$ is a successor ordinal, $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$,*
   (a) if $\mathcal{P}$ is not of lsa type then $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$" and
   (b) if $\mathcal{P}$ is of lsa type then $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$".
3. *There is a sequence $\langle B_i : i < \omega \rangle \subseteq (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma, \mathbb{R})}$ guiding Σ.*
4. *For any $B \in (\mathbb{B}(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}))^{L(\Gamma, \mathbb{R})}$ there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{R}}$ respects B.*

Then $L(\Gamma, \mathcal{R}) \vDash \psi$ and $\mathcal{M}_\infty^{L(\Gamma, \mathbb{R})} = \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$.[1]

[1]Recall that $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ is the direct limit of all Σ-iterates of $\mathcal{P}$

The next theorem is the adaptation of [10, Theorem 2.24] to our current context. It can be proved via exactly the same proof. Because of this, we omit the proof.

Theorem 7.2.2 (Computation of HOD) *Assume AD^+. Suppose $\Gamma \subseteq \wp(\mathbb{R})$ is such that $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$. Then the following holds:*

1. *Suppose $L(\Gamma, \mathbb{R}) \vDash \phi$. Suppose $\beta + 1 < \Omega^\Gamma$. Let $(\mathcal{P}, \Sigma)$ witness ϕ for β. Then letting $\mathcal{M} = \mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$, $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda = \Sigma^{\mathcal{M}}$, for every $\alpha \leq \beta$*

$$\delta^{\mathcal{M}}_\alpha = \theta^\Gamma_\alpha \text{ and } \mathcal{M}|\theta^\Gamma_\alpha = (V^{\mathrm{HOD}^\Gamma}_{\theta^\Gamma_\alpha}, \vec{E} \restriction \theta^\Gamma_\alpha, \Lambda \restriction V^{\mathrm{HOD}^\Gamma}_{\theta^\Gamma_\alpha}, \in).$$

2. *If $L(\Gamma, \mathbb{R}) \vDash \psi$ then letting $\mathcal{M} = \mathcal{M}^{L(\Gamma,\mathbb{R})}_\infty$ $\vec{E} = \vec{E}^{\mathcal{M}}$ and $\Lambda = \Sigma^{\mathcal{M}}$, for every $\alpha \leq \Omega^\Gamma$*

$$\delta^{\mathcal{M}}_\alpha = \theta^\Gamma_\alpha \text{ and } \mathcal{M}|\theta^\Gamma_\alpha = (V^{\mathrm{HOD}^\Gamma}_{\theta^\Gamma_\alpha}, \vec{E} \restriction \theta^\Gamma_\alpha, \Lambda \restriction V^{\mathrm{HOD}^\Gamma}_{\theta^\Gamma_\alpha}, \in).$$

3. *Suppose $\Gamma^* \subseteq \wp(\mathbb{R})$ is such that $\Gamma \subseteq \Gamma^*$, $L(\Gamma^*, \mathbb{R}) \vDash \mathsf{AD}^+$ and there is a hod a pair $(\mathcal{P}, \Sigma) \in \Gamma^*$ such that the following holds:*
   (a) *Σ has strong branch condensation and is strongly Γ-fullness preserving,*
   (b) *$\lambda^{\mathcal{P}}$ is a successor ordinal, $Code(\Sigma_{\mathcal{P}^-}) \in \Gamma$,*
      i. *if $\mathcal{P}$ is not of lsa type then $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$" and*
      ii. *if $\mathcal{P}$ is of lsa type then $L(\Gamma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}^-})$ is a suitable pair such that $\alpha(\mathcal{P}^-, \Sigma_{\mathcal{P}^-}) = \alpha$".*
   (c) *there is a sequence $\langle B_i : i < \omega \rangle \subseteq (\mathbb{B}(\mathcal{P}^-, \Lambda_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ guiding Σ,*
   (d) *for any $B \in (\mathbb{B}(\mathcal{P}^-, \Lambda_{\mathcal{P}^-}))^{L(\Gamma,\mathbb{R})}$ there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma)$ such that $\Sigma_{\mathcal{R}}$ respects B.*

   Then $L(\Gamma, \mathcal{R}) \vDash \psi$ and $\mathcal{M}^{L(\Gamma,\mathbb{R})}_\infty = \mathcal{M}^+_\infty(\mathcal{P}, \Lambda)$.

Thus, working in a model of AD^+, if $\alpha < \Omega$ then to compute $\mathrm{HOD}|\theta_\alpha$ we only need to produce a hod pair $(\mathcal{P}, \Sigma)$ satisfying clauses 3(a)-3(d). In the next chapter, in particular in Theorem 8.1.14 and Section 10.1, we will show that this is indeed true in the minimal model of the Largest Suslin Axiom.

Chapter 8

Models of LSA as derived models

In this chapter, we show that certain derived models satisfy the *Largest Suslin Axiom*. We also prove results that are important elsewhere. The results of Section 10.1 and Theorem 8.1.14 are needed to carry out the computation of HOD (see Theorem 7.2.2). We start with introducing the pointclass $\Gamma(\mathcal{P}, \Sigma)$ where $(\mathcal{P}, \Sigma)$ is an sts hod pair.

8.1 $\Gamma(\mathcal{P}, \Sigma)$ when $\lambda^{\mathcal{P}}$ is a successor

In this section, we translate the results of [10, Section 5.6] to our current context. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor and Σ is strongly fullness preserving and has strong branch condensation. Recall the notation $\mathcal{P}^-$ (see Notation 7.1.2).

Suppose first that $\mathcal{P}$ isn't of lsa type. We now generalize the result of [10, Section 5.6]. Recall the notation $Mice_\Sigma$ (see Notation 4.1.4). Because $\mathcal{P}$ is not of lsa type, it follows that $Code(\Sigma)$ is Suslin, co-Suslin (this can be proved using the proof of [10, Lemma 5.9]). It follows that there is a scaled pointclass closed under continuous images and pre-images and under $\exists^{\mathbb{R}}$, and also contains $Mice_{\Sigma_{\mathcal{P}^-}}$. We then let Γ^*_Σ be the least such pointclass. Also, let

$$\Gamma_\Sigma = (\Sigma^2_1(Code(\Sigma_{\mathcal{P}^-})))^{L(Mice_{\Sigma_{\mathcal{P}^-}}, \mathbb{R})}.$$

Notice that Γ_Σ is a lightface good pointclass. Also $Mice_{\Sigma_{\mathcal{P}^-}}$ belongs to Γ_Σ and is a universal Γ_Σ set. We let

$$\Gamma(\mathcal{P}, \Sigma) = \{A : \text{for cone of } x \in \mathbb{R},\ A \cap C_{\Gamma_\Sigma}(x) \in C_{\Gamma_\Sigma}(C_{\Gamma_\Sigma}(x))\} = Env(\Gamma_\Sigma).$$

Notice that if $(\mathcal{Q}, \Lambda)$ is a tail of $(\mathcal{P}, \Sigma)$ then $\Gamma(\mathcal{Q}, \Lambda) = \Gamma(\mathcal{P}, \Sigma)$. The next theorem is essentialy the conjunction of [10, Lemma 5.13-5.16].

Theorem 8.1.1 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor, $\mathcal{P}$ is not of lsa type and Σ is strongly fullness preserving and has strong branch condensation. Then the following holds.*

1. *There is a tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$ such that $\Gamma^*_\Lambda = \underset{\sim}{\Gamma}_\Lambda$.*
2. *Suppose $\Gamma^*_\Sigma = \underset{\sim}{\Gamma}_\Sigma$. Then for any real x coding $\mathcal{P}^-$,*
$$C_{\Gamma_\Sigma}(x) = Lp^{\Gamma, \Sigma_{\mathcal{P}^-}}(x).$$
3. *Suppose $\Gamma^*_\Sigma = \underset{\sim}{\Gamma}_\Sigma$. Then $Code(\Sigma) \notin \Gamma(\mathcal{P}, \Sigma)$.*
4. *Suppose $\Gamma^*_\Sigma = \underset{\sim}{\Gamma}_\Sigma$. Then there is a tail $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}, \Sigma)$ such that*
$$\Gamma(\mathcal{Q}, \Lambda) = \wp(\mathbb{R}) \cap L(\Gamma(\mathcal{Q}, \Lambda), \mathbb{R}).$$
Because $\Gamma(\mathcal{Q}, \Lambda) = \Gamma(\mathcal{P}, \Sigma)$, it follows that $\Gamma(\mathcal{P}, \Sigma) = \wp(\mathbb{R}) \cap L(\Gamma(\mathcal{P}, \Sigma), \mathbb{R})$.

We spend the rest of this section defining $\Gamma(\mathcal{P}, \Sigma)$ in the case $\mathcal{P}$ is of lsa type. The difficulty with generating LSA pointclasses as $\Gamma(\mathcal{P}, \Sigma)$ is the following. Suppose Γ is an LSA pointclass, i.e., $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{LSA}$. Let α be such that $\alpha + 1 = \Omega^\Gamma$ and set $\Gamma^b = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$[1]. The difficulty is that the pair that generates Γ^b is the same as the pair that generates Γ.

Definition 8.1.2 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $\mathcal{P}$ is of lsa type and Σ has strong branch condensation and is strongly fullness preserving. We then let*
$$\Gamma(\mathcal{P}, \Sigma) = \{A : \textit{for cone of } x \in \mathbb{R},\ A \cap Lp^{\Sigma^{stc}}(x) \in Lp_2^{\Sigma^{stc}}(x)\}.$$

It is not immediately clear that $L(\Gamma(\mathcal{P}, \Sigma)) \cap \wp(\mathbb{R}) = \Gamma(\mathcal{P}, \Sigma)$. Theorem 8.1.13 shows that it is indeed true. Before we prove it, we prove some useful lemmas. The first lemma shows that various Σ-sts mice are internally Σ-closed.

Lemma 8.1.3 *Suppose $(\mathcal{R}, \Phi)$ is an sts hod pair such that Φ is fullness preserving and $\mathcal{M}$ is a Φ-sts mouse over $\mathcal{R}$. Suppose δ is a Woodin cardinal of $\mathcal{M}$ and $(\delta^+)^{\mathcal{M}}$ exists. Then for any $\nu < \delta$, if $\mathcal{S}^{\mathcal{M}}_\nu$ is the output of $(\mathcal{R}, \Sigma^{\mathcal{M}})$-hod pair construction of $\mathcal{M}|\delta$ and $\mathcal{T}$ on $\mathcal{P}$ is the normal tree leading to $\mathcal{S}^{\mathcal{M}}_\nu$ then $\pi^{\mathcal{T}, b}$ exists and $\pi^{\mathcal{T}, b}(\lambda^{\mathcal{R}} - 1) = \lambda^{\mathcal{S}^{\mathcal{M}}_\nu} - 1$.*

[1] The superscript "b" stands for bottom.

Proof. Suppose not. Then we must have that $\pi^{\mathcal{T},b}$ exists and $\mathcal{S}_\nu^{\mathcal{M}} = \mathcal{M}^+(\mathcal{T})(\alpha)$ for some $\alpha < \lambda^{(\mathcal{M}^+(\mathcal{T}))^b}$. It follows that

(1) $(\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta}$ does not have Woodin cardinals.

However, notice that

(2) $\mathcal{M} \vDash$ "$\mathcal{M}^+(\mathcal{T})(\alpha+1)$ is δ^+-iterable via $\Sigma^{\mathcal{M}}_{\mathcal{M}^+(\mathcal{T})(\alpha+1)}$".

(2) simply follows from our indexing scheme and the fact that $\mathcal{M}$ is a Φ-sts premouse. It follows that the comparison of $(\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta}$ and $\mathcal{M}^+(\mathcal{T})(\alpha+1)$ is successful. But because $(\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta}$ is a fully backgrounded construction, in the aforementioned comparison, only $\mathcal{M}^+(\mathcal{T})(\alpha+1)$ moves. It follows from (1) and universality of $(\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta}$ that

(3) if $\mathcal{S} \trianglelefteq (\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta}$ is the iterate of $\mathcal{M}(\mathcal{T})(\alpha+1)$ then $(\mathcal{J}^{\vec{E},\Sigma^{\mathcal{M}}_{\mathcal{S}_\nu^{\mathcal{M}}}})^{\mathcal{M}|\delta} \vDash$ "$\delta^{\mathcal{S}}$ is not a Woodin cardinal"

(3) contradicts the fullness preservation of Φ. $\square$

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is of lsa type and Σ has strong branch condensation and is strongly fullness preserving. Suppose $Code(\Sigma)$ is Suslin, co-Suslin.

Let $\Gamma_0 < \Gamma$ be any two good pointclasses such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma_0}$. Let F be as in Theorem 4.1.6 for Γ. Let $A \in \Gamma$ be a set coding a self-justifying-system $(A_i : i < \omega)$ such that $A_0 = \{(x, y) \in \mathbb{R}^2 : y \in C_{\Gamma_0}(x)\}$. Fix x such that if $F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ then $Code(\Sigma)$ and $\vec{A}$ are Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$.

We then have that the fully backgrounded hod pair construction of $\mathcal{N}_x^*|\delta_x$ reaches a tail of $(\mathcal{P}, \Sigma)$ (see Theorem 4.6.10). Let $(\mathcal{Q}, \Lambda)$ be this tail. Let $\mathcal{N} = (\mathcal{J}^{\vec{E},\Lambda^{stc}})^{\mathcal{N}_x^*|\delta_x}$. Because Σ is fullness preserving we have that $\mathcal{N} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". Let Φ be the strategy of $\mathcal{N}$ induced by Σ_x. Notice that Φ is fullness preserving in the sense of Lp operator, i.e., whenever $\mathcal{M}$ is a Φ-iterate of $\mathcal{N}$ and η is a strong cutpoint of $\mathcal{M}$ then $\mathcal{M}|(\eta^+)^{\mathcal{M}} = Lp^{\Lambda^{stc}}(\mathcal{M}|\eta)$. This can be shown using the proof of Theorem 4.5.3. We now prove several lemmas about $(\mathcal{N}, \Phi)$ leading up to showing that $\Gamma(\mathcal{Q}, \Lambda^{stc})$ can be realized as a derived model of $\mathcal{N}$. Let κ be the least strong cardinal of $\mathcal{N}$. The first lemma is quite standard.

Lemma 8.1.4 *$\mathcal{N} \vDash$ "κ is a limit of Woodin cardinals".*

Proof. It is enough to show that δ_x is a limit of cardinals η such that $Lp^{\Lambda^{stc}}(\mathcal{N}_x^*|\eta) \vDash$ "η is a Woodin cardinal". Fix $\kappa < \delta_x$. Because $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma_0}$, we have that for cone of z, $Lp^{\Sigma^{stc}}(x) \in C_{\Gamma_0}(z)$. We can assume, using absoluteness, that the base of this cone is in $\mathcal{N}_x^*$. Let $T, S \in \mathcal{N}_x^*$ be δ_x-complementing trees witnessing that A is Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$. Let $\pi : M \to H_{(\delta_x^+)^{\mathcal{N}_x^*}}$ be a Skolem hull such that $\mathrm{crit}(\pi) > \kappa$ is an $\mathcal{N}_x^*$-cardinal and $\{T, S\} \in rng(\pi)$. Let $\eta = \mathrm{crit}(\pi)$. Then it follows that $C_{\Gamma_0}(\mathcal{N}_x^*|\eta) \in M$ and hence, $C_{\Gamma_0}(\mathcal{N}_x^*|\eta) \vDash$ "η is a Woodin cardinal". It follows that $Lp^{\Lambda^{stc}}(\mathcal{N}_x^*|\eta) \vDash$ "η is a Woodin cardinal". $\square$

The next lemma shows that

Lemma 8.1.5 *Φ is fullness preserving, i.e., Φ witnesses that $\Gamma(\mathcal{N}|\kappa, \Phi) = \Gamma^b(\mathcal{Q}, \Lambda^{stc})$.*

Proof. Clearly, because Φ witnesses that $\mathcal{N}$ is a Λ^{stc}-sts mouse, $\Gamma(\mathcal{N}|\kappa, \Phi) \subseteq \Gamma^b(\mathcal{Q}, \Lambda^{stc})$. Fix then $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{Q}, \Lambda^{stc})$. We want to see that

(1) there is a Φ-iterate $\mathcal{N}_1$ of $\mathcal{N}|\kappa$ such that for some $t = (\mathcal{Q}, \mathcal{T}, \mathcal{S}, \vec{\mathcal{U}}) \in \mathcal{N}_1$ such that t is according to $\Sigma^{\mathcal{N}_1}$, $\Lambda_{\mathcal{R}} \leq_w \Lambda_{\mathcal{S}^b}$.

Suppose (1) fails. We can then assume, without loss of generality, that for some $\nu < \delta_x$ and some $g \subseteq Coll(\omega, \nu)$, $(\vec{\mathcal{T}}, \mathcal{R}) \in \mathcal{N}_x^*[g]$. Yet again without losing generality we can assume that $\mathcal{R}$ is of successor type. Let now $\mathcal{S}$ be the output of $(\mathcal{Q}, \Sigma^{\mathcal{N}})$-construction of $\mathcal{N}$ in which extenders used have critical point $> \nu$. Let $\mathcal{U}$ be a normal tree on $\mathcal{Q}$ with last model $\mathcal{S}$. We claim that

(2) $\pi^{\mathcal{U},b}$ exists, $\pi^{\mathcal{U},b}(\lambda^{\mathcal{Q}} - 1) = \lambda^{\mathcal{S}^b}$ and for some $\beta < \lambda^{\mathcal{S}^b}$, $\mathcal{S}^b(\beta)$ is a $\Lambda_{\mathcal{R}}$-iterate of $\mathcal{R}$.

The first two clauses of (2) are consequences of Lemma 8.1.3. We prove the third clause of (2). We have that the comparison of $\mathcal{R}$ and $\mathcal{S}$ produces a normal tree $\mathcal{W}^*$ on $\mathcal{R}$ according to $\Lambda_{\mathcal{R}}$ with last model $\mathcal{R}_1$. If $\mathcal{R}_1 \trianglelefteq \mathcal{S}^b$ then because $\mathcal{R}_1$ is of successor type we must have that for some $\beta < \lambda^{\mathcal{S}^b}$, $\mathcal{S}^b(\beta) = \mathcal{R}_1$. Suppose then $\mathcal{R}_1 \ntrianglelefteq \mathcal{S}^b$. We then have $\mathcal{S} \in Y^{\mathcal{R}_1}$. We now have two cases.

(3) $\pi^{\mathcal{U}}$ exists.
(4) otherwise.

Suppose first that (3) holds. It follows that there is an extender $E^* \in \vec{E}^{\mathcal{R}_1}$ such that $\mathrm{crit}(E^*) = \delta^{\mathcal{S}^b}$ and $\mathcal{S} \trianglelefteq Ult(\mathcal{R}_1, E^*)$. Let E be the $\mathcal{R}_1$-least such extender and let γ be such that $\mathcal{S} = Ult(\mathcal{R}_1, E)(\gamma)$. Let $\mathcal{R}_2 = Ult(\mathcal{R}_1, E)$. It follows from the proof of Lemma 8.1.3 that $(\mathcal{J}^{\mathcal{N},\Lambda_{\mathcal{R}_2(\gamma)}})^{\mathcal{N}}$ reaches a Woodin cardinal implying that $\mathcal{S}$ cannot be the output of $(\mathcal{Q},\Sigma^{\mathcal{N}})$-construction in which extenders used have critical point $> \nu$.

Suppose then (4) holds. It follows that $\mathcal{U}$ is of limit length and $\mathcal{M}^+(\mathcal{U}) \vDash$ "$\delta(\mathcal{U})$ is a Woodin cardinal" (otherwise we again have that $\mathcal{S}$ cannot be the output of $(\mathcal{Q},\Sigma^{\mathcal{N}})$-construction in which extenders used have critical point $> \nu$). Because $\mathcal{R}_1$ is lsa small, it follows that

(5) $\mathcal{R}_1 \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal".

Let then $\mathcal{W} \trianglelefteq \mathcal{R}_1$ be the least such that $\mathcal{W} \vDash$ "$\delta(\mathcal{U})$ is a Woodin cardinal" but $\mathcal{J}_1(\mathcal{W}) \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal". Notice that we must have that

(6) the $\mathcal{N}$-authenticated background construction (see Definition 6.2.2) over $\mathcal{M}^+(\mathcal{U})$ does not reach $\mathcal{W}$.

(6) holds because otherwise $\mathcal{S}$ cannot be the last model of $(\mathcal{Q},\Sigma^{\mathcal{N}})$-construction in which extenders used have critical point $> \nu$.

Let then $\mathcal{S}_1$ be the $\mathcal{N}$-authenticated background construction over $\mathcal{M}^+(\mathcal{U})$. If the comparison of the construction producing $\mathcal{S}_1$ and $\mathcal{W}$ halts then, because $\mathcal{S}_1$ side does not move, we must have that $\mathcal{W} \trianglelefteq \mathcal{S}_1$ contradicting (6). Suppose then the comparison of the construction producing $\mathcal{S}_1$ and $\mathcal{W}$ does not halt, implying that it must reach a strategy disagreement. It follows that

(7) there is a normal iteration tree $\mathcal{K}$ on $\mathcal{W}$ according to $\Lambda_{\mathcal{W}}$ with last model $\mathcal{W}_1$, and there is $\mathcal{S}_2$, which is a model appearing in the construction producing $\mathcal{S}_1$, such that letting $o(\mathcal{S}_2) = \beta$, $\mathcal{W}_1|\beta = \mathcal{S}_2|\beta$ but $\mathcal{W}_1||\beta \neq \mathcal{S}_2$.

Let then $t = (\mathcal{M}^+(\mathcal{U}), \mathcal{T}^*, \mathcal{R}^*, \vec{\mathcal{U}}) \in \mathcal{W}_1|\beta \cap \mathcal{S}_2$ be such that the branch of t is indexed at β. Let b be the branch indexed in $\mathcal{W}_1$. Let c be the branch indexed in $\mathcal{S}_2$, if there is such a branch indexed in $\mathcal{S}_2$.

Suppose that c is defined. We claim that $\vec{\mathcal{U}} \neq \emptyset$. Suppose $\vec{\mathcal{U}} = \emptyset$. Then b and c are branches of $\mathcal{T}^*$. We must also have that $\mathcal{T}^*$ is both $\mathcal{W}_1$ and $\mathcal{S}_2$ ambiguous. But indexed branches of such trees just depend on $\mathcal{W}_1|\beta = \mathcal{S}_2|\beta$, implying that $b = c$. Thus, we have that both b and c are the branches of $\vec{\mathcal{U}}$. But we have that

$b = \Lambda_{(\mathcal{R}^*)^b}(\vec{\mathcal{U}}) = c$ (see Lemma 5.5.1), contradiction!

Suppose then c is undefined. In this case, we have that $\mathcal{N}$ never authenticates the branch of $\vec{\mathcal{U}}$, as otherwise c would be defined. It follows that

(8) letting $\xi = o(\mathcal{R}^*)$, the $(\mathcal{Q}, \Sigma^{\mathcal{N}})$-hod pair construction of $\mathcal{N}$ in which extenders used have critical points $> \xi$ does not reach an iterate of $(\mathcal{R}^*)^b$.

Notice that $(\mathcal{R}^*)^b \in pB(\mathcal{R}, \Lambda_{\mathcal{R}})$. Let $\mathcal{S}_3$ be the $(\mathcal{Q}, \Sigma^{\mathcal{N}})$-hod pair construction of $\mathcal{N}$ in which extenders used have critical points $> \xi$. Notice now that (8) asserts that the failure of (2) holds for $((\mathcal{R}^*)^b, \Lambda_{(\mathcal{R}^*)^b})$ and $\mathcal{S}_3$. Let then $\mathcal{R}_2 = ((\mathcal{R}^*)^b)$. Repeating the argument given above we obtain an infinite sequence $(\mathcal{R}_{2k} : k < \omega)$ such that $\mathcal{R}_0 = \mathcal{R}$ and $\mathcal{R}_{2k+2} \in pB(\mathcal{R}_{2k}, \Lambda_{\mathcal{R}_{2k}})$, contradiction! This finishes the proof of Lemma 8.1.5. □

Before we proceed, we record some lemmas that the proof of (7) gives.

Lemma 8.1.6 *Suppose $\pi : \mathcal{N}|(\kappa^+)^{\mathcal{N}} \to \mathcal{M}$ is an iteration via $\Phi_{\mathcal{N}|\kappa}$ and g is $\mathcal{M}$-generic. Then letting F be the function $F(X) = Lp^{\Lambda^{stc}}(X)$, $F \restriction \mathcal{M}[g]$ is uniformly in $\mathcal{M}, g$[2] definable over $\mathcal{M}[g]$.*

Proof. Suppose first $X \in \mathcal{N}|\kappa[g]$ for some generic g. Let δ be a cutpoint Woodin cardinal of $\mathcal{N}|\kappa$ such that g is a $< \delta$-generic and $X \in \mathcal{N}|\delta[g]$. We can now use the proof of (6) in the previous lemma to show that $Lp^{\Lambda^{stc}}(X)$ is the union of all hybrid sts mice over X based on $\mathcal{Q}$ that project to $o(X)$ and appear as models in the $\mathcal{N}|\delta$-authenticated fully backgrounded construction over X. This definition carries over to any Φ-iterate of $\mathcal{M}$ (this is a consequence of absoluteness as the failure of our claim can be reflected inside $\mathcal{N}^*_x$). □

Corollary 8.1.7 *Suppose $\pi : \mathcal{N}|(\kappa^+)^{\mathcal{N}} \to \mathcal{M}$ is an iteration via $\Phi_{\mathcal{N}|(\kappa^+)^{\mathcal{N}}}$ and F is as in Lemma 8.1.6. Then if $h \subseteq Coll(\omega, < \pi(\kappa))$ is $\mathcal{M}$-generic then $F \restriction HC^{\mathcal{M}}[h] \in \mathcal{M}[\mathbb{R}^{\mathcal{M}[h]}]$.*

Lemma 8.1.6 can be used to prove the following lemma.

Lemma 8.1.8 *Suppose $\pi : \mathcal{N}|(\kappa^+)^{\mathcal{N}} \to \mathcal{M}$ is an iteration via $\Phi_{\mathcal{N}|(\kappa^+)^{\mathcal{N}}}$ and δ is a cutpoint Woodin cardinal of $\mathcal{M}$. Let ξ be a cutpoint cardinal of $\mathcal{M}$ such that $\mathcal{M}$ has no Woodin cardinals in the interval (ξ, δ). Let $\eta \in (\xi, \delta)$ be an $\mathcal{M}$-cardinal and let Ψ*

[2] I.e., the definition works for any such $\mathcal{M}$ and g.

be the fragment of Φ that acts on normal non-dropping trees based on $\mathcal{M}|(\eta^+)^{\mathcal{M}}$ that are above ξ. Then letting $h \subseteq Coll(\omega,(\eta^+)^{\mathcal{M}})$ be $\mathcal{M}$-generic, $\Phi \restriction \mathcal{M}|\pi(\kappa)[h] \in \mathcal{M}$ and is $\pi(\kappa)$-universally Baire in $\mathcal{M}[h]$.

Corollary 8.1.9 *Suppose $\pi : \mathcal{N}|(\kappa^+)^{\mathcal{N}} \to \mathcal{M}$ is an iteration via $\Phi_{\mathcal{N}|(\kappa^+)^{\mathcal{N}}}$. Suppose g is $\mathcal{M}|\pi(\kappa)$-generic, $X \in (\mathcal{M}|\pi(\kappa))[g]$ and $\mathcal{R} \in Lp^{\Lambda^{stc}}(X)$ is such that $\rho(\mathcal{R}) = o(X)$. Let $h \subseteq Coll(\omega,|X|)$ be $(\mathcal{M}|\pi(\kappa))[g]$-generic. Then $\mathcal{R} \in \mathcal{M}[g][h]$ and $\mathcal{M}[g][h] \vDash$ "$\mathcal{R}$ has a $\pi(\kappa)$-universally Baire iteration strategy Ψ witnessing that $\mathcal{R}$ is a Λ^{stc}-sts mouse over X based on $\mathcal{Q}$".*

Moreover, if $\mathcal{R} \in (\mathcal{M}|\pi(\kappa))[g]$ is a Λ^{stc}-sts premouse over X such that for some $(\mathcal{M}|\pi(\kappa))[g]]$-generic $h \subseteq Coll(\omega,|X|)$, $(\mathcal{M}|\pi(\kappa))[g][h] \vDash$ "$\mathcal{R}$ has a $\pi(\kappa)$-iteration strategy" then $\mathcal{R} \trianglelefteq Lp^{\Lambda^{stc}}(X)$.

The next lemma shows that $\Gamma(\mathcal{Q},\Lambda^{stc})$ is a derived model of $\mathcal{N}$.

Lemma 8.1.10 *The derived model of $\mathcal{N}|(\kappa^+)^{\mathcal{N}}$ as computed via Φ is $L(\Gamma(\mathcal{Q},\Sigma^{stc}_{\mathcal{Q}}))$. In particular, $\Gamma(\mathcal{Q},\Lambda^{stc}) = \wp(\mathbb{R}) \cap L(\Gamma(\mathcal{Q},\Lambda^{stc}))$.*

Proof. We will use clause 2 of Theorem 6.1.5. First we verify that clause 2 of Theorem 6.1.5 applies. For this we need to verify that

(1) $\mathcal{N}$ is internally Λ^{stc}-closed, and
(2) Φ is a fullness preserving strategy for $\mathcal{N}$.

Notice that (1) is a consequence of Lemma 8.1.3 and (2) is just Lemma 8.1.5. We thus have that clause 2 of Theorem 6.1.5 applies.

To prove Lemma 8.1.10 we need to show that given an $\mathbb{R}$-genericity iteration $\pi : \mathcal{N}|(\kappa^+)^{\mathcal{N}} \to \mathcal{N}_1$ according to $\Phi_{\mathcal{N}|(\kappa^+)^{\mathcal{N}}}$,

(3) if $A \in \Gamma(\mathcal{P},\Sigma)$ then $A \in \mathcal{N}_1(\mathbb{R})$, and
(4) if $A \in \mathcal{N}_1(\mathbb{R})$ is such that $L(A,\mathbb{R}) \vDash \mathsf{AD}^+$ then $A \in \Gamma(\mathcal{P},\Sigma)$.

We start with (3). Towards a contradiction, assume not and let $A \in \Gamma(\mathcal{P},\Sigma)$ witness this. We have that for cone of $z \in \mathbb{R}$, $A \cap Lp^{\Lambda^{stc}}(z) \in Lp_2^{\Lambda^{stc}}(z)$. Let z be some base of the aforementioned cone. Let $\xi > \Theta$ be such that $L_\xi(\wp(\mathbb{R})) \vDash \mathsf{ZF}-$*Replacement* and $\sigma : M \to L_\xi(\wp(\mathbb{R}))$ is a countable hull such that $\mathcal{N}, z \in HC^M$ and $\{\Phi, A\} \in rng(\sigma)$.

Let $g \in L(\wp(\mathbb{R}))$ be M-generic for $Coll(\omega,\mathbb{R}^M)$. Let $(y_i : i < \omega)$ be the generic sequence enumerating $\mathbb{R}^M$ and let $(\delta_i : i < \omega)$ be a sequence of cutpoint Woodin cardinals of $\mathcal{N}|(\kappa^+)$ with sup κ. Let $(\mathcal{N}_i, \mathcal{T}_i : i < \omega)$ be the $\mathbb{R}^M$-genericity iteration.

Thus, $\mathcal{N}_0 = \mathcal{N}|(\kappa^+)^{\mathcal{N}}$, $\mathcal{T}_i$ is a tree on $\mathcal{N}_i$ based on $\mathcal{N}_i|\pi^{\oplus_{j<i}T_j}(\delta_i)$ and $\mathcal{T}_i$ is built according to the rules of y_i-genericity iteration. Let $\pi_{i,k}: \mathcal{N}_i \to \mathcal{N}_k$ be the composition of the iteration embeddings. Let $\mathcal{N}_\omega$ be the direct limit of $\mathcal{N}_i$ under $\pi_{i,k}$.

Because $z \in \mathbb{R}^M$, we have that $A \cap (\mathcal{N}_\omega|\omega_1^M)(\mathbb{R}^M) \in Lp^{\Lambda^{stc}}((\mathcal{N}_\omega|\omega_1^M)(\mathbb{R}^M)))$. Notice that it follows from Lemma 8.1.6 that if $\mathcal{N}_\omega^+$ is the iterate of $\mathcal{N}$ obtained by applying $\oplus_{i<\omega}\mathcal{T}_i$ to $\mathcal{N}$ then

$$Lp^{\Lambda^{stc}}((\mathcal{N}_\omega|\omega_1^M)(\mathbb{R}^M)) \in \mathcal{N}_\omega^+(\mathbb{R}).$$

It follows that $A \in D(\mathcal{N}_\omega, \omega_1^M, h)$ where $h \subseteq Coll(\omega, <\omega_1^M)$ is an $\mathcal{N}_\omega$-generic such that $\mathbb{R}^{\mathcal{N}_\omega[h]} = \mathbb{R}^M$. This finishes the proof of (3).

We keep the notation used to prove (3) and start proving (4). To prove (4), we need to show that if A is as in (4) then

(5) $A \in (\Gamma(\mathcal{P}, \Sigma))^M$.

Suppose that (5) fails. We then have that there is $A \in \mathcal{N}_\omega(\mathbb{R}^M)$ such that $L(A, \mathbb{R}^M) \vDash \mathsf{AD}^+$ and $A \notin (\Gamma(\mathcal{P}, \Sigma))^M$. We first claim that

Claim. in $L(A, \mathbb{R}^M)$, for cone of y, $A \cap Lp^{\Lambda^{stc}}(y) \in Lp_2^{\Lambda^{stc}}(y)$.

Proof. Suppose not. Working in $L(A, \mathbb{R}^M)$, fix $y \in \mathbb{R}^M$ such that for any $y^* \in \mathbb{R}^M$ Turing above y, $A \cap Lp^{\Lambda^{stc}}(y) \notin Lp_2^{\Lambda^{stc}}(y)$. Fix $i < \omega$ such that $y \in \mathcal{N}_\omega[h \cap Coll(\omega, \delta_i)]$. Notice that

(6) for every $y \in \mathbb{R}^M$, $(Lp^{\Lambda^{stc}}(y))^{L(A,\mathbb{R}^M)} = Lp^{\Lambda^{stc}}(y)$.

(6) is a consequence of Corollary 8.1.9. This is because if $\mathcal{R} \trianglelefteq (Lp^{\Lambda^{stc}}(y))^{L(A,\mathbb{R}^M)}$ is such that $\rho(\mathcal{R}) = \omega$ then $\mathcal{R}$ has an iteration strategy in $\mathcal{N}_\omega[y]$ as the iteration strategy of $\mathcal{R}$ is ordinal definable from Λ^{stc} in the derived model of $\mathcal{N}_\omega$.

Let $k < \omega$ be such that there is a name τ for A in $\mathcal{N}_\omega[h \cap Coll(\omega, \delta_k)]$. Let $j = \max(i, k) + 1$. We then have that

(7) in $L(A, \mathbb{R}^M)$, $A \cap (\mathcal{N}_\omega|\delta_j)[h \cap Coll(\omega, \delta_j)] \notin Lp^{\Lambda^{stc}}((\mathcal{N}_\omega|\delta_j)[h \cap Coll(\omega, \delta_j)])$.

However, it follows from Lemma 6.4.6 that

(8) $Lp^{\Lambda^{stc}}((\mathcal{N}_\omega|\delta_j)[h \cap Coll(\omega, \delta_j)]) = \mathcal{N}_\omega|(\delta_j^+)^{\mathcal{N}_\omega}[h \cap Coll(\omega, \delta_j)]$.

(8) and (7) contradict (6) (as $\tau_{h\cap Coll(\omega,\delta_j)} = A\cap(\mathcal{N}_\omega|\delta_j)[h\cap Coll(\omega,\delta_j)]$). □

We will now make use of [15, Theorem 0.1]. It follows from the proof of the aforementioned theorem (applied to all sets of reals in $L(A,\mathbb{R}^M)$) that

(9) in $L(A,\mathbb{R}^M)$, $L(A,\mathbb{R}^M) = Lp^{\Lambda^{stc}}(\mathbb{R}^M)$.

We also have that

(10) if $\Gamma_1 = \{C\in\wp(\mathbb{R}^M)\cap\mathcal{N}_1(\mathbb{R}^M) : L(C,\mathbb{R}^M)\vDash AD^+\}$ then $L(\Gamma_1,\mathbb{R}^M)\vDash AD^+$.

It then follows from (9), (10) and homogeneity of the collapse that

(11) $A\in M$.

(11) and the Claim imply (5). □

The following is a simple corollary of the proof of Lemma 8.1.10.

Corollary 8.1.11 *Suppose $(\eta_i : i<\omega)$ is a sequence of consecutive Woodin cardinals of $\mathcal{N}|\kappa$ and $\lambda = \sup_{i<\omega}\eta_i$. The derived model of $\mathcal{R} =_{def} \mathcal{N}|(\lambda^+)^{\mathcal{N}}$ as computed via $\Phi_{\mathcal{R}}$ is $L(\Gamma(\mathcal{Q},\Sigma^{stc}_{\mathcal{Q}}))$. In particular, $\Gamma(\mathcal{Q},\Lambda^{stc}) = \wp(\mathbb{R})\cap L(\Gamma(\mathcal{Q},\Lambda^{stc}))$.*

Let Ψ be the minimal component of Λ (see Definition 3.9.8). Let $\mathcal{Q}_\infty$ be the direct limit of all Λ-iterates of $\mathcal{Q}$ and let $\pi:\mathcal{Q}\to\mathcal{Q}_\infty$ be the iteration embedding. Notice that $\pi\restriction\mathcal{Q}^b$ depends only on Ψ and hence (by the coding lemma), it is in $L(\Gamma(\mathcal{P},\Sigma))$. Also, because Ψ is fullness preserving, it follows that $\pi[\mathcal{Q}^b]$ can be coded as a subset of $w(\Gamma^b(\mathcal{Q},\Lambda))$. This is because $\mathcal{Q}^b_\infty|\delta^{\mathcal{Q}^b_\infty} = \bigcup\{\mathcal{M}_\infty(\mathcal{R},\Lambda_{\mathcal{R}}) : \mathcal{R}\in pB(\mathcal{Q},\Lambda)\}$ and $\delta^{\mathcal{Q}^b} = w(\Gamma^b(\mathcal{Q},\Lambda))$.

Lemma 8.1.12 $\Psi\in\mathcal{J}_\omega(\pi[\mathcal{Q}^b],\mathcal{Q}^b_\infty,\Gamma^b(\mathcal{Q},\Lambda))$.

Proof. Notice that if $(\vec{\mathcal{T}},\mathcal{S})\in I(\mathcal{Q},\Psi)$ and $\mathcal{W}$ is a tree on $\mathcal{S}$ of limit length according to $\Lambda_{\mathcal{S}}$ such that $\mathcal{W}$ is above $\delta^{\mathcal{S}^b}$ and $\mathcal{W}\in b(\Psi_{\mathcal{S}})$ then letting $b=\Psi_{\mathcal{S}}(\mathcal{W})$, $\mathcal{Q}(b,\mathcal{W})$ exists and has an iteration strategy in $\Gamma^b(\mathcal{Q},\Lambda)$. This is simply because there is an extender $E\in\vec{\mathcal{M}}^{\mathcal{W}}_b$ with critical point $\delta^{\mathcal{S}^b}$ such that $\mathcal{Q}(b,\mathcal{W})\lhd(Ult(\mathcal{M}^{\mathcal{W}}_b,E))^b$. We can the define Ψ in $\mathcal{J}_\omega(\pi[\mathcal{Q}^b],\mathcal{Q}^b_\infty,\Gamma^b(\mathcal{Q},\Lambda))$ with the following procedure. We work in $\mathcal{J}_\omega(\pi[\mathcal{Q}^b],\mathcal{Q}^b_\infty,\Gamma^b(\mathcal{Q},\Lambda))$.

Suppose first X is a transitive set and $\mathcal{R} \in X$ is an lsa type hod mouse. Suppose that there is an embedding $\tau : \mathcal{Q}^b \to \mathcal{R}^b$. Suppose further that $\mathcal{M}$ is an sts mouse over X based on $\mathcal{R}$. We say $\mathcal{M}$ is good if it has an iteration strategy Δ such that if $\mathcal{S}$ is a Δ-iterate of $\mathcal{M}$, $t = (\mathcal{R}, \mathcal{T}, \mathcal{R}_1^*, \vec{\mathcal{U}}) \in \mathcal{S}$ is according to $\Sigma^{\mathcal{M}}$, and $\mathcal{R}_1 = \pi^{\mathcal{T},b}(\mathcal{R}^b)$ then letting Δ^* be the strategy of $\mathcal{R}_1$ induced by Δ,

1. $(\mathcal{R}_1, \Delta)$ is a hod pair such that Δ has strong branch condensation and is strongly fullness preserving,
2. $\mathcal{R}_1 = Hull^{\mathcal{R}_1}(\pi^{\mathcal{T},b} \circ \tau[\mathcal{Q}^b] \cup \delta^{\mathcal{R}_1})$,
3. letting $\sigma : \mathcal{R}_1 \to \mathcal{Q}^b_\infty$ be given by

$$\sigma(x) = \pi(f)(\pi^{\Delta}_{\mathcal{R}_1,\infty}(a)),$$

   where $f \in \mathcal{Q}^b$ and $a \in (\delta^{\mathcal{R}_1^b})^{<\omega}$ are such that $x = \pi^{\mathcal{T},b} \circ \tau(f)(a)$,

$$\pi \restriction \mathcal{Q}^b = \sigma \circ \pi^{\mathcal{T},b} \circ \tau.$$

4. $\vec{\mathcal{U}}$ is according to Δ^*.

We can now define $Lp^{good,sts,\tau}(X)$ which is the stack of good sts mice over X that are based on $\mathcal{R}$. Then we can define $Lp^{good,sts,\tau}_\omega(X)$.

Suppose next that $\mathcal{R}$ is an lsa type hod premouse and $\tau : \mathcal{Q}^b \to \mathcal{R}^b$ is an embedding. Suppose $\vec{\mathcal{U}}$ is a stack on $\mathcal{R}^b$. We say $(\mathcal{R}^b, \vec{\mathcal{U}})$ is a τ-good iteration if there is $k : \mathcal{R}^b \to \mathcal{Q}^b_\infty$ such that $\pi \restriction \mathcal{Q}^b = k \circ \tau$ and for some $(\mathcal{S}, \Delta) \in \Gamma^b(\mathcal{Q}, \Lambda)$ such that Δ has strong branch condensation and is strongly fullness preserving, $k \restriction (\mathcal{R}^b | \delta^{\mathcal{R}^b}) \subseteq \pi^{\Delta}_{\mathcal{S},\infty}[\mathcal{S}]$ and if $\sigma : \mathcal{R} \to \mathcal{S}$ is given by

$$\sigma(x) = (\pi^{\Delta}_{\mathcal{S},\infty})^{-1}(k(x))$$

then $\vec{\mathcal{U}}$ is according to σ-pullback of Δ.

Suppose now that $\vec{\mathcal{T}} = (\mathcal{S}_i, \vec{\mathcal{T}}_i : i \leq m)$ is a stack on $\mathcal{Q}$. We say $\vec{\mathcal{T}}$ is good if the following conditions hold.

1. For every $i \leq m$, $\mathcal{S}_i$ is an lsa type hod premouse such that

$$\mathcal{S}_i = \mathcal{M}^+(\mathcal{S}_i | \delta^{\mathcal{S}_i}).$$

2. For every $i < m$, $\pi^{\vec{\mathcal{T}}_i,b}$ exists.

3. For all cutpoints $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\tau =_{def} \pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists, letting $\mathcal{W}$ be the longest normal initial segment of $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}^b}$, for all limit ordinal $\gamma < lh(\mathcal{W})$ such that $\mathcal{W} \restriction \gamma$ is ambiguous,
   (a) if $Lp^{good,sts,\tau}(\mathcal{M}^+(\mathcal{W} \restriction \gamma)) \models$ "$\delta(\mathcal{W} \restriction \gamma)$ is a Woodin cardinal" then $\mathcal{W}$ doesn't have a branch for $\mathcal{W} \restriction \gamma$ and $\mathcal{M}^{\mathcal{W}}_\gamma = \mathcal{S}_i$ for some $i \leq m$, and
   (b) if $Lp^{good,sts,\tau}(\mathcal{M}^+(\mathcal{W} \restriction \gamma)) \models$ "$\delta(\mathcal{W} \restriction \gamma)$ is not a Woodin cardinal" then $\mathcal{W}$ has a branch b for $\mathcal{W} \restriction \gamma$ such that $\mathcal{Q}(b, \mathcal{W} \restriction \gamma)$ exists and $\mathcal{Q}(b, \mathcal{W} \restriction \gamma) \trianglelefteq Lp^{good,sts,\tau}(\mathcal{M}^+(\mathcal{W} \restriction \gamma))$.

4. For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\tau =_{def} \pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists, letting $\vec{\mathcal{U}}$ be the largest initial segment of $\vec{\mathcal{T}}$ based on $\mathcal{S}^b$, $(\mathcal{S}^b, \vec{\mathcal{U}})$ is a τ-good iteration.

5. For every cutpoint $\mathcal{S}$ of $\vec{\mathcal{T}}$ such that $\tau =_{def} \pi^{\vec{\mathcal{T}}_{\leq \mathcal{S}}, b}$ exists, letting $\mathcal{U}$ be the longest normal initial segment of $\vec{\mathcal{T}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}^b}$ and is such that for some $\eta \in (\delta^{\mathcal{S}^b}, \delta^{\mathcal{S}})$, $\mathcal{U}$ is based on $\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta}$ and is above η, then letting $E \in \vec{E}^{\mathcal{S}}$ be the least extender with critical point $\delta^{\mathcal{S}}$ such that $\mathcal{O}^{\mathcal{S}}_{\eta,\eta,\eta} \triangleleft Ult(\mathcal{S}, E)$, $((Ult(\mathcal{S}, E))^b, \mathcal{U})$ is a $(\pi_E \restriction \mathcal{S}^b) \circ \tau$ good iteration.

Let then Δ be an iteration strategy for $\mathcal{Q}$ such that its domain consists of good stacks and if $\vec{\mathcal{T}} \in dom(\Delta)$ then $\Delta(\vec{\mathcal{T}}) = b$ if and only if $\vec{\mathcal{T}}^\frown\{\mathcal{M}^{\vec{\mathcal{T}}}_b\}$ is a good iteration. It can now be shown that $\Delta = \Psi$. The proof is very much like the proof of clause 2 of Theorem 6.1.5. We leave it to the reader. $\square$

We are now in a position to state the main theorem of this section.

Theorem 8.1.13 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that $\mathcal{P}$ is of lsa type and Σ has strong branch condensation and is strongly fullness preserving. Suppose $Code(\Sigma)$ is Suslin, co-Suslin. Then for some $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$,*

1. *$L(\Gamma(\mathcal{Q}, \Sigma_{\mathcal{Q}})) \cap \wp(\mathbb{R}) = \Gamma(\mathcal{Q}, \Sigma_{\mathcal{Q}})$,*
2. *the set $\{(x, y) : x \in \mathbb{R}$ and $y \notin Lp^{\Sigma^{stc}_{\mathcal{Q}}}(x)\}$ cannot be uniformized in $L(\Gamma(\mathcal{Q}, \Sigma_{\mathcal{Q}}))$, and*
3. *$L(\Gamma(\mathcal{Q}, \Sigma_{\mathcal{Q}})) \models LSA$.*

Proof. Let $\Gamma_0 < \Gamma$ be any two good pointclass such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma_0}$. Let F be as in Theorem 4.1.6 for Γ. Let $A \in \Gamma^\omega$ be a set coding a self-justifying-system $(A_i : i < \omega)$ such that $A_0 = \{(x, y) \in \mathbb{R}^2 : y \in C_{\Gamma_0}(x))\}$. Fix x such that if

$F(x) = (\mathcal{N}_x^*, \mathcal{M}_x, \delta_x, \Sigma_x)$ then $Code(\Sigma)$ and $\vec{A}$ are Suslin, co-Suslin captured by $(\mathcal{N}_x^*, \delta_x, \Sigma_x)$.

We then have that the fully backgrounded hod pair construction of $\mathcal{N}_x^*|\delta_x$ reaches a tail of $(\mathcal{P}, \Sigma)$ (see Theorem 4.6.10). Let $(\mathcal{Q}, \Lambda)$ be this tail. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}, \Lambda^{stc}})^{\mathcal{N}_x^*|\delta_x}$. Because Σ is fullness preserving we have that $\mathcal{N} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". Let Φ be the strategy of $\mathcal{N}$ induced by Σ_x. We now start proving that $(\mathcal{Q}, \Lambda)$ is as desired.

Clause 1 is just Lemma 8.1.11. We prove clause 2 of Theorem 8.1.13, which amounts to showing that the set $B = \{(x, y) : x \in \mathbb{R} \wedge y \notin Lp^{\Lambda^{stc}}(x)\}$ cannot be uniformized in $L(\Gamma(\mathcal{P}, \Sigma))$. Towards a contradiction assume we can uniformize B. It follows that we can find a set of reals $A \in \Gamma(\mathcal{P}, \Sigma)$ such that A codes a sjs $(A_i : i < \omega)$ with the property that $A_0 = B$.

Let $\pi : \mathcal{N}|(\kappa^+) \to \mathcal{M}$ be an $\mathbb{R}$-geneicity iteration. We then have that A is in the derived model of $\mathcal{M}$. Fix then a $< \pi(\kappa)$-generic g over $\mathcal{M}$ such that there is a term relation $\tau \in \mathcal{M}[g]$ realizing A. Let δ be a cutpoint Woodin cardinal of $\mathcal{M}$ such that g is a $< \delta$-generic. Let $\xi < \delta$ be a cutpoint $\mathcal{M}$-cardinal such that $\mathcal{M}$ has no Woodin cardinals in the interval (ξ, δ). Let $\mathcal{M}^* \trianglelefteq \mathcal{M}$ be such that $\tau \in \mathcal{M}^*$ and $\mathcal{M}|\pi(\kappa) \trianglelefteq \mathcal{M}^*$. Let now $\sigma : \mathcal{S} \to \mathcal{M}^*$ be such that $\mathrm{crit}(\sigma) \in (\xi, \delta)$, $\sigma(\mathrm{crit}(\sigma)) = \delta$, $\mathrm{crit}(\sigma)$ is an $\mathcal{M}$-cardinal and $\tau \in rng(\sigma)$. It follows that $Lp^{\Lambda^{stc}}(\mathcal{M}|\mathrm{crit}(\sigma)) \in \mathcal{S}$ and $Lp^{\Lambda^{stc}}(\mathcal{M}|\mathrm{crit}(\sigma)) \vDash$ "$\mathrm{crit}(\sigma)$ is a Woodin cardinal", contradiction! This finishes the proof of clause 2 of Theorem 8.1.13.

To finish the proof of Theorem 8.1.13 we need to show that $L(\Gamma(\mathcal{Q}, \Lambda)) \vDash \mathsf{LSA}$. Suppose first that

(1) for every transitive $X \in HC$ such that $\mathcal{Q} \in X$ and for every $\mathcal{R} \trianglelefteq Lp^{\Lambda^{stc}}(X)$ such that $\rho(\mathcal{R}) = o(X)$, if Φ is the iteration strategy of $\mathcal{R}$ witnessing that $\mathcal{R}$ is a Λ^{stc}-sts mouse then $\Gamma(\mathcal{R}, \Phi) <_w \Gamma^b(\mathcal{Q}, \Lambda)$.

We claim that (1) implies $L(\Gamma(\mathcal{Q}, \Lambda)) \vDash \mathsf{LSA}$. Towards contradiction assume not and set $B = \{(x, y) : x \in \mathbb{R} \wedge y \notin Lp^{\Lambda^{stc}}(x)\}$. We claim that

(2) B is Suslin, co-Suslin in $L(\Gamma(\mathcal{Q}, \Lambda))$.

Clearly (2) contradicts clause 2 of Theorem 8.1.13. To see (2), let Ψ be the minimal component of Λ^{stc} (see Definition 3.9.8). Because $\Gamma^b(\mathcal{Q}, \Lambda) = \Gamma^b(\mathcal{Q}, \Psi)$, it follows from (1) that

(3) B is projective in Ψ.

Let $\mathcal{Q}_\infty$ be the direct limit of all Λ-iterates of $\mathcal{Q}$ and let $\pi : \mathcal{Q} \to \mathcal{Q}_\infty$ be the iteration embedding. Notice that $\pi \restriction \mathcal{Q}^b$ depends only on Ψ and hence, because of Lemma 8.1.12, it is in $L(\Gamma(\mathcal{P},\Sigma))$. Also, because Ψ is fullness preserving, it follows that $\pi[\mathcal{Q}^b]$ can be coded as a subset of $w(\Gamma^b(\mathcal{Q},\Lambda))$. This is because $\mathcal{Q}^b_\infty|\delta^{\mathcal{Q}^b_\infty} = \bigcup\{\mathcal{M}_\infty(\mathcal{R},\Lambda_{\mathcal{R}}) : \mathcal{R} \in pB(\mathcal{Q},\Lambda)\}$ and $\delta^{\mathcal{Q}^b} = w(\Gamma^b(\mathcal{Q},\Lambda))$.

It follows from (3) and Lemma 8.1.12, $B \in \mathcal{J}_\omega(\pi[\mathcal{Q}^b], \mathcal{Q}^b, \Gamma^b(\mathcal{Q},\Lambda))$. Since we are assuming $L(\Gamma(\mathcal{P},\Sigma)) \vDash \neg\mathsf{LSA}$ and since, in $L(\Gamma(\mathcal{P},\Sigma))$, $\delta^{\mathcal{Q}^b_\infty}$ is both $< \Theta$ and is a limit of Suslin cardinals, B must be Suslin, co-Suslin in $L(\Gamma(\mathcal{P},\Sigma))$, implying (2). Thus, it is enough to prove (1).

Suppose (1) fails. We can then assume that the witness is in some $< \delta_x$-generic extension of $\mathcal{N}^*_x$. Moreover, by iterating if necessary, we can assume that X is $< \kappa$-generic over $\mathcal{N}$. Let then $\mathcal{R} \trianglelefteq Lp^{\Lambda^{stc}}(X)$ be least such that $\rho(\mathcal{R}) = o(X)$ yet if Δ is the strategy of $\mathcal{R}$ then $\Gamma(\mathcal{R},\Delta) = \Gamma^b(\mathcal{Q},\Lambda)$. Notice that we have that

(4) $Code(\Delta)$ is Suslin, co-Suslin in $L(\Gamma(\mathcal{Q},\Lambda))$ (this follows from Lemma 8.1.9).

We again let Ψ be the minimal component of Λ^{stc}. It follows that for some $< \kappa$-generic h over $\mathcal{N}$, there is some $(\mathcal{T},\mathcal{S}^*) \in I(\mathcal{Q},\Psi) \cap \mathcal{N}|\kappa[h]$ such that $\Lambda_{\mathcal{S}^*} \in L(\Gamma(\mathcal{Q},\Lambda))$ (this can be shown using Theorem 4.6.8 and the fact that Ψ is Suslin, co-Suslin in $L(\Gamma(\mathcal{Q},\Lambda))$, which follows from (4) and Lemma 8.1.12). It then follows that if $\mathcal{S}$ is such that $(\mathcal{T},\mathcal{S}) \in I(\mathcal{Q},\Lambda) \cap \mathcal{N}|\kappa[h]$ then the fragment of $\Lambda_{\mathcal{S}} \restriction \mathcal{N}|\kappa[h]$ that acts on stacks based on $\mathcal{S}^*$ is in $\mathcal{N}[h]$ (in fact, $\Lambda_{\mathcal{S}} \restriction \mathcal{N}|\kappa[h] \in \mathcal{N}$ because of Lemma 8.1.6).

Let now $\delta > o(\mathcal{S})$ be a cutpoint Woodin cardinal of $\mathcal{N}|\kappa$. Let $\mathcal{S}_1$ be an iterate of $\mathcal{S}$ above $\delta^{\mathcal{S}}$ that is built according to the rules of $\mathcal{N}|\delta$-genericity iteration. We have that $\mathcal{S}_1 \in \mathcal{N}[h]|(\delta^+)^{\mathcal{N}}$. Let $\mathcal{N}_1$ be the output of $(\mathcal{J}^{\vec{E},\Lambda^{stc}_{\mathcal{S}_1}})^{\mathcal{N}}$. It follows from fullness preservation that $\mathcal{N}_1 \vDash$ "$\delta^{\mathcal{S}}$ is a Woodin cardinal".

Let $\mathcal{N}_2$ be the $(\mathcal{N}_1, \pi^{\mathcal{T},b}[\mathcal{Q}^b])$-authenticated backgrounded construction over $\mathcal{N}|\delta$ based on $\mathcal{Q}$ (this makes sense as $\mathcal{N}|\delta$ is generic over $\mathcal{S}_1$ and $\pi^{\mathcal{T},b} \in \mathcal{N}|\delta$, see Definition 6.2.2). Then it follows from universality of $\mathcal{N}_2$ that $\mathcal{N}|(\delta^+)^{\mathcal{N}} \subseteq \mathcal{N}_2 \subseteq \mathcal{N}_1[\mathcal{N}|\delta]$. However, $\delta^{\mathcal{S}_1}$ is not a cardinal of $\mathcal{N}$ yet it is a cardinal of $\mathcal{N}_1[\mathcal{N}|\delta]$, contradiction! This finishes the proof of (1) and hence, the proof of Theorem 8.1.13. $\square$

The next theorem can now be proved using Corollary 8.1.11 and the proof of Theorem 5.20 of [10].

Theorem 8.1.14 *Suppose $(\mathcal{P},\Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a successor ordinal and Σ has a branch condensation and is fullness preserving. Suppose $B \in \mathbb{B}(\mathcal{P}^-,\Sigma_{\mathcal{P}^-})$. There is then $\mathcal{Q} \in pI(\mathcal{P},\Sigma)$ and $\vec{B} = \langle B_i : i < \omega\rangle \subseteq \mathbb{B}(\mathcal{P},\Sigma_{\mathcal{P}^-})$ such that $\vec{B}$ strongly guides $\Sigma_{\mathcal{Q}}$.*

8.2 A hybrid upper bound for LSA

The main theorem of this section, Theorem 8.2.6, is a corollary to the proofs given in the previous section. It can be used in core model induction applications to show that certain hypothesis imply that there is a model of LSA. We give a fairly detailed proof of Theorem 8.2.6.

Definition 8.2.1 *Suppose $(\mathcal{P}, \Sigma)$ is an sts hod pair. We let $\mathcal{N}^{\#}_{\omega,lsa}(\mathcal{P}, \Sigma)$ be the minimal Σ-sts mouse $\mathcal{M}$ over $\mathcal{P}$ such that $\mathcal{M}$ has ω many Woodin cardinals greater than $\delta^{\mathcal{P}}$ such that if λ is their sup then $\mathcal{M} = \mathcal{M}^+(\mathcal{M}|\lambda)$.*

Definition 8.2.2 *We say $\mathcal{N}$ is an active ω Woodin lsa mouse if it has an iteration strategy Σ such that*

1. *$\mathcal{N}$ has a Woodin cardinal δ such that if κ is the least $< \delta$-strong cardinal of $\mathcal{N}$ then letting $\mathcal{P} = \mathcal{N}|((\delta^{\mathcal{P}})^{+\omega})^{\mathcal{N}}$, $(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$ is an sts hod pair such that $\Sigma^{stc}_{\mathcal{P}}$ has strong branch condensation and is strongly $\Gamma^b(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$-fullness preserving,*

2. *$\mathcal{N} = \mathcal{N}^{\#}_{\omega,lsa}(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$,*

3. *for every $\alpha < \lambda^{\mathcal{P}}$ and for every $\xi \in (\delta^{\mathcal{P}}_{\alpha}, o(\delta^{\mathcal{P}}_{\alpha}))$, if $\mathcal{M}^+(\mathcal{P}|\xi) \vDash$ "ξ is a Woodin cardinal" then*

$$\mathcal{N}^{\#}_{\omega,lsa}(\mathcal{M}^+(\mathcal{P}|\xi), \Sigma^{stc}_{\mathcal{M}^+(\mathcal{P}|\xi)}) \vDash \text{“}\xi \textit{ is not a Woodin cardinal”}.$$

We say $\mathcal{P}$ is the lsa part of $\mathcal{N}$. We say $(\mathcal{N}, \Sigma)$ is an active ω Woodin lsa pair. It follows that $\rho(\mathcal{N}) \leq (\kappa^+)^{\mathcal{N}}$ where κ is as in clause 1[3].

In what follows, we let the statement *there is an active ω Woodin lsa pair* be shortening for the statement that there is a pair $(\mathcal{N}, \Sigma)$ such that $\mathcal{N}$ is an active ω Woodin lsa mouse and Σ witnesses the clauses of Definition 8.2.2.

Notice that it follows from Theorem 4.10.4 that if $(\mathcal{N}, \Sigma)$ and $(\mathcal{M}, \Lambda)$ are two active ω Woodin lsa pairs with common lsa part $\mathcal{P}$ such that $\Sigma^{stc} = \Lambda^{stc}$ then $\mathcal{N} = \mathcal{M}$ and $\Sigma = \Lambda$.

[3] The fact that $\rho(\mathcal{N}) \leq (\kappa^+)^{\mathcal{N}}$ can be proved as follows. Suppose that $\rho(\mathcal{N}) > (\kappa^+)^{\mathcal{N}}$. Let $\mathcal{M} = Hull^{\mathcal{N}}((\kappa^+)^{\mathcal{N}})$. Clearly $\mathcal{M}$ is also an active ω Woodin lsa mouse. We would be done if we had $\mathcal{M} \trianglelefteq \mathcal{N}$. To show this, we use the proof of Theorem 4.9.7, and compare $(\mathcal{N}, \mathcal{M}, (\kappa^+)^{\mathcal{N}})$ with $\mathcal{N}$. We need to verify that a version of Lemma 4.9.5 holds for $(\mathcal{N}, \mathcal{M}, (\kappa^+)^{\mathcal{N}})$. However, this can be done via exactly the same proof. We leave the details to the reader.

Lemma 8.2.3 *Suppose $(\bar{\mathcal{N}}, \Sigma)$ is an active ω Woodin lsa pair and $\mathcal{P}$ is the lsa part of $\bar{\mathcal{N}}$. Let $\mathcal{N}$ be the result of iterating the last extender of $\bar{\mathcal{N}}$ through the ordinals. Let $(\delta_i : i < \omega)$ be the Woodin cardinals of $\mathcal{N}$ above $\delta^{\mathcal{P}}$ and let λ be their supremum. Let $\pi : \mathcal{N} \to \mathcal{M}$ be an iteration via Σ that is above $\delta^{\mathcal{P}}$. Suppose g is $< \pi(\lambda)$-generic over $\mathcal{M}$ and $\mathcal{S} \in (\mathcal{M}|\lambda[g]) \cap pB(\mathcal{P}, \Sigma^{stc})$. Then $\mathcal{S}$ is an $\mathcal{M}$-authenticated hod premouse.*

Proof. Towards a contradiction assume not. We assume $\pi = id$ and $g = \emptyset$, the proof of this special case can be easily generalized. We can then find an iteration $\sigma : \mathcal{N} \to \mathcal{N}_1$ above $\delta^{\mathcal{P}}$ and $\mathcal{S} \in pB(\mathcal{P}, \Sigma^{stc}) \cap (\mathcal{N}_1|\sigma(\lambda))$ such that if $k < \omega$ is such that $\mathcal{S} \in \mathcal{N}_1|\sigma(\delta_k)$ then

(1) for every iteration $\pi : \mathcal{N}_1 \to \mathcal{M}$ according to Σ and above $\sigma(\delta_k)$ and for any $\mathcal{Q} \in pB(\mathcal{S}, \Sigma^{stc}_{\mathcal{S}}) \cap (\mathcal{M}|\pi(\lambda))$, for some $\kappa < \pi(\lambda)$ and $l \in (k, \omega)$, $Code(\Sigma_{\mathcal{Q}}) <_w Code(\Sigma_{(\mathcal{S}_\kappa^{\mathcal{M}|\pi(\delta_l)})^b})$ and
(2) for any $\kappa < \pi(\lambda)$ and $l \in (k, \omega)$, $Code(\Sigma_{(\mathcal{S}_\kappa^{\mathcal{N}|\delta_l})^b}) <_w Code(\Sigma_{\mathcal{S}})$.

The strict inequality in (2) is a consequence of Lemma 8.1.3. Without loss of generality we assume $\mathcal{N} = \mathcal{N}_1$. Let k be such that $\mathcal{S} \in \mathcal{N}|\delta_k$. Let $\mathcal{P}_1 = \mathcal{S}_{\delta_k}^{\mathcal{N}|\delta_{k+1}}$ and let $\mathcal{T}$ be the comparison tree on $\mathcal{P}$ such that $\mathcal{P}_1 = \mathcal{M}^+(\mathcal{T})$. Notice that we must have that $\pi^{\mathcal{T},b}$ exists (this is a consequence of Lemma 8.1.3). We can now compare $\mathcal{S}$ with the construction producing $\mathcal{P}_1$ in $\mathcal{N}$. This comparison is done via $\mathcal{N}$-authentication procedure. We outline it below.

Suppose $\mathcal{U}$ is an initial segment of the comparison tree on $\mathcal{S}$ with last model $\mathcal{S}_1$. Suppose $\mathcal{U}$ is of limit length. Let α be largest such that $\mathcal{S}_1(\alpha) = \mathcal{P}_1(\alpha)$. Suppose first that $\mathcal{S}_1(\alpha+1)$ is of successor type. As $\mathcal{P}_1$ is fully backgrounded it follows that $\mathcal{P}_1(\alpha+1)$ is also of successor type. It follows that the rest of $\mathcal{U}$ is a stack on $\mathcal{S}_1(\alpha+1)$ and is a result of comparing $\mathcal{S}_1(\alpha+1)$ with $\mathcal{P}_1(\alpha+1)$.

Suppose that $\mathcal{M}(\mathcal{U}) \lhd \mathcal{P}_1(\alpha+1)$. Then let $\mathcal{W} \trianglelefteq \mathcal{P}_1(\alpha+1)$ be the least such that $\mathcal{W} \vDash$ "$\delta(\mathcal{U})$ is a Woodin cardinal" but $\mathcal{J}_1(\mathcal{W}) \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal". Because $\mathcal{S}$ is a Σ-iterate of $\mathcal{P}$ it follows there is a branch b of $\mathcal{U}$ (the branch chosen by $\Sigma_{\mathcal{S}}$) such that $\mathcal{Q}(b, \mathcal{U})$ exists and $\mathcal{Q}(b, \mathcal{U}) = \mathcal{W}$. Then clearly $b \in \mathcal{N}$ and we let II play b. Next suppose that $\mathcal{M}(\mathcal{U}) = \mathcal{P}_1(\alpha+1)|\delta^{\mathcal{P}_1(\alpha+1)}$. In this case we look for a branch b of $\mathcal{U}$ such that for some $\beta \in b$, $s(\mathcal{T}, \alpha+1) \subseteq \pi^{\mathcal{U}}_{\beta,b}$. Again the branch chosen by Σ is the unique branch with this property, and so there is such a branch in $\mathcal{N}$ and we can extend $\mathcal{U}$ by letting II play such a branch.

Next suppose that $\mathcal{S}_1(\alpha+1)$ is of limit type. It follows that $\delta_\alpha^{\mathcal{S}_1(\alpha+1)}$ is a measurable cardinal in $\mathcal{S}_1(\alpha+1)$. Suppose then there is $\mathcal{W} \trianglelefteq \mathcal{P}_1(\alpha+1)$ such that $\mathcal{W} \vDash$ "$\delta(\mathcal{U})$ is a Woodin cardinal" but $\mathcal{J}_1(\mathcal{W}) \vDash$ "$\delta(\mathcal{U})$ is not a Woodin cardinal". We can then identify $\Sigma_{\mathcal{S}}(\mathcal{U})$ inside $\mathcal{N}$ as above and extend $\mathcal{U}$ accordingly.

Assume then there is no such $\mathcal{W}$. Let $b = \Sigma_{\mathcal{S}}(\mathcal{U})$. Because $\mathcal{S} \in pB(\mathcal{P}, \Sigma^{stc})$, we have that $\mathcal{Q}(b, \mathcal{U})$ exists and is a $\Sigma^{stc}_{\mathcal{M}^+(\mathcal{U})}$-sts mouse over $\mathcal{M}^+(\mathcal{U})$. It follows that

(3) $\mathcal{P}_1(\alpha+1) = \mathcal{M}^+(\mathcal{T}) = \mathcal{M}^+(\mathcal{U})$ and $\mathcal{T}$ is an $\mathcal{N}$-ambiguous tree.

We now work towards showing that $\mathcal{N}$ has a branch indexed for $\mathcal{T}$. Let $\mathcal{K}$ be the $\mathcal{N}$-authenticated background construction over $\mathcal{M}^+(\mathcal{T})$ in which extenders used have critical point $> \delta_k$.

Claim 1. $\mathcal{K}$ has ω Woodin cardinals.

Proof. Suppose not. This can only happen if the construction stops at some stage $\mathcal{K}^*$ and this can happen only if we encounter some stack $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{T}_1, \mathcal{P}_2, \vec{\mathcal{U}}) \in \mathcal{K}^*$ of length 2 such that according to our indexing scheme (see Definition 3.8.2), we have to index a branch of t in $\mathcal{K}^*$ yet we cannot find an $\mathcal{N}$-authenticated branch of t. Notice, however, that because $\mathcal{P}_2^b \in pB(\mathcal{S}, \Sigma_{\mathcal{S}})$, we have that $\mathcal{P}_2$ is $\mathcal{N}$-authenticated and so, we must have that $(\mathcal{P}_2^b, \vec{\mathcal{U}})$ is an $\mathcal{N}$-authenticated iteration. Also, notice that if $\vec{\mathcal{U}} = \emptyset$ then the branch of t just depends on $\mathcal{K}^*$ and not our authentication procedure. $\square$

Our goal now is to compare the construction producing $\mathcal{K}$ and $\mathcal{Q}(b, \mathcal{U})$. Let Ψ be the strategy of $\mathcal{Q}(b, \mathcal{U})$ induced by $\Sigma_{\mathcal{S}}$ and acting on trees above $\delta(\mathcal{U})$

Claim 2. The comparison of the construction producing $\mathcal{K}$ and $\mathcal{Q}(b, \mathcal{U})$ is successful.

Proof. Suppose not. We can then find a normal tree $\mathcal{U}_1$ on $\mathcal{Q}(b, \mathcal{U})$ with last model $\mathcal{Q}_1$ and a normal tree $\mathcal{T}_1$ on $\mathcal{N}$ with last model $\mathcal{N}_1$ such that $\mathcal{U}_1$ is according to Ψ, $\mathcal{T}_1$ is according to Σ and for some $\beta \notin dom(\vec{E})^{\mathcal{Q}_1}$, letting $\mathcal{K}_1 = \pi^{\mathcal{T}_1}(\mathcal{K})$, $\mathcal{Q}_1|\beta = \mathcal{K}_1|\beta$ and $\mathcal{Q}_1||\beta \neq \mathcal{K}_1||\beta$. Let then $t = (\mathcal{M}^+(\mathcal{T}), \mathcal{W}, \mathcal{R}, \vec{\mathcal{W}}_1) \in \mathcal{Q}_1|\beta$ be a stack of length 2 whose branch is indexed at β. It follows that t is a stack whose branch should be indexed at β in $\mathcal{K}_1$. Let c be the branch of t in $\mathcal{Q}_1$. Let e, if it exists, be the branch of t in $\mathcal{K}_1$. Notice that if $\vec{\mathcal{W}}_1$ is undefined then both c and e exists and are equal as such branches just depend on $\mathcal{Q}_1|\beta = \mathcal{K}_1|\beta$.

We thus have that c is a branch of $\vec{\mathcal{W}}_1$. Notice that if e exists then $e = \Sigma_{\mathcal{R}^b}(\vec{\mathcal{W}}_1)$. It follows that $e = c$. We thus have that e doesn't exist. It follows that in $\mathcal{N}_1$, $(\mathcal{R}^b, \vec{\mathcal{W}}_1)$ is not an $\mathcal{N}_1$-authenticated iteration. Since $\text{crit}(\pi^{\mathcal{T}_1}) > \delta_k$ and since $\mathcal{R}^b \in pB(\mathcal{S}, \Sigma^{stc}_{\mathcal{S}})$, we get a contradiction to (1). □

Because $\mathcal{K}$ has ω Woodin cardinals and is a proper class model, it follows from Claim 2 and clause 3 of Definition 8.2.2 that $\mathcal{Q}(b,\mathcal{U}) \trianglelefteq \mathcal{K}$. We thus have that $\mathcal{Q}(b,\mathcal{U}) \in \mathcal{N}$. It follows that to show that $\mathcal{N}$ has a branch indexed for $\mathcal{T}$, it is enough to show that clause 4 of Definition 3.8.2 holds for $\mathcal{Q}(b,\mathcal{U})$ and c where $c = \Sigma_{\mathcal{P}}(\mathcal{T})$. Let $\mathcal{W} = \mathcal{Q}(c,\mathcal{T}) = \mathcal{Q}(b,\mathcal{U})$. To do this, we need to show that

(4) there is $\mathcal{M} \trianglelefteq \mathcal{N}$ such that $c \in \mathcal{M}$ is a cofinal branch through $\mathcal{T}$ such that for some pair (β,γ) such that $\gamma < \alpha$ and $\beta < o(\mathcal{M})$,

1. $\mathcal{M}|\beta$ is unambiguous (see Definition 3.6.1) and $\mathcal{M}|\beta \vDash \mathsf{ZFC}+$"there are infinitely many Woodin cardinals $> \delta(\mathcal{T})$",
2. $b \in \mathcal{M}|\beta$ and $\mathcal{M}|\beta \vDash$ "b is well-founded branch",
3. $\mathcal{M}|\beta \vDash$ "$\mathcal{Q}(b,\mathcal{T})$ exists and is an sts ψ_γ-premouse over $\mathcal{M}(\mathcal{T})$" and
4. letting $(\delta_i : i < \omega)$ be the first ω Woodin cardinals $> \delta(\mathcal{T})$ of $\mathcal{M}|\beta$, $\mathcal{M}|\beta \vDash$ "$\mathcal{W}$ is $< Ord$-iterable above $\delta(\mathcal{T})$ via a strategy Σ such that letting $\lambda = \sup_{i<\omega} \delta_i$, for every generic $g \subseteq Coll(\omega, <\lambda)$, Σ has an extension $\Sigma^+ \in D(\mathcal{M}|\beta,\lambda,g)$ such that $D(\mathcal{M},\lambda,g) \vDash$ "Σ^+ is an ω_1-iteration strategy" and whenever $\mathcal{R} \in D(\mathcal{M}|\beta,\lambda,g)$ is a Σ^+-iterate of $\mathcal{W}$ and $t \in \mathcal{R}$ is a stack on $\mathcal{M}^+(\mathcal{T})$ of length 2 then t is $(\mathcal{P},\Sigma^{\mathcal{M}})$-authenticated".

To show the existence of such an $\mathcal{M}$, it is enough to show that $\mathcal{N}$ satisfies clauses 1-4 and first three clauses are straightforward. We show that clause 4 holds with $(\delta_i : i \in (k+2,\omega))$ as our sequence of Woodin cardinals. We next identify the model $\mathcal{R}$ in the construction producing $\mathcal{K}$ such that $\mathcal{C}(\mathcal{R}) = \mathcal{W}$. We first claim that

Claim 3. if $\mathcal{K}_1$ is the $\mathcal{N}$-authenticated construction of $\mathcal{N}|\delta_{k+2}$ over $\mathcal{M}^+(\mathcal{T})$ using extenders with critical point $> \delta^{\mathcal{N}}_{k+1}$ then $\mathcal{K}_1 \trianglelefteq \mathcal{K}$.

Proof. Suppose not. It follows from the proof of Claim 2 that $\mathcal{K}_1$ has height δ_{k+2}. If $\mathcal{K}_1 \ntrianglelefteq \mathcal{K}$ then there is some model $\mathcal{Q}$ appearing in the construction producing $\mathcal{K}$ such that $\rho(\mathcal{Q}) < \delta_{k+2}$. Let p be the standard parameter of $\mathcal{Q}$. Let $X \prec \mathcal{Q}$ be such that $\rho(\mathcal{Q}) < X \cap \delta_{k+2} \in \delta_{k+2}$ is a cardinal in $\mathcal{N}$[4] and $\bar{\mathcal{Q}}$ be the transitive collapse of X. By condensation (using the fact that X contains solidity witnesses for p), $\bar{\mathcal{Q}} \triangleleft \mathcal{Q}$. Since

[4]This is possible because δ_{k+2} is strongly inaccessible in $\mathcal{N}$.

$\bar{\mathcal{Q}}$ is sound and $\rho(\bar{\mathcal{Q}}) = \rho(\mathcal{Q}) < X \cap \delta$, $X \cap \delta$ is not a cardinal in $\mathcal{N}$. Contradiction. $\square$

It follows from Claim 3 that $\mathcal{W} \trianglelefteq \mathcal{K}_1$. To complete the proof of Clause 4 of (4), it is now enough to show the following claim.

Claim 4. Suppose $\eta \in (\delta_{k+1}, \delta_{k+2})$ is an $\mathcal{N}$-cardinal and $g \subseteq Coll(\omega, (\eta^+)^{\mathcal{N}})$. Let Φ be the fragment of Σ that acts on non-dropping trees that are based on $\mathcal{N}|(\eta^+)^{\mathcal{N}}$ and are above δ_{k+1}. Then $\Phi \restriction \mathcal{N}|\lambda[g] \in \mathcal{N}|\lambda[g]$ and if $\Lambda = \Phi \restriction HC^{\mathcal{N}|\lambda[g]}$ then in $\mathcal{N}[g]$, Λ is a $< \lambda$-universally Bair iteration strategy such that for any poset $\mathbb{P} \in \mathcal{N}|\lambda[g]$, if $k \subseteq \mathbb{P}$ is $\mathcal{N}[g]$-generic and Λ^k is the canonical extension of Λ to $HC^{\mathcal{N}|\lambda[g*k]}$ then $\Lambda^k = \Psi \restriction HC^{\mathcal{N}|\lambda[g*k]}$.

Proof. We only prove that $\Phi \restriction \mathcal{N}|\lambda[g] \in \mathcal{N}|\lambda[g]$ and leave the rest to the reader. Let $\mathcal{Q} = \mathcal{N}|(\eta^+)^{\mathcal{N}}$ and let $\mathcal{W}_1 \in \mathcal{N}[g]$ be a tree on $\mathcal{Q}$ of limit length and according to Φ. Let $e = \Phi(\mathcal{W}_1)$. We want to show that $e \in \mathcal{N}[g]$ and $\mathcal{N}[g]$ has uniform way of identifying e. Notice that $\mathcal{Q}(e, \mathcal{W}_1)$ exists. Let $\mathcal{K}_2$ be the $\mathcal{N}$-authenticated background construction over $\mathcal{M}(\mathcal{W}_1)$. The proof of Claim 1 and Claim 2 show that $\mathcal{Q}(e, \mathcal{W}_1) \trianglelefteq \mathcal{K}_2$. It is now easy to find the uniform definition of e. $\square$

Claim 4 finishes the proof of Lemma 8.2.3. $\square$

Corollary 8.2.4 *Suppose $(\bar{\mathcal{N}}, \Sigma)$ is an active ω Woodin lsa pair and $\mathcal{P}$ is the lsa part of $\bar{\mathcal{N}}$. Let $\mathcal{N}$ be the result of iterating the last extender of $\bar{\mathcal{N}}$ through the ordinals. Let Φ be the fragment of Σ that acts on stacks above $\delta^{\mathcal{P}}$. Then Φ is $\Gamma^b(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$-fullness preserving.*

Proof. Given $\mathcal{S} \in pB(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$, let $\pi : \mathcal{N} \to \mathcal{M}$ be a Σ-iterate of $\mathcal{N}$ above $\delta^{\mathcal{P}}$ such that $\mathcal{S}$ is generic over $\mathcal{M}$ for the extender algebra at the first Woodin of $\mathcal{M}$ that is larger than $\delta^{\mathcal{P}}$. It follows from Lemma 8.2.3 that $\mathcal{S}$ is $\mathcal{M}$-authenticated. $\square$

Lemma 8.2.5 *Suppose $(\bar{\mathcal{N}}, \Sigma)$ is an active ω Woodin lsa pair and $\mathcal{P}$ is the lsa part of $\bar{\mathcal{N}}$. Let $\mathcal{N}$ be the result of iterating the last extender of $\bar{\mathcal{N}}$ through the ordinals. Let $\delta < \eta$ be two consecutive Woodin cardinals of $\mathcal{N}$ such that $\delta > \delta^{\mathcal{P}}$. Let $\mathcal{N}^*$ be the output of $\mathcal{N}$-authenticated background construction of $\mathcal{N}|\eta$ in which extenders used have critical point $> \delta$. Then*

1. *$\mathcal{N}^*$ has height η and*

2. *if $\mathcal{N}_1$ is the result of translating $\mathcal{N}$ onto a structure over $\mathcal{N}^*$ via S-constructions then $\mathcal{N}_1$ is a normal iterate of $\mathcal{N}$ via a tree that is based on $\mathcal{N}|\delta_0$ where δ_0 is the least Woodin cardinal of $\mathcal{N}$ above $\delta^{\mathcal{P}}$.*

Proof. We start by verifying clause 1. Suppose $\mathcal{N}^*$ fails to reach height η. This can only happen if at some stage of the construction we reach a model $\mathcal{M}$ such that there is some $t = (\mathcal{P}, \mathcal{T}, \mathcal{P}_1, \vec{\mathcal{U}}) \in \mathcal{M}$ such that $t \in dom(\Sigma^{\mathcal{M}})$, $t \notin dom(\Sigma^{\mathcal{N}})$, and it is required by the rules of sts indexing scheme that we add a branch of t to $\mathcal{M}$. It follows that if $\vec{\mathcal{U}} = \emptyset$ then the branch of $\mathcal{T}$ just depends on $\mathcal{M}$. So $\vec{\mathcal{U}} \neq \emptyset$, and hence $(\mathcal{P}_1^b, \vec{\mathcal{U}})$ is not an $\mathcal{N}$-authenticated iteration. It then follows that $\mathcal{P}_1^b$ is not an $\mathcal{N}$-authenticated hod premouse, contradicting Lemma 8.2.3.

We verify clause 2. Notice that $\mathcal{N}_1[\mathcal{N}|\eta] = \mathcal{N}$. Thus $\mathcal{N}_1$ is η-sound ω Woodin mouse. It is then enough to show that there is a tree $\mathcal{U} \in \mathcal{N}$ on $\mathcal{N}|\delta_0$ such that $\mathcal{M}(\mathcal{U}) = \mathcal{N}^*$.

Suppose not. Let $\mathcal{U} \in \mathcal{N}$ be the tree on $\mathcal{N}|\delta_0$ that is a result of comparing $\mathcal{N}|\delta_0$ with the construction producing $\mathcal{N}^*$. Since comparison fails, we must have that $\Sigma(\mathcal{U}) \notin \mathcal{N}$. Let $b = \Sigma(\mathcal{U})$. We must have that $\mathcal{Q}(b, \mathcal{U})$ exists and $\mathcal{Q}(b, \mathcal{U}) \ntrianglelefteq \mathcal{N}^*$. It follows that $\mathcal{N}^* \vDash$ "$\delta(\mathcal{U})$ is Woodin". It follows from Lemma 6.4.6 that $\mathcal{Q}(b, \mathcal{U}) \in \mathcal{N}$ ($\mathcal{Q}(b, \mathcal{U})$ can be obtained as via an S-construction). Thus, in the further comparison of $\mathcal{Q}(b, \mathcal{U})$ and the construction producing $\mathcal{N}^*$, $\mathcal{N}^*$ side does not move.

Let $\mathcal{W}_0 = \mathcal{Q}(b, \mathcal{U})$. We can then successivly produce a sequence $(\mathcal{W}_i, \mathcal{U}_i, b_i)$ such that

1. $\mathcal{U}_i$ is a tree on $\mathcal{W}_i$ that is a result of comparing $\mathcal{W}_i$ with the construction of producing $\mathcal{N}^*$,

2. $b_i = \Sigma_{\mathcal{W}_i}(\mathcal{U}_i)$,

3. $\mathcal{W}_{i+1} = \mathcal{Q}(b_i, \mathcal{U}_i)$,

It then follows that $\mathcal{N}^* \vDash$ "$\delta(\mathcal{U}_i)$ are Woodin cardinals" and if $\eta = \sup_{i<\omega} \delta(\mathcal{U}_i)$ then $\mathcal{M}^+(\mathcal{N}^*|\eta) \trianglelefteq \mathcal{N}^*$. This contradicts the minimality of $\mathcal{N}$. $\square$

Theorem 8.2.6 *Suppose $(\bar{\mathcal{N}}, \Sigma)$ is an active ω Woodin lsa pair and $\mathcal{P}$ is the lsa part of $\bar{\mathcal{N}}$. Let $\mathcal{N}$ be the result of iterating the last extender of $\bar{\mathcal{N}}$ through the ordinals. Then the derived model of $\mathcal{N}$ computed via Σ using the Woodin cardinals above $\delta^{\mathcal{P}}$ is a model of* LSA.

Proof. Let $(\delta_i : i < \omega)$ be the Woodin cardinals of $\mathcal{N}$ that are greater than $\delta^{\mathcal{P}}$. Let λ be their sup. It follows from Lemma 8.1.3 that $\mathcal{N}|\lambda$ is internally Σ-closed. It follows from Corollary 8.2.4 that Σ is $\Gamma^b(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$-fullness preserving.

Suppose X is a transitive countable set such that $\mathcal{P} \in X$. Let for $i \in 2$, $\pi_i : \mathcal{N} \to \mathcal{M}_i$ be an iteration according to Σ such that $\mathrm{crit}(\pi_i) > \delta^{\mathcal{P}}$ and X is $< \pi(\lambda)$-generic over $\mathcal{M}_i$.

Claim 1. $Lp^{\mathcal{M}_0, sts}(X, \mathcal{P}) = Lp^{\mathcal{M}_1, sts}(X, \mathcal{P})$.

Proof. Let $\mathcal{K}_0$ be the $\mathcal{M}_0$-authenticated background construction over X based on $\mathcal{P}$ and $\mathcal{K}_1$ be the $\mathcal{M}_1$-authenticated background construction over X based on $\mathcal{P}$. We compare the construction producing $\mathcal{K}_0$ with the one producing $\mathcal{K}_1$. Notice that it follows from the proof of Claim 1 of Lemma 8.2.3 that both constructions reach proper class models. It then follows from the proof of Claim 2 of Lemma 8.2.3 that the aforementioned comparison produces $\sigma_0 : \mathcal{M}_0 \to \mathcal{M}_2$ and $\sigma_1 : \mathcal{M}_1 \to \mathcal{M}_3$ such that $\mathrm{crit}(\sigma_i) > o(X)$ and $\sigma_0(\mathcal{K}_0)$ and $\sigma_1(\mathcal{K}_1)$ are lined up (i.e. one is an initial segment of the other). Because they both have exactly ω Woodin cardinals it follows from our minimality assumption on $\mathcal{N}$ that $\sigma_0(\mathcal{K}_0) = \sigma_1(\mathcal{K}_1)$. The claim now follows. □

Given a transitive $X \in HC$, we let $\mathcal{W}(X) = Lp^{\mathcal{M}, sts}(X, \mathcal{P})$ where $\mathcal{M}$ is such that there is an iteration $\pi : \mathcal{N} \to \mathcal{M}$ according to Σ such that $\mathrm{crit}(\pi) > \delta^{\mathcal{P}}$ and X is $< \pi(\lambda)$-generic over $\mathcal{M}$. Suppose $\mathcal{S} \in pI(\mathcal{P}, \Sigma)$, $\alpha < \lambda^{\mathcal{S}}$, $\eta \in [\delta^{\mathcal{S}}_{\lambda^{\mathcal{S}}-1}, \delta^{\mathcal{S}})$ is such that $\mathcal{M}^+(\mathcal{S}|\eta) \vDash$ "η is a Woodin cardinal". We then claim that

Claim 2. $\mathcal{W}(\mathcal{M}^+(\mathcal{S}|\eta)) \vDash$ "η is not a Woodin cardinal".

Proof. Suppose otherwise. Notice that $\mathcal{S} \vDash$ "η is not a Woodin cardinal". Let $\mathcal{Q} \trianglelefteq \mathcal{S}$ be the least such that $\mathcal{Q} \vDash$ "η is a Woodin cardinal" but $\mathcal{J}[\mathcal{Q}] \vDash$ "η is not a Woodin cardinal". Then $\mathcal{Q}$ is a $\Sigma^{stc}_{\mathcal{M}^+(\mathcal{S}|\eta)}$-sts mouse. Let now $\pi : \mathcal{N} \to \mathcal{M}$ be an iteration according to Σ above $\delta^{\mathcal{P}}$ such that $\mathcal{S}$ is $< \pi(\lambda)$-generic over $\mathcal{M}$. Let $\mathcal{K}$ be the $\mathcal{M}$-authenticated background construction over $\mathcal{M}^+(\mathcal{S}|\eta)$. Because we are assuming that the claim fails, we must have that $\mathcal{K} \vDash$ "η is a Woodin cardinal".

We now compare $\mathcal{Q}$ with the construction of $\mathcal{M}$ producing $\mathcal{K}$. Notice that this comparison halts (this follows from the proof of Claim 2 that appears in the proof of Lemma 8.2.3). Now, $\mathcal{Q}$ has to win this comparison. Since $\mathcal{K}$ is proper class and has ω Woodin cardinals, the fact that $\mathcal{Q}$ wins contradicts the minimality assumption on

$\mathcal{N}$ (more precisely, contradicts (3) of Definition 8.2.2). □

Suppose next that $\eta = \delta^{\mathcal{S}}$. Let $\mathcal{W}_\alpha(X)$ be the αth iterate of $\mathcal{W}$. We then have that

Claim 3. $\mathcal{W}_\omega(\mathcal{M}^+(\mathcal{S}|\eta)) = \mathcal{S}$.

Proof. Let $\sigma : \mathcal{N} \to \mathcal{S}^+$ be the result of applying the iteration producing $\mathcal{S}$ to the entire model $\mathcal{N}$. Thus $\mathcal{S}$ is the lsa part of $\mathcal{S}^+$. Let now $\pi : \mathcal{N} \to \mathcal{M}$ be an iteration according to Σ above $\delta^{\mathcal{P}}$ such that $\mathcal{S}$ is $< \pi(\lambda)$-generic over $\mathcal{M}$. Let $\mathcal{K}$ be the $\mathcal{M}$-authenticated background construction over $\mathcal{M}^+(\mathcal{S}|\eta)$. We now compare the construction producing $\mathcal{K}$ with $\mathcal{S}^+$. As before this construction has to halt. It then follows from our minimality condition on $\mathcal{N}$ that $\mathcal{W}_\omega(\mathcal{M}^+(\mathcal{S}|\eta)) = \mathcal{S}$. □

The next claim computes the powerset of the Woodin cardinals of $\mathcal{N}$. The proof is very similar to the proof of Claim 3 and we omit it.

Claim 4. Let $\pi : \mathcal{N} \to \mathcal{M}$ be an iteration according to Σ above $\delta^{\mathcal{P}}$. Then for any $k < \omega$, $\mathcal{M}|(\delta_k^+)^{\mathcal{M}} = \mathcal{W}(\mathcal{M}|\delta_k)$.

The next claim can be proved using the proof of Claim 3 and the proof of Lemma 8.1.9. Also see the proof of Claim 4 of Lemma 8.2.3.

Claim 5. Suppose $X \in HC$ is a transitive set and $\mathcal{R} \trianglelefteq \mathcal{W}(X)$ is such that $\rho(\mathcal{R}) = o(X)$. Let $\pi : \mathcal{N} \to \mathcal{M}$ be an iteration according to Σ above $\delta^{\mathcal{P}}$ such that X is $< \pi(\lambda)$-generic over $\mathcal{M}$. Let k be such that for some $g \subseteq Coll(\omega, < \pi(\delta_k))$, $X \in HC^{\mathcal{M}|\pi(\delta_k)[g]}$. Then $\mathcal{R}$ has a $< \pi(\lambda)$-universally Baire iteration strategy in $\mathcal{M}[g]$.

Suppose $g \subseteq Coll(\omega, \mathbb{R})$ generic. Let $(x_i : i < \omega)$ be an enumeration of $\mathbb{R}$ in $V[g]$. Let $\pi : \mathcal{N} \to \mathcal{M}$ be $\mathbb{R}$-genericity iteration according to Σ and guided by $(x_i : i < \omega)$. The next claim is a corollary to Claim 5 and clause 2 of Theorem 6.1.5.

Claim 6. Then the set $B = \{(x, y) \in \mathbb{R}^2 : y \notin \mathcal{W}(x)\}$ and $\Sigma^{stc}_{\mathcal{P}}$ are both in $\mathcal{M}(\mathbb{R})$.

Let Ψ be the minimal component of $\Sigma^{stc}_{\mathcal{P}}$ (see Definition 3.9.8). Let $\mathcal{P}_\infty$ be the direct limit of all Σ-iterates of $\mathcal{P}$ and let $\pi : \mathcal{P} \to \mathcal{P}_\infty$ be the iteration embedding. Notice that $\pi \restriction \mathcal{P}^b$ depends only on Ψ. Also, because Ψ is strongly $\Gamma^b(\mathcal{P}, \Sigma^{stc}_{\mathcal{P}})$-fullness preserving, it follows that $\pi[\mathcal{P}^b]$ can be coded as a subset of $w(\Gamma^b(\mathcal{P}, \Sigma^{stc}))$. This is because $\mathcal{P}^b_\infty|\delta^{\mathcal{P}^b_\infty} = \bigcup\{\mathcal{M}_\infty(\mathcal{R}, \Lambda_{\mathcal{R}}) : \mathcal{R} \in pB(\mathcal{P}, \Sigma^{stc})\}$ and $\delta^{\mathcal{P}^b} = w(\Gamma^b(\mathcal{P}, \Sigma^{stc}))$. It follows from Lemma 8.1.12 that

Claim 7. $\Psi \in \mathcal{J}_\omega(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma^{stc}))$.

Next we establish a crucial claim.

Claim 8. $\mathcal{J}(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma^{stc})) \vDash \mathsf{AD}^+$.

Proof. Suppose not. $A \in \mathcal{J}(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma^{stc}))$ be a set of reals that is not determined. We have that $A \in \mathcal{M}(\mathbb{R})$. Let $X = \pi[\mathcal{P}^b]$. Fix $x \in \mathbb{R}$ and $\mathcal{Q} \in pB(\mathcal{P}, \Sigma^{stc})$ such that A is definable from $X, x, (\mathcal{Q}, \Sigma_\mathcal{Q}), \mathcal{P}^b_\infty$ and a finite sequence of ordinals over $\mathcal{J}(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma^{stc}))$. By minimizing the sequence of ordinals we can suppose that A is definable without ordinal parameters.

Let $(\mathcal{M}_i, \mathcal{T}_i : i < \omega)$ be the $\mathbb{R}$-genericity iteration of $\mathcal{N}$ guided by $(x_i : i < \omega)$. For $i < \omega$ let $\pi_i = \pi^{\oplus_{j \leq i} \mathcal{T}_j}$ and for $i < j \leq \omega$ let $\pi_{i,j} : \mathcal{M}_i \to \mathcal{M}_j$ be the composition of iteration emebddings. Let i be large enough so that $x, \mathcal{Q} \in HC^{\mathcal{M}_i[(x_j : j \leq i)]}$ and $\Sigma_\mathcal{Q} \restriction HC^{\mathcal{M}_i[(x_j : j \leq i)]}$ is $< \pi_i(\lambda)$-universally Baire. Let $\tau \in M_i[(x_j : j \leq i)]$ be a name such that $\pi_{i,\omega}(\tau)$ is a term relation for A. We claim that if $\mathcal{R} = (\mathcal{M}_i | (\pi_i(\delta^+_{i+1})^{\mathcal{M}}))[(x_j : j \leq i)]$ then letting Φ be the fragment of Σ that acts on trees based on $\mathcal{R}$ that are above $\pi_i(\delta_i)$, $(\mathcal{R}, \Phi, \tau)$ term captures A. It then follows from a result of Neeman that A is determined (see [9]).

Let then $\mathcal{T}$ be an iteration tree on $\mathcal{M}_i$ based on $\mathcal{R}$ according to Φ. Let $\eta = \pi_i(\delta_{i+1})$. Let $\mathcal{S}$ be the last model of $\mathcal{T}$. We want to see that if $h \subseteq Coll(\omega, \pi^\mathcal{T}(\eta))$ is $\mathcal{S}$-generic then $(\pi^\mathcal{T}(\tau))_h = A \cap \mathcal{S}[h]$. Let $k > i$ be large enough that $\mathcal{S} \in \mathcal{M}_k[(x_j : j \leq k)]$. Let $\mathcal{S}^*$ be the output of $\mathcal{M}_k | \pi_k(\delta_{k+1})$-authenticated backgrounded construction over $\mathcal{S} | \pi^\mathcal{T}(\eta)$. We then have that $\mathcal{S}^*$ is an iterate of $\mathcal{S} | \pi^\mathcal{T}(\pi_i(\delta_{i+2}))$ (see Lemma 8.2.5). Let $\mathcal{S}^{**} = \pi_{k,k+1}(\mathcal{S}^*)$. Finally, let $\mathcal{S}_1$ be the result of translating $\mathcal{M}_{k+1}$ over $\mathcal{S}^{**}$ via S-constructions. We then have that $\mathcal{S}_1[\mathcal{M}_{k+1} | \pi_{k+1}(\delta_{k+1})] = \mathcal{M}_{k+1}$.

It follows that we can think of $(\mathcal{T}_j : j \in (k+1, \omega))$ as a $\mathbb{R}$-genericity iteration on $\mathcal{S}_1$ guided by $(x_j : j \in (k+1, \omega))$. Let then $\mathcal{S}_2$ be the last model of this genericity iteration. We then have that $\mathcal{S}_2[\mathcal{M} | \pi(\delta_{k+1})] = \mathcal{M}$. Let $\sigma : \mathcal{M}_i \to \mathcal{S}_2$ be the iteration embedding. It then follows that in $\mathcal{S}_2[(x_j : j \leq i)]$, $\sigma(\tau)$ is the term relation that denotes the least set in $\mathcal{J}(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma^{stc}))$ which is not determined and is definable from x and $(\mathcal{Q}, \Sigma_\mathcal{Q})$. It then follows that $\sigma(\tau)$ is realized as A. $\square$

The proof of the next claim is exactly like the proof of (1) that appeared in the proof of Theorem 8.1.13 and Lemma 8.2.3. We leave it to the reader.

Claim 9. For any transitive $X \in HC$ such that $\mathcal{P} \in X$ and for any $\mathcal{R} \trianglelefteq \mathcal{W}(X)$ such that $\rho(\mathcal{R}) = o(X)$, $\mathcal{R}$ has an iteration strategy in $\Gamma^b(\mathcal{P}, \Sigma^{stc})$.

It follows from Claim 9 that the set $B = \{(x, y) \in \mathbb{R}^2 : y \notin \mathcal{W}(x)\}$ is projective in Ψ and hence, $B \in \mathcal{J}(\mathcal{P}^b_\infty, \pi[\mathcal{P}^b], \Gamma^b(\mathcal{P}, \Sigma))$. It follows from Claim 9 that $\mathcal{J}(B) \vDash \mathsf{AD}^+$. We now have that

Claim 10. In $\mathcal{M}(\mathbb{R})$, let $\Gamma = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}^+\}$. Then $\Psi, B \in L(\Gamma, \mathbb{R})$.

It follows from the proof of clause 2 of Theorem 8.1.13 that B cannot be uniformized in $L(\Gamma, \mathbb{R})$. Hence, $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$. □

8.3 Strong Γ-fullness preservation reviseisted

Theorem 8.3.1 *Suppose $(\mathcal{Q}, \Lambda)$ is a pair appearing on the Γ-hod pair construction of $\mathcal{N}^*_y$. Then Λ is strongly Γ-fullness preserving.*

Chapter 9

Condensing sets

The goal of this chapter is to introduce the theory of condensing sets. Such sets were first considered in [11, Section 10, 11.1], where they were presented in the form of a condensation property for elementary embeddings (see [11, Definition 11.14]). The current presentation dates back to an unpublished note by the first author.

Prior to this work, condensing sets have been used in the context of the core model induction. As a convenience to the reader, we recap some of the basic machinery used in the core model induction. We model our presentation on [11] but we will also use the set up of [31]. A typical situation is as follows. We have an embedding $j : M \to N$ with critical point κ and such that $H^M_{\kappa^+} = H^N_{\kappa+}$. In M, we consider the maximal model of determinacy that has been built via core model induction. While the exact definition of the maximal model is somewhat case specific, it can be essentially described as follows.

Let $g \subseteq Coll(\omega, < j(\kappa))$ be N-generic. For $\nu < \kappa$ let $g_\nu = g \cap Coll(\omega, < \nu)$. We then can extend j to act on $M[g_\kappa]$. We denote this extension by j again and we have that $j : M[g_\kappa] \to N[g]$.

Working in $M[g_\kappa]$, consider the set of hod pairs $(\mathcal{Q}, \Lambda)$ such that

1. $\mathcal{Q} \in HC^{M[g]}$,
2. for some $\nu < \kappa$ such that $\mathcal{Q} \in M[g_\nu]$, letting $\Psi = \Lambda \restriction HC^{M[g_\nu]}$, $\Psi \in M[g_\nu]$ and $M[g_\nu] \vDash$ "$Code(\Psi)$ is κ-uB" and
3. if $T, S \in M[g_\nu]$ witness that $Code(\Psi)$ is κ-uB then $Code(\Lambda) = p[T]^{M[g_\kappa]}$.

Let Γ be the set of such pairs $(\mathcal{Q}, \Lambda)$. An additional requirement is that Λ is fullness preserving and has branch condensation. While the branch condensation is the same as before, fullness preservation is not the same as the definition given in this paper.

We refer the interested reader to [11] for more details on how to define Γ. It is in fact somewhat more involved.

The goal of a core model induction is to show that Γ is rich. This is done as follows. First a target theory is fixed. The theory used in [11] is "$\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular". In Chapter 12, our target is LSA. Suppose then there is no lsa type hod pair $(\mathcal{Q}, \Lambda) \in \Gamma$. Preliminary arguments, such as those used in [12, Theorem 4.1], show that Γ is of limit type, i.e., for any $(\mathcal{Q}, \Lambda) \in \Gamma$ there is $(\mathcal{R}, \Psi) \in \Gamma$ such that $\Gamma(\mathcal{Q}, \Lambda) <_w \Gamma(\mathcal{R}, \Psi)$.

Next we let $\mathcal{P}^- = \bigcup_{(\mathcal{Q},\Lambda)\in\Gamma} \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. Fixing $\alpha < \lambda^{\mathcal{P}^-}$ and $(\mathcal{Q}, \Lambda) \in \Gamma$ such that $\mathcal{P}^-(\alpha) = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$, we let $\Sigma_\alpha = \Lambda_{\mathcal{P}^-(\alpha)}$. It follows from comparison that Σ_α is independent of $(\mathcal{Q}, \Lambda)$. Let $\Sigma = \oplus_{\alpha<\lambda^{\mathcal{P}^-}} \Sigma_\alpha$. Suppose next that there is $\mathcal{M} \trianglelefteq Lp^\Sigma(\mathcal{P}^-)$ such that $\rho(\mathcal{M}) < o(\mathcal{P}^-)$. We then let $\mathcal{P}$ be the least such $\mathcal{M}$. Otherwise we let $\mathcal{P} = Lp^\Sigma_\omega(\mathcal{P}^-)$.

The next major step is to build an iteration strategy for $\mathcal{P}$ that extends Σ. We let Σ^+ be this new strategy. Σ^+ is constructed as follows.

Definition 9.0.2 (The construction of the strategy) *Suppose $\vec{\mathcal{T}} \in HC^{N[g]}$ is a stack on $\mathcal{P}$. Working in $N[g]$, we say $\vec{\mathcal{T}}$ is j-realizable if there is a sequence $(\sigma_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$ and a sequence $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}})) \subseteq j(\Gamma)$ such that*

1. *$\sigma_{\mathcal{P}} = \sigma$, for all terminal nodes $\mathcal{R}$ of $\vec{\mathcal{T}}$, $\sigma_{\mathcal{R}} : \mathcal{R} \to j(\mathcal{P})$ and whenever $\mathcal{R} \prec^{\vec{\mathcal{T}},s} \mathcal{Q}$, $\sigma_{\mathcal{R}} = \sigma_{\mathcal{Q}} \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{R},\mathcal{Q}}$.*

2. *For every non-trivial terminal node $\mathcal{R}$ of $\vec{\mathcal{T}}$, $\sigma_{\mathcal{R}}[\mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}} + 1)] \subseteq rng(\pi^{\Lambda_{\mathcal{R}}}_{\mathcal{S}_{\mathcal{R}},\infty})$.*

3. *For every non-trivial terminal node $\mathcal{R}$, letting $k_{\mathcal{R}} : \mathcal{R}(\xi^{\vec{\mathcal{T}},\mathcal{R}} + 1) \to \mathcal{S}_{\mathcal{R}}$ be given by $k_{\mathcal{R}}(x) = y$ if and only if $\sigma_{\mathcal{R}}(x) = \pi^{\Lambda_{\mathcal{R}}}_{\mathcal{S}_{\mathcal{R}},\infty}(y)$, $k_{\mathcal{R}}\vec{\mathcal{T}}_{\mathcal{R}}$ is according to $\Lambda_{\mathcal{R}}$.*

4. *Suppose $\mathcal{R}$ is a non-trivial terminal node of $\vec{\mathcal{T}}$. Let $\mathcal{S}^*_{\mathcal{R}}$ be the last model of $k_{\mathcal{R}}\vec{\mathcal{T}}_{\mathcal{R}}$. Suppose $\vec{\mathcal{T}}_{\mathcal{R}}$ has a last model $\mathcal{Q}_{\mathcal{R}}$ and that $\pi^{\vec{\mathcal{T}}_{\mathcal{R}}}$ is defined. It then follows that $\mathcal{Q}_{\mathcal{R}} \in tn(\vec{\mathcal{T}})$ and $\mathcal{R} \prec^{\vec{\mathcal{T}},s} \mathcal{Q}_{\mathcal{R}}$. Let $k^*_{\mathcal{R}} : \mathcal{Q}_{\mathcal{R}} \to \mathcal{S}^*_{\mathcal{R}}$ come from the copying construction. Then for all $x \in \mathcal{Q}_{\mathcal{R}}$, $\sigma_{\mathcal{Q}_{\mathcal{R}}}(x) = \sigma_{\mathcal{R}}(f)(\pi^{\Lambda_{\mathcal{R}}}_{\mathcal{S}^*_{\mathcal{R}},\infty,j(\eta)}(k^*_{\mathcal{R}}(a))$ where $f \in \mathcal{R}$ and $a \in [\mathcal{Q}_{\mathcal{R}}(\pi^{\vec{\mathcal{T}}}_{\mathcal{R},\mathcal{Q}_{\mathcal{R}}}(\xi^{\vec{\mathcal{T}},\mathcal{R}}) + 1)]^{<\omega}$ are such that $x = \pi^{\vec{\mathcal{T}}}_{\mathcal{R},\mathcal{Q}_{\mathcal{R}}}(f)(a)$.*

5. *Suppose $\mathcal{R}$ is a trivial terminal node of $\vec{\mathcal{T}}$. Then for every $\xi < \lambda^{\mathcal{R}}$, there is $(\mathcal{S}, \Lambda) \in j(\Gamma)$ such that $\sigma_{\mathcal{R}}[\mathcal{R}(\xi + 1)] \subseteq rng(\pi^{\Lambda}_{\mathcal{S},\infty,j(\eta)})$.*

We say that $(\sigma^{\vec{\mathcal{T}}}_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$ are the j-realizable embeddings of $\vec{\mathcal{T}}$ and $(\mathcal{S}_{\mathcal{R}}, \Lambda_{\mathcal{R}} : \mathcal{R} \in tn(\vec{\mathcal{T}}))$ are the j-realizable pairs of $\vec{\mathcal{T}}$.

Given a stack $\vec{\mathcal{T}} \in HC^{N[g]}$ on $\mathcal{P}$ such that either there is a strongly linear closed and cofinal set $C \subseteq tn(\vec{\mathcal{T}})$ or $\vec{\mathcal{T}}_{\mathcal{S}_{\vec{\mathcal{T}}}}$ is of limit length, we set $\vec{\mathcal{T}} \in dom(\Sigma^+)$ if $\vec{\mathcal{T}}$ is j-realizable. We set $\Sigma^+(\vec{\mathcal{T}}) = b$ if $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is j-realizable.

Σ^+ may not be a total strategy simply because we may not be able to find $(\mathcal{S}, \Lambda)$ as in the last clause of Definition 9.0.2. However, the proof of [11, Lemma 11.6] gives the following.

Theorem 9.0.3 *Suppose $|\mathcal{P}| < (\kappa^+)^M$. Then Σ^+ is a total (ω_1, ω_1)-strategy in $N[g]$.*

Then there are two arguments that we run as part of the proof of Theorem 9.0.3. First we show that $\mathcal{P} = Lp_\omega^\Sigma(\mathcal{P}^-)$. The reader can see, for example [31, Lemma 3.78], for an argument. Roughly, if not, suppose n is such that $\rho_{n+1}(\mathcal{P}) < \delta^{\mathcal{P}} \leq \rho_n(\mathcal{P})$, then in $j(\Gamma)$, we can define an $OD_{\Sigma_\alpha}^{j(\Gamma)}$ set $A \subseteq \delta_\alpha^{\mathcal{P}}$ such that $A \notin \mathcal{P}$. By fullness of $\mathcal{P}(\alpha)$ and SMC in $j(\Gamma)$, $A \in \mathcal{P}(\alpha) \in \mathcal{P}$. Contradiction.

The next argument attempts to show that $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is regular". Showing this finishes the proof of the main theorem of [11]. In this book we present two arguments for obtaining a model of LSA from PFA (see Theorem **??** and Theorem 12.0.22). In both cases, we need to do more in order to finish the argument. It is in this step that the theory of condensing sets is used. A reader interested in many details should consult [11, Section 10, 11.1] and [31, Lemma 3.81].

9.1 Condensing sets

We introduce the notion of condensing set in the most general setting. Suppose ϕ is a formula in the language of set theory and A is a set. We let $\mathcal{F}_{\phi,A}$ be a collection of hod pairs $(\mathcal{Q}, \Lambda)$ such that $\mathcal{Q}$ is countable, Λ is an $(\omega_2, \omega_2, \omega_2)$-iteration strategy having strong branch condensation and such that $\phi[A, (\mathcal{Q}, \Lambda)]$ holds.

Terminology 9.1.1

1. *We say (ϕ, A) is **bottom part closed** if whenever $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\phi,A}$ and $\mathcal{R} \in pB(\mathcal{Q}, \Lambda)$ then $(\mathcal{R}, \Lambda_{\mathcal{R}}) \in \mathcal{F}_{\phi,A}$.*
2. *We say (ϕ, A) is of **limit type** if for every $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\phi,A}$, there is $(\mathcal{R}, \Psi) \in \mathcal{F}_{\phi,A}$ such that $\mathcal{R}$ is of limit type and $Code(\Lambda) \in \Gamma^b(\mathcal{R}, \Psi)$.*
3. *Let $\Gamma_{\phi,A} = \bigcup\{\Gamma(\mathcal{R}, \Psi) : (\mathcal{R}, \Psi) \in \mathcal{F}_{\phi,A} \wedge \mathcal{R}$ is of limit type$\}$. We say (ϕ, A) is **stable** if whenever $(\mathcal{R}, \Psi) \in \mathcal{F}_{\phi,A}$, Ψ is strongly $\Gamma_{\phi,A}$-fullness preserving.*
4. *We say (ϕ, A) is **directed** if whenever $(\mathcal{Q}, \Lambda), (\mathcal{P}, \Sigma) \in \mathcal{F}_{\phi,A}$, there are $\mathcal{R} \in pI(\mathcal{Q}, \Lambda)$ and $\mathcal{S} \in pI(\mathcal{P}, \Sigma)$ such that either*

(a) $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Sigma_{\mathcal{R}} = \Lambda_{\mathcal{R}}$ or

(b) $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Lambda_{\mathcal{S}} = \Sigma_{\mathcal{S}}$.

Notation 9.1.2 *Suppose (ϕ, A) is bottom part closed, is of limit type, is stable and is directed.*

1. *Let $\mathcal{P}^-_{\phi,A} = \bigcup_{(\mathcal{Q},\Lambda)\in\mathcal{F}_{\phi,A}} \mathcal{M}_\infty(\mathcal{Q},\Lambda)$.*

2. *Fix $\alpha < \lambda^{\mathcal{P}^-_{\phi,A}}$ and $(\mathcal{Q},\Lambda) \in \mathcal{F}_{\phi,A}$ such that $\mathcal{P}^-_{\phi,A}(\alpha) = \mathcal{M}_\infty(\mathcal{Q},\Lambda)$. Let $\Sigma_{\alpha,\phi,A} = \Lambda_{\mathcal{P}^-_{\phi,A}(\alpha)}$ and let $\Sigma_{\phi,A} = \oplus_{\alpha<\lambda^{\mathcal{P}^-_{\phi,A}}} \Sigma_{\alpha,\phi,A}$.*

3. *Suppose there is $\mathcal{M} \trianglelefteq Lp^{\Gamma_{\phi,A},\Sigma_{\mathcal{F}}}(\mathcal{P}^-_{\phi,A})$ such that $\rho(\mathcal{M}) < o(\mathcal{P}^-_{\phi,A})$. Then let $\mathcal{P}_{\phi,A}$ be the least such $\mathcal{M}$. Otherwise let $\mathcal{P}_{\phi,A} = Lp_\omega^{\Gamma_{\phi,A},\Sigma_{\phi,A}}(\mathcal{P}^-_{\phi,A})$.*

Definition 9.1.3 *Suppose (ϕ, A) is bottom-part closed, is of limit type, is stable and is directed. We say (ϕ, A) is* ***full*** *if $\mathcal{P}_{\phi,A} = Lp_\omega^{\Gamma_{\phi,A},\Sigma_{\phi,A}}(\mathcal{P}^-_{\phi,A})$.*

Definition 9.1.4 *Suppose (ϕ, A) is full. We say* ***lower part*** *(ϕ, A)**-covering*** *holds if $cf(o(\mathcal{P}_{\phi,A})) \geq \omega_1$.*

Suppose now that (ϕ, A) is full and lower part (ϕ, A)-covering fails. We let $\Gamma = \Gamma_{\phi,A}$, $\mathcal{P} = \mathcal{P}_{\phi,A}$ and $\Sigma = \Sigma_{\phi,A}$. Given $X \in \wp_{\omega_1}(\mathcal{P})$, we let $\mathcal{Q}_X$ be the transitive collapse of $Hull^{\mathcal{P}}(X)$ and $\tau_X : \mathcal{Q}_X \to \mathcal{P}$ be the inverse of the transitive collapse. We let Σ_X be the τ_X-pullback of Σ.

Definition 9.1.5 (Weakly condensing set) *We say that $X \in \wp_{\omega_1}(\mathcal{P})$ is a (ϕ, A)-**weakly condensing set** if $\mathcal{P} = Hull^{\mathcal{P}}(X \cup \delta^{\mathcal{P}})$ and whenever $X \subseteq Y \in \wp_{\omega_1}(\mathcal{P})$, Σ_Y is a strongly Γ-fullness preserving iteration strategy with strong branch condensation.*

Let $X \subseteq Y \in \wp_{\omega_1}(\mathcal{P})$. We say that Y *extends* X or Y is an *extension of* X if

1. $\tau_{X,Y} \restriction (\mathcal{Q}_X|\delta^{\mathcal{Q}_X})$ is the iteration map via Σ_X and

2. $\mathcal{Q}_Y = Hull_1^{\mathcal{Q}_Y}(\delta^{\mathcal{Q}_Y} \cup \tau_{X,Y}[\mathcal{Q}_X])$.

Let $\delta^{\mathcal{Q}_Y} = \tau_Y^{-1}(\delta)$. Let $\tau_{X,Y} : \mathcal{Q}_X \to \mathcal{Q}_Y$ be $\tau_Y^{-1} \circ \tau_X$. Let $\sigma_Y^{X,-} = \bigcup_{\alpha+1<\lambda^{\mathcal{Q}_Y}} \pi^{\Sigma_Y}_{\mathcal{Q}_Y(\alpha),\infty}$ and $\sigma_Y^X : \mathcal{Q}_Y \to \mathcal{P}$ be given by: for any $f \in \mathcal{Q}_X$ and any $a \in (\mathcal{Q}_Y|\delta^{\mathcal{Q}_Y})^{<\omega}$, and $x = \tau_{X,Y}(f)(a)$,

$$\sigma_Y^X(a) = \sigma_X(f)(\pi^{\Sigma_Y}_{\mathcal{Q}_Y,\infty}(a)) = \sigma_X(f)(\sigma_Y^{X,-}(a)).$$

Definition 9.1.6 *Suppose Y is an extension of a weakly condensing set X. Let $\delta_Y = \delta^{\mathcal{Q}_Y}$. We say that Y is **an honest extension** of X if*

(a) $\tau_X = \sigma_Y^X \circ \tau_{X,Y}$, and

(b) $\pi^{\Sigma_Y}_{\mathcal{Q}_Y,\infty} \restriction (\mathcal{Q}_Y|\delta_Y) = \sigma_Y^X \restriction (\mathcal{Q}_Y|\delta_Y)$.

Remark 9.1.7 *X is obviously an honest extension of itself, but there are other (non-trivial) honest extensions of X. For example, if $X = X' \cap \mathcal{P}$ where $X' \prec H^V_\lambda$ for some regular λ (this will be the case for our intended X) and $Y = Y' \cap \mathcal{P}$ for some $X' \prec Y'$, then Y is an honest extension of X.*

Definition 9.1.8 (Condensing set) *Suppose $X \in \wp_{\omega_1}(\mathcal{P})$ is a (ϕ, A)-weakly condensing set. We say that X is a (ϕ, A)**-condensing set** if whenever Y extends X, Y is an honest extension of X.*

We expect that under many hypothesis such as PFA lower part (ϕ, A)-covering fails. We also expect that under many hypothesis, failure of lower part (ϕ, A)-covering implies the existence of (ϕ, A)-condensing sets. In the next few chapters, we explore some specific situations where we know how to prove the existence of (ϕ, A)-condensing sets.

We finish by remarking that (ϕ, A) depends on the specific situation we are in. For instance, in [11], ϕ isolates those hod pairs that have certain extendability and self-determining properties (see [11, Definition 3.1, 3.5, 3.8]).

We finish here by showing that below LSA, pullback strategies are unique.

Lemma 9.1.9 (Uniqueness of strategies) *Suppose (ϕ, A, X) is such that ϕ is a formula in the language of set theory, (ϕ, A) is full, lower part (ϕ, A)-covering fails and X is a (ϕ, A)-condensing set. Suppose further that whenever $(\mathcal{Q}, \Lambda) \in \Gamma_{\phi,A}$, $\mathcal{Q}$ is not of lsa type. Then whenever Y and Z are two honest extensions of X such that $\mathcal{Q}_Y = \mathcal{Q}_Z$, $\Sigma_Y = \Sigma_Z$.*

Proof. Suppose that $\Sigma_Y \neq \Sigma_Z$. Let $\Phi = \Sigma_Y$ and $\Psi = \Sigma_Z$. Because we can trace disagreement of strategies to minimal disagreements, we can find a stack $\vec{\mathcal{T}}$ on $\mathcal{Q} =_{def} \mathcal{Q}_Y (= \mathcal{Q}_Z)$ according to both Σ_Y and Σ_Z with last model $\mathcal{R}$ such that

(1) for some $\alpha < \lambda^{\mathcal{R}}$, $\delta(\vec{\mathcal{T}}) \subseteq \delta^{\mathcal{R}}_\alpha$[1], $\mathcal{R}(\alpha+1)$ is of successor type, $\Phi_{\mathcal{R}(\alpha),\vec{\mathcal{T}}} = \Psi_{\mathcal{R}(\alpha),\vec{\mathcal{T}}}$ but $\Phi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}} \neq \Psi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}$.

[1]Recall that $\delta(\vec{\mathcal{T}})$ is the sup of generators of $\vec{\mathcal{T}}$

We claim that $\tau_{X,Y} = \tau_{X,Z}$. Because both Y and Z are extensions of X, we have that both $\tau_{X,Y} \restriction (\mathcal{Q}_X|\delta_X)$ and $\tau_{X,Z}(\mathcal{Q}_X|\delta_X)$ are the iteration embedding according to Σ_X. Because Σ_X has strong branch condensation and is strongly $\Gamma_{\phi,A}$-fullness preserving, we have that $\tau_{X,Y} \restriction (\mathcal{Q}_X|\delta_X) = \tau_{X,Z} \restriction (\mathcal{Q}_X|\delta_X)$. Because $\mathcal{Q}_X = Hull^{\mathcal{Q}_X}(\delta_X \cup X)$ and $X \subseteq Y \cap Z$, we have that $\tau_{X,Y} = \tau_{X,Z}$. Let then $\tau =_{def} \tau_{X,Y} = \tau_{X,Z}$.

Next, because of the smallness assumption on hod pairs in $\Gamma_{\phi,A}$, it follows from (ϕ, A)-condensation of X that

(2) $\sup(Hull^{\mathcal{R}}(\delta^{\mathcal{R}}_\alpha, \pi^{\vec{\mathcal{T}}} \circ \tau[\mathcal{Q}_X])) = \delta^{\mathcal{R}}_{\alpha+1}$.

We can now find, using the normal comparison, a normal tree $\mathcal{U}$ on $\mathcal{R}(\alpha+1)$ according to both $\Phi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}$ and $\Psi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}$ such that if $b = \Phi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}(\mathcal{U})$, $c = \Psi_{\mathcal{R}(\alpha+1),\vec{\mathcal{T}}}(\mathcal{U})$, $\mathcal{R}_b = \mathcal{M}^{\mathcal{U}}_b$ and $\mathcal{R}_c = \mathcal{M}^{U}_c$ then

(3) $b \neq c$ and $\pi^{\mathcal{U}}_b(\delta^{\mathcal{R}}_{\alpha+1}) = \pi^{\mathcal{U}}_c(\delta^{\mathcal{R}}_{\alpha+1})$
(4) letting $\nu = \pi^{\mathcal{U}}_b(\alpha+1)$, $\mathcal{R}_b(\nu) = \mathcal{R}_c(\nu)$ and $\Phi_{\mathcal{R}_b(\nu),\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{R}_b\}} = \Psi_{\mathcal{R}_c(\nu),\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{R}_c\}}$.

Let $k_b : \mathcal{R}_b \to \mathcal{P}$ and $k_c : \mathcal{R}_c \to \mathcal{P}$ be the realizabilty maps according to $\Phi_{\mathcal{R}_b,\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{R}_b\}}$ and $\Psi_{\mathcal{R}_c,\vec{\mathcal{T}}^\frown\mathcal{U}^\frown\{\mathcal{R}_c\}}$. Notice that it follows from (4) that $k_b \restriction \mathcal{R}_b(\nu) = k_c \restriction \mathcal{R}_c(\nu)$. Notice that we also have that

(5) $k_b \restriction (Hull^{\mathcal{R}_b}(\delta^{\mathcal{R}_b}_{\nu-1}, \pi^{\mathcal{U}}_b \circ \pi^{\vec{\mathcal{T}}} \circ \tau[\mathcal{Q}_X])) = k_c \restriction (Hull^{\mathcal{R}_c}(\delta^{\mathcal{R}_c}_{\nu-1}, \pi^{\mathcal{U}}_c \circ \pi^{\vec{\mathcal{T}}} \circ \tau[\mathcal{Q}_X]))$.

Combining (2) and (5) we get that (recall that $\delta(\mathcal{U}) = \delta^{\mathcal{R}_b}_\nu$)

(6) $rng(\pi^{\mathcal{U}}_b) \cap rng(\pi^{\mathcal{U}}_c) \cap \delta(\mathcal{U})$.

Clearly (6) implies that $b = c$.

□

9.2 Condensing sets from elementary embeddings

The following two theorems can be proved using the proof of [11, Lemma 11.15]. First we introduce some terminology. Suppose κ is an inaccessible cardinal and

$G \subseteq Col(\omega, <\kappa)$ is V-generic. Suppose (ϕ, A) is such that $V[G] \vDash$ "(ϕ, A) is full and lower part (ϕ, A)-covering fails".

Terminology 9.2.1 *We say (ϕ, A) is* ***homogenous*** *if $\mathcal{P}_{\phi,A} \in V$, $\Sigma_{\phi,A} \restriction V \in V$ and for any $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\phi,A}$, there is $(\mathcal{R}, \Psi) \in \mathcal{F}_{\phi,A}$ such that $\mathcal{R} \in V$, $\Psi \restriction H^V_\kappa \in V$ and $\Gamma(\mathcal{Q}, \Lambda) \subseteq \Gamma(\mathcal{R}, \Psi)$.*

Theorem 9.2.2 *Suppose $N \subseteq M$ are transitive models of set theory and $j : M \to N$ is an elementary embedding with critical point κ. Suppose $g \subseteq Coll(\omega, < j(\kappa))$ is N-generic and $h = g \cap Coll(\omega, < \kappa)$. Let $j : M[h] \to N[g]$ be the extension of j. Suppose ϕ is a formula in the language of set theory and $A \in M[g]$. Suppose further that $M[g] \vDash$ "(ϕ, A) is full, (ϕ, A) is homogenous and lower part (ϕ, A)-covering fails". Then $j[\mathcal{P}_{\phi,A}]$ is a $(\phi, j(A))$-condensing set in $N[g]$. Hence, $M[g] \vDash$ "there is a (ϕ, A)-condensing set".*

Terminology 9.2.3 *We say (ϕ, A) is* ***maximal*** *if there is no hod pair or an sts hod pair $(\mathcal{Q}, \Lambda)$ such that $\mathcal{Q}$ is of limit type, Λ has strong branch condensation and is strongly $\Gamma_{\phi,A}$-fullness preserving and $\Gamma(\mathcal{Q}, \Lambda) = \Gamma_{\phi,A}$.*

Theorem 9.2.4 *Suppose (ϕ, A) is maximal and full, lower part (ϕ, A)-covering fails and X is a (ϕ, A)-condensing set. Then $\mathcal{P}_{\phi,A} \vDash$ "$\delta^{\mathcal{P}_{\phi,A}}$ is regular".*

We will not prove Theorem 9.2.2. However, in what follows we will outline a proof of another existence theorem, Theorem 9.2.7, that is somewhat harder to prove than Theorem 9.2.2. Theorem 9.2.7 will be applied in situations where there are no large cardinals (e.g. measurables) in V; one intended application is in the construction of models of LSA from instances of threadability in Chapter 12. There the embedding j is replaced by a kind of uncollapse maps of some hull that is countably closed; also, the hull is transitive past the size of the collapse forcing.

Suppose κ is a cardinal such that $\kappa^\omega = \kappa$. Let $G \subseteq Col(\omega, < \kappa)$ be V-generic. Suppose (ϕ, A) is such that $V[G] \vDash$ "(ϕ, A) is full and lower part (ϕ, A)-covering fails". Working in $V[G]$, let $\mathcal{P}^- = \mathcal{P}^-_{\phi,A}$, $\mathcal{P} = \mathcal{P}_{\phi,A}$, $\Sigma = \Sigma_{\phi,A}$, $\mathcal{F} = \mathcal{F}_{\phi,A}$ and $\Gamma = \Gamma_{\phi,A}$. Let $\delta = \delta^{\mathcal{P}} = o(\mathcal{P}^-)$, $\gamma = \lambda^{\mathcal{P}}$ and $\lambda = \max\{(|\Gamma|^{++}), (2^\kappa)^+\}^{V[G]}$. In this case, too, much like Terminology 9.2.1, we can define what it means to say that (ϕ, A) is homogenous.

We continue by assuming that (ϕ, A) is homogenous. Working in V, we say that $X \prec H^V_\lambda$ is *good* if $\kappa \subset X$, $|X| = \kappa$, $X^\omega \subset X$ and $\{\mathcal{P}^-, \Gamma, \mathcal{F}\} \subset X[G]$. Let $\pi_X : M_X \to H^V_\lambda$ be the uncollapse map (π_X naturally extends to $M_X[G]$ and we also denote the extension π_X). Let $(\Gamma_X, \mathcal{P}^-_X, \delta_X, \mathcal{F}_X, \Sigma_X, \gamma_X) = \pi_X^{-1}(\Gamma, \mathcal{P}^-, \delta, \mathcal{F}, \Sigma, \gamma)$. Let

$$\mathcal{P}_X = \mathrm{Lp}^{\Sigma_X,\Gamma}(\mathcal{P}_X^-),$$

For any hod pair $(\mathcal{Q},\Lambda) \in \mathcal{F}_X$, $(\mathcal{Q},\pi_X(\Lambda)) \in \mathcal{F}$. For notational convenience, we will also use Λ to denote its extension $\pi_X(\Lambda)$.

For a good X, using the embedding π_X we can define a strategy Σ_X^+ for $\mathcal{P}_X$ using the construction of Definition 9.0.2. We have that Σ_X^+ is such that

- Σ_X^+ extends Σ_X;
- for any Σ_X^+ iterate $\mathcal{Q}$ of $\mathcal{P}_X$ via stack $\vec{\mathcal{T}}$ such that the iteration embedding $\pi^{\vec{\mathcal{T}}}$ exists, there is an embedding $\sigma : \mathcal{Q} \to \mathcal{P}$ such that $\pi_X = \sigma \circ \pi^{\vec{\mathcal{T}}}$. Furthermore, letting $\Psi = (\Sigma_X^+)_{\vec{\mathcal{T}},\mathcal{Q}}$, for all $\alpha < \lambda^{\mathcal{Q}}$, $\Psi_{\mathcal{Q}(\alpha)}$ has branch condensation.
- Σ_X^+ is $\Gamma(\mathcal{P}_X, \Sigma_X^+)$-fullness preserving.

We call Σ_X^+ the π_X-pullback strategy for $\mathcal{P}_X$. By the theory developed above, for any $(\vec{\mathcal{U}},\mathcal{R}) \in I(\mathcal{P}_X,\Sigma_X)$, letting $\Lambda = (\Sigma_X)_{\vec{\mathcal{U}},\mathcal{R}}$, there is $(\vec{\mathcal{W}},\mathcal{S}) \in I(\mathcal{R},\Lambda)$ such that $\Lambda_{\vec{\mathcal{W}},\mathcal{S}}$ has branch condensation.

Terminology 9.2.5 *Given a good X, we say X **captures** (ϕ, A) if $\mathcal{P}_X \in M_X$, $\Sigma_X^+ \in \Gamma$, $\pi_X(\mathcal{P}_X) = \mathcal{P}$, and π_X is cofinal in $o(\mathcal{P})$. We say (ϕ, A) is **captured** if for a stationary set of good X, X captures (ϕ, A). We call this set $\mathfrak{S}_{\phi,A}$.*

Below, we use the notation "$\forall^* X \in \mathfrak{S}_{\phi,A}$" to mean "$\forall X \in \mathfrak{C} \cap \mathfrak{S}_{\phi,A}$ for some club $\mathfrak{C}$".

Theorem 9.2.6 *Suppose (ϕ, A) is captured. Then $\forall^* X \in \mathfrak{S}_{\phi,A}$, $X \cap \mathcal{P}$ is a weakly condensing set.*

Proof. Suppose not. Fix a good X' such that X' captures (ϕ, A) but $X = X' \cap \mathcal{P}$ is not a weakly condensing set. Note that $\pi_X \restriction \mathcal{P}_X$ is cofinal in $\mathcal{P}$. Let Y be an extension of X such that $(\mathcal{Q}_Y, \Sigma_Y) \notin \mathcal{F}$ (Σ_Y is the τ_Y-pullback of Σ). This means there is $(\mathcal{R}, \vec{\mathcal{T}}) \in I(\mathcal{Q}_Y, \Sigma_Y)$ such that $\pi^{\vec{\mathcal{T}}}$ exists and a strong cut point γ such that letting $\alpha \leq \lambda^{\mathcal{R}}$ be the largest such that $\delta_\alpha^{\mathcal{R}} \leq \gamma$, then in Γ, there is a mouse $\mathcal{M} \lhd \mathrm{Lp}^{(\Sigma_Y)_{\vec{\mathcal{T}},\mathcal{R}(\alpha)}}(\mathcal{R}|\gamma)$[2] such that $\mathcal{M} \notin \mathcal{R}$. By definition, $\tau_X = \tau_Y \circ \tau_{X,Y}$ ($\tau_X = \pi'_X \restriction \mathcal{P}_X$ here). We use i to denote $\pi^{\vec{\mathcal{T}}} \circ \tau_{X,Y}$ and $k : \mathcal{R} \to \mathcal{P}$ to denote the τ_Y-realization map in the definition of Σ_Y.

Let $(\mathcal{P}_X^+, \Lambda_X) \in V$ be a Σ_X-hod pair such that

[2]The other cases of Γ-fullness preservation are handled similarly.

- $\Gamma(\mathcal{P}_X^+, \Lambda_X) \vDash \mathcal{R}$ is not full as witnessed by $\mathcal{M}$.
- $\Lambda_X \in \Gamma$ is Γ-fullness preserving and has branch condensation.
- $\lambda^{\mathcal{P}_X^+}$ is limit and $\mathrm{cof}^{\mathcal{P}_X^+}(\lambda^{\mathcal{P}_X^+})$ is not measurable in $\mathcal{P}_X^+$.

Such a pair $(\mathcal{P}_X^+, \Lambda_X)$ exists by boolean comparisons. In particular, $\mathcal{P}_X^+$ is a Σ_X-hod premouse over $\mathcal{P}_X$.

By arguments similar to that used in [31, Lemma 3.78], $\forall^* X' \in \mathfrak{S}_{\phi,A}$, letting $X = X' \cap \mathcal{P}$, no levels of $\mathcal{P}_X^+$ projects across $\mathcal{P}_X$ and in fact, $o(\mathcal{P}_X)$ is a cardinal of $\mathcal{P}_X^+$. The second clause follows from the following argument. Suppose not and let $\mathcal{N}_X \lhd \mathcal{P}_X^+$ be least such that $\rho_\omega(\mathcal{N}_X) = \delta_X$ for stationary many good $X' \in \mathfrak{S}_{\phi,A}$. Fix such an X'. Let $f : \kappa \to \delta_X$ be an increasing and cofinal map in $\mathcal{P}_X$, where $\kappa = \mathrm{cof}^{\mathcal{P}_X}(\delta_X)$. We can construe $\mathcal{N}_X$ as a sequence $g = \langle \mathcal{N}_\alpha \mid \alpha < \kappa \rangle$, where $\mathcal{N}_\alpha = \mathcal{N}_X \cap \delta_{f(\alpha)}^{\mathcal{P}_X}$. Note that $\mathcal{N}_\alpha \in \mathcal{P}_X$ for each $\alpha < \kappa$. Now let $\mathcal{R}_0 = \mathrm{Ult}_0(\mathcal{P}_X, \mu)$, $\mathcal{R}_1 = \mathrm{Ult}_\Gamma(\mathcal{N}_X, \mu)$, where $\mu \in \mathcal{P}_X$ is the (extender on the sequence of $\mathcal{P}_X$ coding a) measure on κ with Mitchell order 0.[3] Let $i_0 : \mathcal{P}_X \to \mathcal{R}_0$, $i_1 : \mathcal{N}_X \to \mathcal{R}_1$ be the ultrapower maps. Letting $\delta = \delta_X$, it's easy to see that $i_0 \restriction (\delta + 1) = i_1 \restriction (\delta + 1)$, $i_0(\delta) = i_1(\delta) = \delta$, and $\wp(\delta)^{\mathcal{R}_0} = \wp(\delta)^{\mathcal{R}_1}$. This means $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa \rangle \in \wp(\delta)^{\mathcal{R}_0}$. By fullness of $\mathcal{P}_X$ in Γ,[4] $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa \rangle \in \mathcal{P}_X$. Using i_0, $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa \rangle \in \mathcal{P}_X$, and the fact that $i_0 \restriction \mathcal{P}_X | \delta_X = i_1 \restriction \mathcal{N}_X | \delta_X \in \mathcal{P}_X$, we can get $\mathcal{N}_X \in \mathcal{P}_X$ as follows. For any $\alpha, \beta < \delta_X$,

$$\alpha \in \mathcal{N}_\beta \text{ if and only if } i_0(\alpha) \in i_1(\mathcal{N}_\beta) = i_0(\mathcal{N}_\beta).$$

Since $\mathcal{P}_X$ can compute the right hand side of the equivalence, it can compute the sequence $\langle \mathcal{N}_\alpha \mid \alpha < \kappa \rangle$. Contradiction.

By the above argument, $\mathcal{P}_X^+$ thinks $\mathcal{P}_X$ is full. Let

$$\tau_X^* : \mathcal{P}_X^+ \to \mathcal{P}^+$$

be the ultrapower map by the $(\mathrm{crt}(\tau_X), \delta)$-extender E induced by π_X. Note that τ_X^* extends $\tau_X \restriction \mathcal{P}_X$ (since τ_X is cofinal in $\mathcal{P}$) and $\mathcal{P}^+$ is wellfounded since X is closed under ω-sequences. Let

$$i^* : \mathcal{P}_X^+ \to \mathcal{R}^+$$

be the ultrapower map by the $(\mathrm{crt}(i), \delta^{\mathcal{R}})$-extender induced by i. Note that $\mathcal{R} \lhd \mathcal{R}^+$ and $\mathcal{R}^+$ is wellfounded since there is a natural map

[3] The case κ is not measurable in $\mathcal{P}_X$ is easier and we leave it to the reader.

[4] Any $A \subset \delta$ in $\mathcal{R}_0$ is $OD^{\Gamma}_{\Sigma_X^-}$ and so by Strong Mouse Capturing (SMC), $A \in \mathcal{P}_X$.

$$k^* : \mathcal{R}^+ \to \mathcal{P}^+$$

extending k such that $\tau_X^* = k^* \circ i^*$. Without loss of generality, we may assume $\mathcal{M}$'s unique strategy $\Sigma_{\mathcal{M}} \leq_w \Lambda_X$. Also, let $(\dot{\mathcal{R}}, \dot{\vec{\mathcal{T}}})$ be the canonical $Col(\omega, \kappa)$-names for $(\mathcal{R}, \vec{\mathcal{T}})$. Let K be the transitive closure of $H_\kappa^V \cup (\dot{\mathcal{R}}, \dot{\vec{\mathcal{T}}})$.

Let $\mathcal{W} = \mathcal{M}_\omega^{\Lambda_X, \sharp}$ and Λ be the unique strategy of $\mathcal{W}$. Let $\mathcal{W}^*$ be a Λ-iterate of $\mathcal{W}$ below its first Woodin cardinal that makes K-generically generic. Then in $\mathcal{W}^*[K]$, the derived model $D(\mathcal{W}^*[K])$ satisfies

$$L(\Gamma(\mathcal{P}_X^+, \Lambda_X), \mathbb{R}) \vDash \dot{\mathcal{R}} \text{ is not full.}$$ [5]

So the above fact is forced over $\mathcal{W}^*[K]$ for $\dot{\mathcal{R}}$.

Let $H \prec H_\lambda$ be countable (in V) such that all relevant objects are in H. Let $\pi : M \to H$ invert the transitive collapse and for all $a \in H$, let $\overline{a} = \pi^{-1}(a)$. By the countable completeness of E, there is a map $\overline{\pi} : \overline{\mathcal{R}^+} \to \mathcal{P}_X^+$ such that

$$\pi \restriction \overline{\mathcal{P}_X^+} = \overline{\pi} \circ \overline{i^*}.$$ [6]

Let Λ_0 be the π-pullback of Λ_0 and Λ_1 be the $\overline{\pi}$-pullback of Λ_X. Note that Λ_0 extends $\pi^{-1}(\Lambda_0)$ and Λ_0 is also the $\overline{i^*}$-pullback of Λ_1; so in particular, $\Lambda_0 \leq_w \Lambda_1$. We also confuse $\overline{\Lambda}$ with the π-pullback of Λ. Hence $\Gamma(\overline{\mathcal{P}_X^+}, \Lambda_0)$ witnesses that $\overline{\mathcal{R}}$ is not full and this fact is forced over $\bar{\mathcal{W}}^*[\bar{K}]$ for the name $\dot{\bar{\mathcal{R}}}$. This means if we further iterate $\overline{\mathcal{W}^*}$ to $\mathcal{Y}$ such that $\mathbb{R}^{V[G]}$ can be realized as the symmetric reals over $\mathcal{Y}$ then in the derived model $D(\mathcal{Y})$,

$$L(\Gamma(\overline{\mathcal{P}_X^+}, \Lambda_0)) \vDash \overline{\mathcal{R}} \text{ is not full.} \tag{9.1}$$

In the above, we have used the fact that the interpretation of the UB-code of the strategy for $\overline{\mathcal{P}_X^+}$ in $\mathcal{Y}$ to its derived model is $\Lambda_0 \restriction \mathbb{R}^{V[G]}$; this key fact is proved in [10, Theorem 3.26] and Chapter 6.

Now we iterate $\overline{\mathcal{R}^+}$ to $\mathcal{S}$ via Λ_1 to realize $\mathbb{R}^{V[G]}$ as the symmetric reals for the collapse $Col(\omega, < \delta^{\mathcal{S}})$, where $\delta^{\mathcal{S}}$ is the sup of $\mathcal{S}$'s Woodin cardinals. By the fact that $\Lambda_0 \leq_w \Lambda_1$, we get that in the derived model $D(\mathcal{S})$,

$$\overline{\mathcal{R}} \text{ is not full as witnessed by } \bar{\mathcal{M}}.$$

[5] This is because we can continue iterating $\mathcal{W}^*$ above the first Woodin cardinal to $\mathcal{W}^{**}$ such that letting λ be the sup of the Woodin cardinals of $\mathcal{W}^{**}$, then there is a $Col(\omega, < \lambda)$-generic h such that $\mathbb{R}^{V[G]}$ is the symmetric reals for $\mathcal{W}^{**}[h]$. And in $\mathcal{W}^{**}(\mathbb{R}^{V[G]})$, the derived model satisfies that $L(\Gamma(\mathcal{P}_X^+, \Lambda_X)) \vDash \mathcal{R}$ is not full.

[6] This is because $i^* \circ \bar{\pi} = \pi \restriction \overline{\mathcal{R}^+}$ (by countable completeness of E) and $i^* \circ \pi \restriction \overline{\mathcal{P}_X^+} = \pi \restriction \bar{\mathcal{R}}^+ \circ \overline{i^*}$.

So $\Sigma_{\bar{\mathcal{M}}}$ is $\mathrm{OD}_{\Sigma_{\overline{\mathcal{R}}}}$ in $D(\mathcal{S})$ and hence $\overline{\mathcal{M}} \in \overline{\mathcal{R}}$. This contradicts internal fullness of $\overline{\mathcal{R}}$ in $\overline{\mathcal{R}^+}$.

□

The main theorem of this chapter is.

Theorem 9.2.7 *Suppose (ϕ, A) is captured. Then $\forall^* X' \in \mathfrak{S}_{\phi,A}$, $X' \cap \mathcal{P}$ is a condensing set.*

Proof. To prove the theorem, we need the following definition, due to the first author (cf. [11] or [31]). The proof is based on [31, Lemma 3.82]. For completeness, we give a fairly detailed argument here.

Suppose X is a weakly condensing set and $B \in \mathcal{P}_X \cap \wp(\delta_X)$.[7] We say that τ_X has *B-condensation* if whenever $\mathcal{Q} = \mathcal{Q}_Y$ (where Y is an extension of X) is such that there are elementary embeddings $\upsilon : \mathcal{P}_X \to \mathcal{Q}$, $\tau : \mathcal{Q} \to \mathcal{P}$ such that $\mathcal{Q}$ is countable in $V[G]$ and $\tau_X = \tau \circ \upsilon$, then $\upsilon(T_{\mathcal{P}_X,B}) = T_{\mathcal{Q},\tau,B}$, where

$$T_{\mathcal{P}_X,B} = \{(\phi, s) \mid s \in [\delta_X]^{<\omega} \wedge \mathcal{P}_X \vDash \phi[s, B]\},$$

and

$$T_{\mathcal{Q},\tau,B} = \{(\phi, s) \mid s \in [\delta^{\mathcal{Q}}_\alpha]^{<\omega} \text{ for some } \alpha < \lambda_{\mathcal{Q}} \wedge \mathcal{P} \vDash \phi[\pi^{\Sigma^{\tau,-}_{\mathcal{Q}}}_{\mathcal{Q}(\alpha),\infty}(s), \tau_X(B)]\},$$

where $\Sigma^\tau_{\mathcal{Q}}$ is the τ-pullback strategy and $\Sigma^{\tau,-}_{\mathcal{Q}} = \oplus_{\alpha<\lambda^{\mathcal{Q}}} \Sigma^\tau_{\mathcal{Q}(\alpha)}$. We say τ_X has *condensation* if it has B-condensation for every $B \in \mathcal{P}_X \cap \wp(\delta_X)$.

To prove that a weakly condensing set X is condensing, it is enough to prove that τ_X has condensation.

Suppose for contradiction that the set T of $X' \in \mathfrak{S}_{\phi,A}$ such that $X = X' \cap \mathcal{P}$ is cofinal in $\mathcal{P}$ and is not a condensing set is stationary. For each $X' \in T$, let $X = X' \cap \mathcal{P}$ (we will use this type of notations throughout this proof without mentioning again) and A_X be the $\lesssim_X$-least such that τ_X fails to have A_X-condensation, where $\lesssim_X$ is the canonical well-ordering of $\mathcal{P}_X$. We say that a tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$ is a *bad tuple* if

1. $Y \in T$;
2. $\mathcal{P}_i = \mathcal{P}_{X_i}$ for all i, where $X'_i \in T$ and $\mathcal{Q}_i = \mathcal{Q}_{Y_i}$ for Y_i an extension of X_i;
3. for all $i < j$, $X_i \prec Y_i \prec X_j \prec Y$;

[7] For the rest of this proof, whenever X is weakly condensing, we automatically assume that $X = X' \cap \mathcal{P}$ for some good X'.

4. $\mathcal{M}_{\infty,Y}$ be the direct limit of iterates $(\mathcal{Q}, \Lambda)$ of $(\mathcal{P}_Y, \Sigma_Y)$ such that Λ has branch condensation;

5. for all i, $\xi_i : \mathcal{P}_i \to \mathcal{Q}_i$, $\sigma_i : \mathcal{Q}_i \to \mathcal{M}_{\infty,Y}$, $\tau_i : \mathcal{P}_{i+1} \to \mathcal{M}_{\infty,Y}$, and $\pi_i : \mathcal{Q}_i \to \mathcal{P}_{i+1}$;

6. for all i, $\tau_i = \sigma_i \circ \xi_i$, $\sigma_i = \tau_{i+1} \circ \pi_i$, and $\tau_{X_i,X_{i+1}} \restriction \mathcal{P}_i =_{\text{def}} \phi_{i,i+1} = \pi_i \circ \xi_i$;

7. $\phi_{i,i+1}(A_{X_i}) = A_{X_{i+1}}$;

8. for all i, $\xi_i(T_{\mathcal{P}_i,A_{X_i}}) \neq T_{\mathcal{Q}_i,\sigma_i,A_{X_i}}$.

In (8), $T_{\mathcal{Q}_i,\sigma_i,A_{X_i}}$ is computed relative to $\mathcal{M}_{\infty,Y}$, that is

$$T_{\mathcal{Q}_i,\sigma_i,A_{X_i}} = \{(\phi, s) \mid s \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega} \text{ for some } \alpha < \lambda^{\mathcal{Q}_i} \wedge \mathcal{M}_{\infty,Y} \vDash \phi[\pi^{\Sigma^{\sigma_i,-}_{\mathcal{Q}_i}}_{\mathcal{Q}_i(\alpha),\infty}(s), \tau_i(A_{X_i})]\}$$

Claim 9.2.8 *There is a bad tuple.*

Proof. For brevity, we first construct a bad tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{P}\}$ with $\mathcal{P}$ playing the role of $\mathcal{M}_{\infty,Y}$. We then simply choose a sufficiently large, good Y and let $i_Y : \mathcal{P}_Y \to \mathcal{M}_{\infty,Y}$ be the direct limit map, $m_Y : \mathcal{M}_{\infty,Y} \to \mathcal{P}$ be the natural factor map, i.e. $m_Y \circ i_Y = \pi_Y$. It's easy to see that for all sufficiently large Y, the tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, m_Y^{-1} \circ \tau_i, m_Y^{-1} \circ \xi_i, m_Y^{-1} \circ \pi_i, m_Y^{-1} \circ \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$ is a bad tuple.

The key point is (6). Let $A^*_X = \tau_X(A_X)$ for all $X \in T$. By Fodor's lemma, there is an A^* such that $\exists^* X \in T$ $A^*_X = A^*$.[8] So there is an increasing and cofinal sequence $\{X_\alpha \mid \alpha < \kappa^+\} \subseteq T$ such that for $\alpha < \beta$, $\tau_{X_\alpha,X_\beta}(A_{X_\alpha}) = A_{X_\beta} = \tau^{-1}_{X_\beta}(A)$. This easily implies the existence of such a tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{P}\}$. $\square$

Fix a bad tuple $\mathcal{A} = \{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$. Let $(\mathcal{P}_0^+, \Pi)$ be a (g-organized) $\Sigma^-_{\mathcal{P}_0}$-hod pair (cf. [20]) such that

$$\Gamma(\mathcal{P}_0^+, \Pi) \vDash \mathcal{A} \text{ is a bad tuple.}$$

We may also assume $(\mathcal{P}_0^+, \Pi \restriction V) \in V$, $\lambda^{\mathcal{P}_0^+}$ is limit of nonmeasurable cofinality in $\mathcal{P}_0^+$ and there is some $\alpha < \lambda^{\mathcal{P}_0^+}$ such that $\Sigma_Y \leq_w \Pi_{\mathcal{P}_0^+(\alpha)}$. This type of reflection is possible because we replace $\mathcal{H}^+$ by $\mathcal{M}_{\infty,Y}$. Let $\mathcal{W} = \mathcal{M}_\omega^{\sharp,\Sigma_Y,\Pi,\oplus_{n<\omega}\Sigma_{X_n}}$ and Λ be the unique strategy of $\mathcal{W}$. If $\mathcal{Z}$ is the result of iterating $\mathcal{W}$ via Λ to make $\mathbb{R}^{V[G]}$ generic, then letting h be $\mathcal{Z}$-generic for the Levy collapse of the sup of $\mathcal{Z}$'s Woodin cardinals to ω such that $\mathbb{R}^{V[G]}$ is the symmetric reals of $\mathcal{Z}[h]$, then in $\mathcal{Z}(\mathbb{R}^{V[G]})$,

[8] "$\exists^* X \in T$" means "stationarily many $X \in T$".

$$\Gamma(\mathcal{P}_0^+, \Pi) \vDash \mathcal{A} \text{ is a bad tuple.}$$

Now we define by induction $\xi_i^+ : \mathcal{P}_i^+ \to \mathcal{Q}_i^+$, $\pi_i^+ : \mathcal{Q}_i^+ \to \mathcal{P}_{i+1}^+$, $\phi_{i,i+1}^+ : \mathcal{P}_i^+ \to \mathcal{P}_{i+1}^+$ as follows. $\phi_{0,1}^+ : \mathcal{P}_0^+ \to \mathcal{P}_1^+$ is the ultrapower map by the $(\text{crt}(\pi_{X_0,X_1}), \Theta_{X_1})$-extender derived from π_{X_0,X_1}. Note that $\phi_{0,1}^+$ extends $\phi_{0,1}$. Let $\xi_0^+ : \mathcal{P}_0^+ \to \mathcal{Q}_0^+$ extend ξ_0 be the ultrapower map by the $(\text{crt}(\xi_0), \delta^{\mathcal{Q}_0})$-extender derived from ξ_0. Finally let $\pi_0^+ = (\phi_{0,1}^+)^{-1} \circ \xi_0^+$. The maps $\xi_i^+, \pi_i^+, \phi_{i,i+1}^+$ are defined similarly. Let also $\mathcal{M}_Y = \text{Ult}(\mathcal{P}_0^+, E)$, where E is the (λ_X, Θ_Y)-extender derived from $\pi_{X,Y}$. There are maps $\epsilon_{2i} : \mathcal{P}_i^+ \to \mathcal{M}_Y$, $\epsilon_{2i+1} : \mathcal{Q}_i^+ \to \mathcal{M}_Y$ for all i such that $\epsilon_{2i} = \epsilon_{2i+1} \circ \xi_i^+$ and $\epsilon_{2i+1} = \epsilon_{2i+2} \circ \pi_i^+$. When $i = 0$, ϵ_0 is simply i_E. Letting $\Sigma_i = \Sigma_{\mathcal{P}_i}^-$ and $\Psi = \Sigma_{\mathcal{Q}_i}^-$, $A_i = A_{X_i}$, there is a finite sequence of ordinals t and a formula $\theta(u, v)$ such that in $\Gamma(\mathcal{P}_0^+, \Pi)$

9. for every $i < \omega$, $(\phi, s) \in T_{\mathcal{P}_i, A_i} \Leftrightarrow \theta[\pi^{\Sigma_i}_{\mathcal{P}_i(\alpha),\infty}, t]$, where α is least such that $s \in [\delta_\alpha^{\mathcal{P}_i}]^{<\omega}$;

10. for every i, there is $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \xi_i(A_i)}$ such that $\neg\theta[\pi^{\Psi_i}_{\mathcal{Q}_i(\alpha)}(s_i), t]$ where α is least such that $s_i \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega}$.

The pair (θ, t) essentially defines a Wadge-initial segment of $\Gamma(\mathcal{P}_0^+, \Pi)$ that can define the pair $(\mathcal{M}_{\infty,Y}, A^*)$, where $\tau_i(A_i) = A^*$ for some (any) i.

Now let $X \prec H_\lambda$ be countable that contains all relevant objects and $\pi : M \to X$ invert the transitive collapse. For $a \in X$, let $\overline{a} = \pi^{-1}(a)$. By countable completeness of the extender E, there is a map $\pi^* : \overline{\mathcal{M}_Y} \to \mathcal{P}_0$ such that $\pi \restriction \overline{\mathcal{M}_Y} = \epsilon_0 \circ \pi^*$. Let $\overline{\Pi_i}$ be the $\pi^* \circ \overline{\epsilon_i}$-pullback of Π. Note that in $V[G]$, $\overline{\Sigma_Y} \leq_w \overline{\Pi_0} \leq_w \overline{\Pi_1} \cdots \leq_w \Pi^{\pi^*}$.

Let $\dot{\mathcal{A}} \in (H_{\bar{\kappa}})^M$ be the canonical name for $\bar{\mathcal{A}}$. It's easy to see (using the assumption on $\mathcal{W}$) that if $\mathcal{W}^*$ is a result of iterating $\bar{\mathcal{W}}$ via $\bar{\Lambda}$ (we confuse $\bar{\Lambda}$ with the π-pullback of Λ; they coincide on M) in M below the first Woodin of $\bar{\mathcal{W}}$ to make H-generically generic, where H is the transitive closure of $H^M_{\omega_2} \cup \dot{A}$, then in $\mathcal{W}^*[H]$, the derived model of $\mathcal{W}^*[H]$ at the sup of $\mathcal{W}^*$'s Woodin cardinals satisfies:

$$L(\bar{\mathcal{P}}_0, \mathbb{R}) \vDash \dot{\mathcal{A}} \text{ is a bad tuple.}$$

Now we stretch this fact out to $V[G]$ by iterating $\mathcal{W}^*$ to $\mathcal{W}^{**}$ to make $\mathbb{R}^{V[G]}$-generic. In $\mathcal{W}^{**}(\mathbb{R}^{V[G]})$, letting $i : \mathcal{W}^* \to \mathcal{W}^{**}$ be the iteration map then

$$\Gamma(\bar{\mathcal{P}}_0{}^+, \bar{\Pi}) \vDash i(\bar{\mathcal{A}})^{9} \text{ is a bad tuple.}$$

[9] We abuse the notation slightly here. Technically, $\bar{\mathcal{A}}$ is not in $\mathcal{W}^*$ but $\mathcal{W}^*$ has a canonical name $\dot{\mathcal{A}}$ for $\bar{\mathcal{A}}$. Hence by $i(\bar{\mathcal{A}})$, we mean the interpretation of $i(\dot{\mathcal{A}})$.

By a similar argument as in [30, Theorem 3.1.25], we can use the strategies $\overline{\Pi_i}^+$'s to simultanously execute a $\mathbb{R}^{V[G]}$-genericity iterations. The last branch of the iteration tree is wellfounded. The process yields a sequence of models $\langle\overline{\mathcal{P}_{i,\omega}^+}, \overline{\mathcal{Q}_{i,\omega}^+} \mid i < \omega\rangle$ and maps $\overline{\xi_{i,\omega}^+} : \overline{\mathcal{P}_{i,\omega}^+} \to \overline{\mathcal{Q}_{i,\omega}^+}$, $\overline{\pi_{i,\omega}^+} : \overline{\mathcal{Q}_{i,\omega}^+} \to \overline{\mathcal{P}_{i+1,\omega}^+}$, and $\overline{\phi_{i,i+1,\omega}^+} = \overline{\pi_{i,\omega}^+} \circ \overline{\pi_{i,\omega}^+}$. Furthermore, each $\overline{\mathcal{P}_{i,\omega}^+}, \overline{\mathcal{Q}_{i,\omega}^+}$ embeds into a Π^{π^*}-iterate of $\overline{\mathcal{M}_Y}$ and hence the direct limit $\mathcal{P}_\infty$ of $(\overline{\mathcal{P}_{i,\omega}^+}, \overline{\mathcal{Q}_{j,\omega}^+} \mid i,j < \omega)$ under maps $\overline{\pi_{i,\omega}^+}$'s and $\overline{\xi_{i,\omega}^+}$'s is wellfounded. We note that $\overline{\mathcal{P}_{i,\omega}^+}$ is a (g-organized) Σ_i^π-premouse and $\overline{\mathcal{Q}_{i,\omega}^+}$ is a ${}^g\Psi_i^\pi$-premouse because the genericity iterations are above $\overline{\mathcal{P}_i}$ and $\overline{\mathcal{Q}_i}$ for all i and by [10, Theorem 3.26], the interpretation of the strategy of $\bar{\mathcal{P}}_i$ ($\bar{\mathcal{Q}}_i$ respectively) in the derived model of $\bar{\mathcal{P}}_{i,\omega}^+$ ($\bar{\mathcal{P}}_{i,\omega}^+$, respectively) is (g-organized) Σ_i^π (Ψ_i^π, respectively). Let C_i be the derived model of $\overline{\mathcal{P}_{i,\omega}^+}$, D_i be the derived model of $\overline{\mathcal{Q}_{i,\omega}^+}$ (at the sup of the Woodin cardinals of each model), then $\mathbb{R}^{V[G]} = \mathbb{R}^{C_i} = \mathbb{R}^{D_i}$. Furthermore, $C_i \cap \wp(\mathbb{R}) \subseteq D_i \cap \wp(\mathbb{R}) \subseteq C_{i+1} \cap \wp(\mathbb{R})$ for all i.

(9), (10) and the construction above give us that there is a $t \in [\mathrm{OR}]^{<\omega}$, a formula $\theta(u,v)$ such that

11. for each i, in C_i, for every (ϕ, s) such that $s \in \delta^{\overline{\mathcal{P}_i}}$, $(\phi, s) \in T_{\overline{\mathcal{P}_i}, \overline{A_i}} \Leftrightarrow \theta[\pi^{\overline{\Sigma_i}}_{\overline{\mathcal{P}_i}(\alpha),\infty}(s), t]$ where α is least such that $s \in [\delta_\alpha^{\overline{\mathcal{P}_i}}]^{<\omega}$.

Let n be such that for all $i \geq n$, $\overline{\xi_{i,\omega}^+}(t) = t$. Such an n exists because the direct limit $\mathcal{P}_\infty$ is wellfounded as we can arrange that $\mathcal{P}_\infty$ is embeddable into a Π^{π^*}-iterate of $\bar{\mathcal{M}}_Y$. By elementarity of $\overline{\xi_{i,\omega}^+}$ and the fact that $\overline{\xi_{i,\omega}^+} \restriction \mathcal{P}_i = \overline{\xi_i}$,

12. for all $i \geq n$, in D_i, for every (ϕ, s) such that $s \in \delta^{\overline{\mathcal{Q}_i}}$, $(\phi, s) \in T_{\overline{\mathcal{Q}_i}, \overline{\xi_i}(\overline{A_i})} \Leftrightarrow \theta[\pi^{\overline{\Psi_i}}_{\overline{\mathcal{Q}_i}(\alpha),\infty}(s), t]$ where α is least such that $s \in [\delta_\alpha^{\overline{\mathcal{Q}_i}}]^{<\omega}$.

However, using (10), we get

13. for every i, in D_i, there is a formula ϕ_i and some $s_i \in [\delta^{\overline{\mathcal{Q}_i}}]^{<\omega}$ such that $(\phi_i, s_i) \in T_{\overline{\mathcal{Q}_i}, \overline{\xi_i}(\overline{A_i})}$ but $\neg\phi[\pi^{\overline{\Psi_i}}_{\overline{\mathcal{Q}_i}(\alpha),\infty}(s_i), t]$ where α is least such that $s \in [\delta_\alpha^{\overline{\mathcal{Q}_i}}]^{<\omega}$.

Clearly (12) and (13) give us a contradiction. This completes the proof of the lemma. $\square$

The following is a useful corollary of the proof of the previous theorem. We will apply this corollary in many applications later.

Corollary 9.2.9 *Suppose $Y \prec Z$ are honest extensions of a (ϕ, A)-condensing set X. Suppose $B \in \wp(\delta^{\mathcal{P}}) \cap \mathcal{P}$ and B is in the range of τ_Y, τ_Z. Let $a \in (\delta^{\mathcal{Q}_Y})^{<\omega}$. Then $\pi^{\Sigma_Y}_{\mathcal{Q}_Y,\infty}(a) \in B$ if and only if $\pi^{\Sigma_Z}_{\mathcal{Q}_Z,\infty}(\tau_{Y,Z}(a)) \in B$.*

9.3 Condensing sets in models of AD^+

Thus far we have build condensing sets while working in models of ZFC. In this section, we prove their existence in models of AD^+. The material presented in this section will be used in the proof of generation of pointclasses (see Theorem 10.1.1). Throughout this section we assume $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$. Recall the notation $\Gamma_1 \trianglelefteq_{mouse} \Gamma_2$ (see [10, Page 82] or Section 5.3).

Suppose Γ is a mouse full pointclass (Definition 5.3.2) such that:

$(*)_\Gamma$ there is a good pointclass Γ^* containing Γ and there is a sequence $(\Gamma_\alpha : \alpha < \Omega)$ with the property that

1. Ω is a limit ordinal,
2. $\Gamma_\alpha \triangleleft_{mouse} \Gamma$,
3. for $\alpha < \beta < \Omega$, $\Gamma_\alpha \triangleleft_{mouse} \Gamma_\beta$,
4. there is no completely mouse-full pointclass $\Psi \triangleleft_{mouse} \Gamma$ such that for some α, $\Gamma_\alpha \triangleleft_{mouse} \Psi \triangleleft_{mouse} \Gamma_{\alpha+1}$,
5. if $\alpha < \Omega$ is a limit ordinal then $\Gamma_\alpha = \bigcup_{\beta<\alpha} \Gamma_\beta$,
6. $\Gamma = \bigcup_{\alpha<\Omega} \Gamma_\alpha$.

Recall the definitions of HP^Γ and $Mice^\Gamma$ (see Notation 4.1.4). Let $\mathcal{F} = \{(\mathcal{P}, \Sigma) \in HP^\Gamma : \Sigma$ is strongly Γ-fullness preserving and has strong branch condensation$\}$. We then let $\mathcal{M}^- = \bigcup_{(\mathcal{P},\Sigma)\in\mathcal{F}} \mathcal{M}_\infty(\mathcal{P}, \Sigma)$. It follows from AD^+ theory that if $(\mathcal{P}, \Sigma) \in \mathcal{F}$ then Σ can be extended to a (Θ, Θ, Θ)-iteration strategy. To see this, consider HOD_Σ and use generic interpretability (see Theorem 5.2.5) along with the fact that if $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ coded as a set of reals then it can be added to HOD_Σ generically. In what follows, we assume that if $(\mathcal{P}, \Sigma) \in \mathcal{F}$ then Σ is a (Θ, Θ, Θ)-iteration strategy.

Given $\alpha < \lambda^{\mathcal{M}^-}$, we let Σ_α be the strategy of $\mathcal{M}^-(\alpha)$ such that whenever $(\mathcal{P}, \Sigma) \in \mathcal{F}$ is such that $\mathcal{M}_\infty(\mathcal{P}, \Sigma) = \mathcal{M}^-(\alpha)$ then $\Sigma_{\mathcal{M}^-(\alpha)} = \Sigma_\alpha$. Next we let $Lp^{\Gamma, \oplus_{\alpha<\lambda^{\mathcal{M}^-}} \Sigma_\alpha}(\mathcal{M}^-)$ be the stack of all sound $\oplus_{\alpha<\lambda^{\mathcal{M}^-}} \Sigma_\alpha$-premice $\mathcal{N}$ over $\mathcal{M}^-$ such that $\rho(\mathcal{N}) \leq o(\mathcal{M}^-)$ and whenever $\pi : \mathcal{S} \to \mathcal{N}$ is elementary and $\mathcal{S}$ is countable then $\mathcal{S}$, as a $\oplus_{\alpha<\lambda^{\pi^{-1}(\mathcal{M}^-)}} \Sigma^\pi_{\pi(\alpha)}$-mouse, has an ω_1-iteration strategy in Γ. Finally, if there is $\mathcal{N} \trianglelefteq Lp^{\Gamma, \oplus_{\alpha<\lambda^{\mathcal{M}^-}} \Sigma_\alpha}(\mathcal{M}^-)$ such that $\rho(\mathcal{N}) < o(\mathcal{M}^-)$ then let $\mathcal{M}$ be the least such $\mathcal{N}$ and otherwise let $\mathcal{M} = Lp_\omega^{\Gamma, \oplus_{\alpha<\lambda^{\mathcal{M}^-}} \Sigma_\alpha}(\mathcal{M}^-)$.

We let $\phi(U, V)$ be the formula that expresses the fact that U is a mouse full pointclass such that $(*)_U$ holds and V is a hod pair $(\mathcal{Q}, \Lambda)$ such that $Code(\Lambda) \in U$ and Λ has strong branch condensation and is strongly U-fullness preserving.

Theorem 9.3.1 *One of the following holds.*

1. *There is a hod pair $(\mathcal{P}, \Sigma)$ such that Σ has strong branch condensation and is strongly Γ-fullness preserving, and $\Gamma(\mathcal{P}, \Sigma) = \Gamma$ (i.e., (ϕ, Γ) is not maximal).*
2. *$\mathcal{M} = Lp^{\Gamma, \oplus_{\alpha<\lambda^{\mathcal{M}^-}} \Sigma_\alpha}(\mathcal{M}^-)$, lower part (ϕ, Γ)-covering fails and there is a (ϕ, Γ)-condensing set $X \in \wp_{\omega_1}(\mathcal{M})$.*

Proof. Towards a contradiction assume that both clauses are false. We drop (ϕ, Γ) from our terminology. Let $A_0 \subseteq \mathbb{R}$ be such that $w(A_0) = \Gamma$. We enlarge the set HP^Γ. Let HP be the set of hod pairs in HP^Γ and also the pairs $(\mathcal{P}, \Sigma)$ such that for some limit ordinal λ there is a sequence of hod pairs $((\mathcal{P}_\alpha, \Sigma_\alpha) : \alpha < \lambda)$ with the property that

1. for all $\alpha < \lambda$, $(\mathcal{P}_\alpha, \Sigma_\alpha) \in HP^\Gamma$,
2. for $\alpha < \beta < \lambda$, $\mathcal{P}_\alpha \lhd_{hod} \mathcal{P}_\beta$ and $\Sigma_\alpha = (\Sigma_\beta)_{\mathcal{P}_\alpha}$, and
3. $\mathcal{P}|\delta^{\mathcal{P}} = \bigcup_{\alpha<\lambda} \mathcal{P}_\alpha$ and $\Sigma = \oplus_{\alpha<\lambda}\Sigma_\alpha$.

We also change $Mice^\Gamma$. Let $Mice$ be the sets in $Mice^\Gamma$ and also triples $(a, \Sigma, \mathcal{M})$ such that if M_Σ is the structure Σ iterates then $(M_\Sigma, \Sigma) \in HP$, $a \in HC$ and $\mathcal{M}$ is a Σ-mouse over a. Let $(A_0)_\Gamma$ be the set of reals σ that code a pair $\langle \sigma_l, \sigma_r \rangle$ of continuous functions such that $\sigma_l^{-1}[A_0]$ is a code for a set in $(\mathcal{Q}, \Lambda) \in HP$ and $\sigma_r^{-1}[A_0]$ is a code for a triple $(a, \mathcal{M}, \Psi)$ such that $(a, \Lambda, \mathcal{M}) \in Mice$ and Ψ is the unique strategy of $\mathcal{M}$. Let $A_1 = (A_0)_\Gamma$.

For each pair $(\mathcal{P}, \Sigma) \in HP$, there is a sjs $\langle B_i : i < \omega \rangle$ such that $Mice^\Gamma_\Sigma = B_0$ and for every $i < \omega$, $B_i \in \Gamma$. We then let A_2 be the set of reals (σ_l, σ_r) such that

1. $\sigma_l^{-1}[A_0]$ codes a hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that $Code(\Sigma) \in \Gamma$ and Σ has strong branch condensation and is strongly Γ-fullness preserving,
2. $\sigma_r^{-1}[A_0]$ codes a sjs $\langle B_i : i < \omega \rangle$ such that $B_i \in \Gamma$ for all i and $Mice^\Gamma_\Sigma = B_0$.

Finally let A_3 be the set of reals σ such that $\sigma_l^{-1}[A_0]$ is a code for a set $(\mathcal{Q}, \Lambda) \in HP$ and σ_r is a real coding a pair of reals (u, v) such that v codes $Lp^{\Gamma, \Lambda}(u)$

Let $\Gamma_0 <_w \Gamma_1$ be a two good pointclasses such that for each $i < 4$, $A_i \in \Delta_{\Gamma_0}$ and sets that are projective in a universal Γ_0 set are contained in $\utilde{\Delta}_{\Gamma_1}$. Let $F_0, (N_0, \Phi_0)$

and $F_1, (N_1, \Phi_1)$ be as in Theorem 4.1.6 for Γ_0 and Γ_1 respectively. Let $x \in dom(F_0)$ be such that if $F_0(x) = (\mathcal{N}, \mathcal{M}, \delta, \Psi)$ then $(\mathcal{N}, \delta, \Psi)$ Suslin, co-Suslin captures $\oplus_{i<4} A_i$. Let $y \in dom(F_1)$ be such that if $F_1(y) = (\mathcal{N}^*_y, \mathcal{M}_y, \delta_y, \Sigma_y)$ then $(\mathcal{N}^*_y, \delta_y, \Sigma_y)$ Suslin, co-Suslin captures $Code(\Psi^*)$ where Ψ^* is the ω_1-strategy of $\mathcal{M}_2^{\#,\Psi}$ (and hence, also Suslin co-Suslin captures $\oplus_{i<4} A_i$).

Let κ be the least $< \delta_y$-strong cardinal of $\mathcal{N}^*_y$. Let $g \subseteq Coll(\omega, <\kappa)$ be $\mathcal{N}^*_y$-generic. Let $A_2^g = A_2 \cap \mathcal{N}^*_y[g]$. Let $\psi(u, v)$ be a formula such that $\mathcal{N}^*_y \vDash \psi[A_2^g, v]$ if and only if v is a hod pair $(\mathcal{Q}, \Lambda) \in \mathcal{N}^*_y[g]$ such that Λ is an $(\omega_2, \omega_2, \omega_2)$-strategy in $\mathcal{N}^*_y[g]$ and there is $\sigma \in A_2^g$ such that $\mathcal{N}^*_y[g] \vDash \sigma_l^{-1}[A_0] = Code(\Lambda \restriction HC^{\mathcal{N}^*_y[g]})$.

Given $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\psi, A_2^g}$ and $\sigma \in A_2^g$, we let Λ^σ be the iteration strategy of $\mathcal{Q}$ coded by $\sigma_l^{-1}[A_0]$. Notice that $(\mathcal{Q}, \Lambda^\sigma) \in \mathcal{F}$ and $(\mathcal{Q}, \Lambda)$ is independent of σ (this later claim is a consequence of the fact that A_2 is Suslin, co-Suslin captured by $(\mathcal{N}^*_y, \delta_y, \Sigma_y)$). We then abuse our notation and let Λ also stand for Λ^σ. Notice that if h is any $< \delta_y$-generic over $\mathcal{N}^*_y[g]$ then

$$Code(\Lambda) \cap \mathcal{N}^*_y[g * h] = (\sigma_l^{-1}[A_0])^{\mathcal{N}^*_y[g*h]}.$$

We clearly have that (ψ, A_2^g) is lower part closed. Examining the definition of A_2, it is also clear that (ψ, A_2^g) is stable. The next claim shows that it is directed.

Claim 1. $\mathcal{N}^*_y[g] \vDash$ "(ψ, A_2^g) is directed".

Proof. Fix $(\mathcal{Q}_0, \Lambda_0), (\mathcal{Q}_1, \Lambda_1) \in \mathcal{F}_{\phi, A_2^g}$. Fix for $i < 2$, $\sigma_i \in \mathcal{N}^*_y[g]$ such that for each $i < 2$, $\mathcal{N}^*_y[g] \vDash (\sigma_i)_l^{-1}[A_0] = Code(\Lambda_i)$.

We now compare $(\mathcal{Q}_i, \Lambda_i)$ with the hod pair construction of $\mathcal{N}^*_y$. It follows from Theorem 4.6.10 that for each $i < 2$, $\mathcal{Q}_i$ iterates, via Λ_i, to some model $\mathcal{Q}_i^+$ in the aforementioned hod pair construction such that $(\Lambda_i)_{\mathcal{Q}_i^+}$ is the strategy $\mathcal{Q}_i$ inherits from the background construction. Let $\nu_i < \kappa$ be such that $\mathcal{Q}_i, \sigma_i \in \mathcal{N}^*_y[g \cap Coll(\omega, \nu_i)]$, and let $g_i = g \cap Coll(\omega, \nu_i)$. To complete the proof it is enough to show that

(1) for each i, $(\mathcal{Q}_i^+, (\Lambda_i)_{\mathcal{Q}_i^+})$ appears in the Γ-hod pair construction of $\mathcal{N}^*_y|\kappa[g_i]$ in which all extenders used have critical point $> \max(\nu_0, \nu_1)$.

Suppose (1) fails. Let $\eta \in (\kappa, \delta_y)$ be such that $(\mathcal{Q}_i^+, \Lambda_i)$ appears in the Γ-hod pair construction of $\mathcal{N}^*_y|\eta[g_i]$. Let then $E \in \vec{E}^{\mathcal{N}^*_y}$ be such that $\mathrm{crit}(E) = \kappa$ and $\nu_E > \nu_i$. It follows that in $Ult(\mathcal{N}^*_y, E)[g_i]$, $(\mathcal{Q}_i^+, \Lambda_i)$ appears in the Γ-hod pair construction of $(Ult(\mathcal{N}^*_y, E)|\pi_E(\kappa))[g_i]$. Using elementarity we get a contradiction. $\square$

Working in $\mathcal{N}^*_y$, let $\mathcal{P}^- = \mathcal{P}^-_{\psi, A_2^g}$. Our next claim implies that (ψ, A_2^g) is of limit type.

Claim 2. $\lambda^{\mathcal{P}^-}$ is a limit ordinal.

Proof. Suppose not. It follows that there is $(\mathcal{Q}, \Lambda) \in \mathcal{F}^g$ such that $\mathcal{P}^- = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. Let $\sigma \in \mathbb{R}^{\mathcal{N}_y^*[g]}$ be such that $Code(\Lambda) = (\sigma_l^{-1}[A_0])^{\mathcal{N}_y^*[g]}$. Let $\nu < \kappa$ be a cutpoint cardinal of $\mathcal{N}_y^*$ such that $\mathcal{Q}, \sigma \in \mathcal{N}_y^*|\nu[g]$. It follows from the proof of (1) above that the Γ-hod pair construction of $\mathcal{N}_y^*|\kappa$ in which extenders used have critical point $> \nu$ reaches a pair $(\mathcal{R}, \Phi)$ such that $\mathcal{R}$ is a Λ-iterate of $\mathcal{Q}$ and $\Phi = \Lambda_{\mathcal{R}}$.

Because of our condition on Γ (namely that Ω is a limit ordinal) there is $\alpha+1 < \Omega$ such that $\Gamma_\alpha = \Gamma(\mathcal{Q}, \Lambda)$. It follows that the Γ-hod pair construction of $\mathcal{N}_y^*$ using extenders with critical point $> \nu$ reaches $(\mathcal{S}, \Delta) \in \mathcal{F}$ such that $\Gamma(\mathcal{S}, \Delta) = \Gamma_{\alpha+1}$. It follows from the proof of (1) above that the Γ-hod pair construction of $\mathcal{N}_y^*|\kappa$ in which extenders used have critical point $> \nu$ reaches such a pair $(\mathcal{S}, \Delta)$. It is now enough to show that there is $\tau \in \mathbb{R}^{\mathcal{N}_y^*[g]}$ such that $Code(\Delta) = \tau_l^{-1}[A_0]$. Let $\nu_1 \in (\nu, \kappa)$ be an $\mathcal{N}_y^*$-cutpoint cardinal such that $\mathcal{S} \in \mathcal{N}_y^*|\nu_1$.

Let $\eta \in (\nu_1, \kappa)$ be the least $\mathcal{N}_y^*$-cardinal such that $\mathcal{M}_1^{\#,\Psi}(\mathcal{N}_y^*|\eta) \vDash$ "η is a Woodin cardinal". Let $\mathcal{N}_1$ be the output of $\mathcal{J}^{\vec{E},\Psi}$ construction of $\mathcal{N}_y^*|\eta$ done using extenders with critical points $> \nu_1$. We now compare $(\mathcal{S}, \Delta)$ with the Γ-hod pair construction of $\mathcal{N}_1$. Notice that all extenders of $\mathcal{N}_1$ have critical points $> \nu_1$. Let $\mathcal{S}_1$ be the output of the aforementioned Γ-hod pair construction.

We claim that some proper initial segment of $\mathcal{S}_1$ is a Δ-iterate of $\mathcal{S}$. Suppose not. Let $\tau \in \mathbb{R}$ be such that $Code(\Delta) = (\tau)_l^{-1}[A_0]$. Let $z \in dom(F_1)$ be such that $y, \tau <_T z$. Let $F_1(z) = (\mathcal{N}_z^*, \mathcal{M}_z, \delta_z, \Sigma_z)$.

Working in $\mathcal{N}_z^*$, let η_1 be the least $\mathcal{N}_z^*$-cardinal such that $\mathcal{M}_1^{\#,\Psi}(\mathcal{N}_z^*|\eta_1) \vDash$ "η_1 is a Woodin cardinal". Let $\mathcal{N}^*$ be the output of the fully backgrounded $\mathcal{J}^{\vec{E},\Psi}$-construction of $\mathcal{N}_z^*|\eta_1$ done over $\mathcal{N}_y^*|\nu_1$. Comparing $\mathcal{N}_y^*$ with the construction producing $\mathcal{N}^*$ we get a tree $\mathcal{T}$ on $\mathcal{N}_y^*$ according to Σ_y such that $\mathcal{T}$ is based on $\mathcal{N}_y^*|\eta$ and if $\mathcal{T}^-$ is $\mathcal{T}$ without its last branch then $\mathcal{M}(\mathcal{T}^-) = \mathcal{N}^*|\eta_1$.

We now have that $\mathcal{M}_1^{\#,\Psi}(\mathcal{N}^*|\eta_1[\tau]) \vDash$ "η_1 is a Woodin cardinal" (this can be shown by considering S-constructions). Yet, by elementarity $(\mathcal{S}, \Delta)$ wins the comparison against the Γ-hod pair construction of $\mathcal{N}^*|\eta_1$, contradicting universality of the latter. This contradiction implies that some initial segment of $\mathcal{S}_1$ is a Δ-iterate of $\mathcal{S}$. Let $\mathcal{S}_2$ be this initial segment.

We now show that there is a real $q \in \mathbb{R}^{\mathcal{N}_y^*[g]}$ such that $Code(\Delta_{\mathcal{S}_2}) = q_l^{-1}[A_0]$. Fix $r \in V$ such that $Code(\Delta_{\mathcal{S}_2}) = \tau_l^{-1}[A_0]$. Let ξ be a cutpoint of $\mathcal{N}_1$ such that $\mathcal{S}_2 \in \mathcal{N}_1|\xi$. Let $\mathcal{N}_1^+ = \mathcal{M}_1^{\#,\Psi}(\mathcal{N}_1|\eta)$ and let Ψ^+ be the strategy of $\mathcal{N}_1^+$. Let $\pi : \mathcal{N}_1^+ \to \mathcal{N}_2$ be an iteration of $\mathcal{N}_1^+$ via Ψ^+ such that τ is generic over $\mathcal{N}_1^+$ for the extender algebra at $\pi(\eta)$. Let δ be the second Woodin cardinal in $\mathcal{N}_1^+$ (so $\delta > \eta$). We now have that

(2) for every $h \subseteq Coll(\omega, \pi(\delta))$ that is $\mathcal{N}_2[\tau]$-generic, $\mathcal{N}_2[\tau][h] \vDash$ "$Code(\Delta_{\mathcal{S}_2}) = \tau_l^{-1}[A_0]$".

It follows from elementarily of π that

(3) $\mathcal{N}_1^+ \vDash$ "it is forced by $Coll(\omega, \eta)$ that there is a real s such that in any further $Coll(\omega, \delta)$-generic extension $Code(\Delta_{\mathcal{S}_2}) = s_l^{-1}[A_0]$".

Because $\mathcal{N}_1^+$ is countable in $\mathcal{N}_y^*[g]$, by absoluteness, we can fix $q \in \mathbb{R}^{\mathcal{N}_y^*[g]}$ such that

(4) q is generic over $\mathcal{N}_1^+$ for $Coll(\omega, \eta)$ and whenever $h \subseteq Coll(\omega, \delta)$ is $\mathcal{N}_1^+[q]$-generic, $\mathcal{N}_1^+[q][h[\vDash$ "$Code(\Delta_{\mathcal{S}_2}) = q_l^{-1}[A_0]$".

Now δ is a Woodin cardinal in $\mathcal{N}_1^+[q]$. It follows from genericity iterations that

$$Code(\Delta_{\mathcal{S}_2}) = q_l^{-1}[A_0].$$

This finishes the proof of Claim 2. □

Our discussion before Claim 1, Claim 1 and Claim 2 show that (ψ, A_2^g) is lower part closed, is of limit type, is stable and is directed. We now work in $\mathcal{N}_y^*[g]$. Let

1. $\Sigma = \Sigma_{\psi, A_2^g}$ (see clause 2 of Notation 9.1.2) and
2. $\mathcal{P} = \mathcal{P}_{\psi, A_2^g}$.

Notice that if h is $Coll(\omega, \mathbb{R}^{\mathcal{N}_y^*[g]})$-generic over $\mathcal{N}_y^*[g]$ and z is the real coding $A_2 \cap \mathbb{R}^{\mathcal{N}_y^*[g]}$ then z also codes a real τ such that $Code(\Sigma) = \tau_l^{-1}[A_0]$.

Claim 3. $Code(\Sigma) \in \Gamma$.

Proof. Towards a contradiction assume not. Suppose that for some α, $\rho(\mathcal{J}_\alpha[\mathcal{P}^-]) < o(\mathcal{P}^-)$. Let then $\mathcal{S} = \mathcal{J}_\alpha[\mathcal{P}^-]$ where α is the least such. If there is no such α then let $\mathcal{S} = \mathcal{M}^+[\mathcal{P}^-]$.

Suppose now that $\mathcal{S} \vDash$ "cf$(\delta^{\mathcal{S}})$ is not a measurable cardinal". It then follows that $\Gamma(\mathcal{S}, \Sigma) = \Gamma$, contradicting that clause 1 of our theorem fails. Suppose $\mathcal{S} \vDash$ "cf$(\delta^{\mathcal{S}})$

is a measurable cardinal". Let $E \in \vec{E}^{\mathcal{S}}$ be the Mitchell order 0 extender in $\mathcal{S}$ such that $\mathrm{crit}(E) = \mathrm{cf}^{\mathcal{S}}(\delta^{\mathcal{S}})$. Notice that $Code(\Sigma) \leq_w Ult(\mathcal{S}, E)(\lambda^{\mathcal{S}})$; this implies that if $\mathcal{S}^* = Ult(\mathcal{S}, E)(\lambda^{\mathcal{S}})$ and $\Sigma^* = \oplus_{\alpha<\lambda^{\mathcal{S}}} \Sigma_{\mathcal{S}^*(\alpha)}$ then $\Gamma(\mathcal{S}^*, \Sigma^*) = \Gamma$. This is again a contradiction, as we now have that clause 1 of our theorem holds. $\square$

Since $\Gamma \neq \wp(\mathbb{R})$ is a mouse full pointclass, there is a $C \subseteq \mathbb{R}$ such that $\Gamma_1 \in L(C, \mathbb{R})$. We then have that $L(C, \mathbb{R}) \vDash \mathsf{DC}$. Work in $W = L(C, \mathbb{R})$ and let $G \subseteq Coll(\omega_1, \mathbb{R})$ be W-generic. Notice that $W[G] \vDash \mathsf{ZFC}$. Let $((\mathcal{Q}_\alpha, \Lambda_\alpha) : \alpha < \omega_1)$ be an enumeration of $\mathcal{F}$ and $(z_\alpha : \alpha < \omega_1)$ be an enumeration of $\mathbb{R}$. In $W[G]$, choose a sequence $(y_\alpha : \alpha < \omega_1)$ of reals such that

1. $y_0 = y$,
2. for all $\alpha < \omega_1$, letting $F_1(y_\alpha) = (\mathcal{N}^*_{y_\alpha}, \mathcal{M}_{y_\alpha}, \delta_{y_\alpha}, \Sigma_{y_\alpha})$, $(z_\beta : \beta \leq \alpha) \in \mathcal{N}^*_{y_\alpha}$ and $\oplus_{\beta\leq\alpha}\Lambda_\alpha$ is Suslin, co-Suslin captured by $(\mathcal{N}^*_{y_\alpha}, \delta_{y_\alpha}, \Sigma_{y_\alpha})$, and
3. for $\alpha < \beta$, $(\mathcal{N}^*_{y_\alpha} : \alpha < \beta) \in HC^{\mathcal{N}^*_{y_\beta}}$.

We now construct a sequence of Φ_1-mice $(\mathcal{M}_\alpha, \mathcal{N}_\alpha : \mathcal{N}_\alpha \lhd \mathcal{M}_\alpha \wedge \alpha < \omega_1)$ and a sequence of commuting embedding $\pi_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ such that $\pi_{\alpha,\beta}(\mathcal{N}_\alpha) = \mathcal{N}_\beta$ and if $\kappa_\alpha = \mathrm{crit}(\pi_{\alpha,\beta})$ then $\mathcal{N}_\alpha = \mathcal{M}_\alpha|\kappa_\alpha$. For $\alpha > 0$ we will have that $\mathcal{M}_\alpha$ is the output of a fully backgrounded construction of $\mathcal{N}^*_{y_\alpha}$ relative to Φ_1 and also that $\mathcal{N}_\alpha \trianglelefteq \mathcal{M}_\alpha$. We then let Δ_α be the strategy of $\mathcal{M}_\alpha$ induced by Σ_{y_α}. Let $\mathcal{M}_0 = \mathcal{N}^*_{y_0}$ and $\mathcal{N}_0 = \mathcal{M}_0|\kappa$. Given $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$, let $\mathcal{M}_{\alpha+1} = (\mathcal{J}^{\vec{E},\Phi_1}[\mathcal{N}_\alpha])^{\mathcal{N}^*_{y_\alpha}}$. Let $\pi_{\alpha,\alpha+1} : \mathcal{M}_\alpha \to \mathcal{M}_{\alpha+1}$ be the iteration embedding according to Δ_α. Let $\kappa_{\alpha+1}$ be the least $\delta_{y_{\alpha+1}}$-strong cardinal of $\mathcal{M}_{\alpha+1}$ and let $\mathcal{N}_{\alpha+1} = \pi_{\alpha,\alpha+1}(\mathcal{N}_\alpha) = \mathcal{M}_{\alpha,\alpha+1}|\kappa_{\alpha+1}$.

Suppose now that we have constructed a sequence $(\mathcal{M}_\alpha, \mathcal{N}_\alpha : \mathcal{N}_\alpha \lhd \mathcal{M}_\alpha \wedge \alpha < \lambda)$ and a sequence of commuting embedding $\pi_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for $\alpha < \beta < \lambda$. Let $\mathcal{M}^*_\lambda$ be the direct limit of $\mathcal{M}_\alpha$ under $\pi_{\alpha,\beta}$. Let $\pi^*_{\alpha,\lambda} : \mathcal{M}_\alpha \to \mathcal{M}^*_\lambda$ be the embedding given by the direct limit construction. Let then $\mathcal{N}_\lambda = \pi_{0,\lambda}(\mathcal{N}_0)$ and let $\mathcal{M}_\lambda$ be the output of $\mathcal{J}^{\vec{E},\Phi_1}[\mathcal{N}_\lambda]$ construction of $\mathcal{N}^*_{y_\lambda}$ done over $\mathcal{N}_\lambda$. Letting $k : \mathcal{M}^*_\lambda \to \mathcal{M}_\lambda$ be the iteration embedding according to $(\Sigma_y)_{\mathcal{M}^*_\lambda}$, we set $\pi_{\alpha,\lambda} = k \circ \pi^*_{\alpha,\lambda}$.

Finally let $\mathcal{M}_{\omega_1}$ be the direct limit of $\mathcal{M}_\alpha$ under $\pi_{\alpha,\beta}$ and let $\pi_{\alpha,\omega_1} : \mathcal{M}_\alpha \to \mathcal{M}_{\omega_1}$ be the direct limit embedding. Let $\mathcal{P}_\infty = \pi_{0,\omega_1}(\mathcal{P})$ and $\mathcal{P}^-_\infty = \pi_{0,\omega_1}(\mathcal{P}^-)$.

Claim 4. Fix $\alpha < \omega_1$ and let $h \subseteq Coll(\omega, <\kappa_\alpha)$ be $\mathcal{N}^*_{y_\alpha}$-generic. Then $\pi_{0,\alpha}(\mathcal{P}) = (\mathcal{P}_{\psi,A_2^h})^{\mathcal{N}^*_{y_\alpha}[h]}$ and $\Delta_\alpha = (\Sigma_{\psi,A_2^h})^{\mathcal{N}^*_{y_\alpha}[h]}$ where $A_2^h = A_2 \cap \mathcal{N}^*_{y_\alpha}[h]$.

We leave the proof of Claim 4 to the reader. To prove it use the proof of (1) and show that given $(\mathcal{Q}, \Lambda) \in \mathcal{F}_{\phi, A_2^h}$, some Γ-hod pair construction of $\mathcal{M}_\alpha$ reaches a Λ-iterate of $\mathcal{Q}$. We let $\mathcal{P}_\alpha = \pi_{0,\alpha}(\mathcal{P}_\alpha)$, $\mathcal{P}_\alpha^- = \pi_{0,\alpha}(\mathcal{P}^-)$ and $\Sigma^\alpha = \pi_{0,\alpha}(\Sigma)$.

Claim 5. $\mathcal{P}_\infty = \mathcal{M}$.

Proof. Notice that

(5) for $\alpha < \beta < \omega_1$ and for $\xi < \lambda^{\mathcal{P}_\alpha}$, $\pi_{\alpha,\beta} \restriction \mathcal{P}_\alpha(\xi)$ is the iteration embedding according to $(\Sigma^\alpha)_{\mathcal{P}_\alpha(\xi)}$, and
(6) if $\alpha < \omega_1$ and $\mathcal{Q}$ is a $(\Sigma^\alpha)_{\mathcal{P}_\alpha(\xi)}$-iterate of $\mathcal{P}_\alpha(\xi)$ then there is $\beta < \omega_1$ such that $\mathcal{P}_\beta(\pi_{\alpha,\beta}(\xi))$ is a $(\Sigma^\alpha)_{\mathcal{Q}}$-iterate of $\mathcal{Q}$.

To see (6), let β be large enough such that $(\mathcal{Q}_\beta, \Lambda_\beta) = (\mathcal{Q}, (\Sigma^\alpha)_{\mathcal{Q}})$. It then follows that $\mathcal{P}_\beta(\pi_{\alpha,\beta}(\xi))$ is a $(\Sigma^\alpha)_{\mathcal{Q}}$-iterate of $\mathcal{Q}$. It follows from (5) and (6) that $\mathcal{P}_\infty|\delta^{\mathcal{M}} = \mathcal{M}|\delta^{\mathcal{M}}$.

If $\rho(\mathcal{P}_\infty) < o(\mathcal{P}_\infty^-)$ then we must have that $\mathcal{P}_\infty = \mathcal{M}$. Suppose then $\rho(\mathcal{P}_\infty) > o(\mathcal{P}_\infty^-)$. Clearly $\mathcal{P}_\infty \trianglelefteq Lp_\omega^{\Gamma,\Sigma}(\mathcal{M}^-)$. Suppose then $\mathcal{P}_\infty \triangleleft Lp^{\Gamma,\Sigma}(\mathcal{M}^-)$. By a standard Skolem hull argument, it follows that for some $\alpha < \omega_1$, $\mathcal{P}_\alpha \triangleleft Lp^{\Gamma,\Sigma^\alpha}(\pi_{0,\alpha}(\mathcal{P}^-))$. However, because $\rho(\mathcal{P}_\infty) > o(\mathcal{P}_\infty^-)$, $\mathcal{N}_{y_\alpha}^* \vDash$ "$\mathcal{P}_\alpha = Lp^{\Gamma,\Sigma^\alpha}(\pi_{0,\alpha}(\mathcal{P}^-))$", contradiction. □

Claim 6. $\rho(\mathcal{M}) > o(\mathcal{M}^-)$.

Proof. Assume $\rho(\mathcal{M}) < o(\mathcal{M}^-)$ (it follows from the definition of $\mathcal{M}$ that equality is impossible). Using Definition 9.0.2, we can construct an iteration strategy Σ^+ extending Σ such that Σ^+ is the π_{0,ω_1}-realizable strategy. We have Σ^+ is strongly $\Gamma(\mathcal{P}, \Sigma^+)$-fullness preserving (see Theorem 5.4.6). It follows from Theorem 5.4.6 that Σ^+ has (lower-level?) strong branch condensation. Because Σ^+ is π_{0,ω_1}-realizable, we have that $\Gamma(\mathcal{P}, \Sigma^+) \subseteq \Gamma$. Because clause 1 fails, we must have that $\Gamma(\mathcal{P}, \Sigma^+) <_w \Gamma$. Because Σ is strongly Γ-fullness preserving, we can finish by using the argument given on page 143 of [10].

The argument proceeds by considering the set $\mathcal{G}$ of all hod pairs $(\mathcal{S}, \Phi)$ such that $\Gamma(\mathcal{S}, \Phi) = \Gamma(\mathcal{P}, \Sigma^+)$ and Φ has strong branch condensation and is strongly $\Gamma(\mathcal{S}, \Phi)$-fullness preserving. Let $\nu = \sup \pi_{\mathcal{P},\infty}^{\Sigma^+}[\mathcal{P}]$. We have that $\rho(\mathcal{M}_\infty(\mathcal{P}, \Sigma^+)) < \nu$. However, because for any $(\mathcal{S}, \Phi) \in \mathcal{G}$, $\mathcal{M}_\infty(\mathcal{S}, \Phi) = \mathcal{M}_\infty(\mathcal{P}, \Sigma^+)$, we have that if $(\mathcal{Q}, \Lambda) \in \mathcal{F}$ is such that $(\mathcal{P}, \Sigma^+) \in L(\Lambda, \mathbb{R})$ then $V_\nu^{\mathrm{HOD}^{L(\Lambda,\mathbb{R})}} \subseteq \mathcal{M}_\infty(\mathcal{P}, \Sigma^+)|\nu$. We leave the details to the reader. □

Suppose then $\mathcal{P} = Lp_{\omega}^{\Gamma,\Lambda}(\mathcal{P}^-)$.

Claim 7. $\mathcal{N}_y^* \vDash |\mathcal{P}| = \kappa$.

Proof. Recall the real z introduced before the statement of Claim 3. We have that $z \in \mathcal{N}_y^*[g][h]$ where h is $Coll(\omega, \mathbb{R}^{\mathcal{N}_y^*[g]})$-generic and z codes A_2^g. It then also follows that z codes a real τ such that $Code(\Sigma) = \tau_l^{-1}[A_0]$. Because $(\mathcal{N}_y^*, \delta_y, \Sigma_y)$ Suslin, co-Suslin capture A_3, it follows that there is a real $u \in \mathcal{N}_y^*[g][h]$ that codes $\mathcal{P}$. □

Notice that Claim 7 implies that lower part (ϕ, Γ)-covering fails as it implies that $\mathrm{cf}(o(\mathcal{P}_\infty)) = \omega$.

It follows from Theorem 9.2.7 that $X =_{def} \pi_{0,1}[\mathcal{P}] \in \wp(\mathcal{P}_1) \cap \mathcal{M}_1$ is such that

(7) for any $\mathcal{M}_1[g]$-generic $h \subseteq Coll(\omega, <\kappa_1)$, $\mathcal{M}_1[g*h] \vDash$ "X is countable and is a (ψ, A_2^{g*h})-condensing set" where $A_2^{g*h} = A_2 \cap \mathcal{M}_1[g*h]$.

It follows from Claim 7 that

(8) for every $\alpha \in [1, \omega_1)$ and for every $\mathcal{M}_{y_\alpha}[g]$-generic $h \subseteq Coll(\omega, <\kappa_\alpha)$, $\mathcal{M}_{y_\alpha}[g*h] \vDash$ "$\pi_{1,\alpha}[X]$ is a (ψ, A_2^{g*h})-condensing set" where $A_2^{g*h} = A_2 \cap \mathcal{M}_\alpha[g*h]$.

We claim that $X^+ = \pi_{1,\omega_1}[X]$ is a condensing set in W. Notice that X^+ is countable in $W[G]$ and hence, it follows from ω-completeness of $Coll(\omega_1, \mathbb{R})$ and DC that X^+ is in W. To prove that X^+ is a condensing set in W we need to show that (8) holds for $\mathcal{N}_{y_\alpha}^*$, not just $\mathcal{M}_{y_\alpha}$.

Claim 8. For every $\alpha \in [1, \omega_1)$ and for every $\mathcal{N}_{y_\alpha}^*[g]$-generic $h \subseteq Coll(\omega, <\kappa_\alpha)$, $\mathcal{N}_{y_\alpha}[g*h] \vDash$ "$\pi_{1,\alpha}[X]$ is a (ψ, A_2^{g*h})-condensing set" where $A_2^{g*h} = A_2 \cap \mathcal{N}_\alpha[g*h]$.

Proof. We give the proof for $\alpha = 1$ and leave the rest to the reader. Let $h \subseteq Coll(\omega, <\kappa_1)$ be $\mathcal{N}_{y_1}^*[g]$-generic and let $Y \subseteq \mathcal{P}_1$ be an extension of X. In what follows we will use the notation used to defining condensing sets all localized to $\mathcal{N}_{y_1}^*$. Thus, $\Sigma_Y \in \mathcal{N}_{y_1}^*$ is the τ_Y-pullback of $\pi_{0,1}(\Sigma)$. However, we will also confuse Σ_Y and $\pi_{0,1}(\Sigma)$ with their canonical extensions that act on all stacks.

First we need to show that

(9) Σ_Y is a strongly Γ-fullness preserving iteration strategy.

The proof follows the steps of Lemma 9.2.6. Recall that in that proof the key step is to find a universal model extending $\mathcal{P}$ such that π_{0,ω_1} acts on it. Here, we describe how to find this universal model and leave the rest, which is just like the proof of Lemma 9.2.6, to the reader. To simplify, we only show that if $\mathcal{Q}_Y^- = \tau_Y^{-1}(\mathcal{P}_1^-)$ then $\mathcal{Q}_Y = Lp_\omega^{\Gamma,\Sigma_Y}(\mathcal{Q}_Y^-)$. The rest of the proof is very similar.

Suppose then that $\mathcal{Q}_Y \triangleleft Lp_\omega^{\Gamma,\Sigma_Y}(\mathcal{Q}_Y^-)$ and let $\mathcal{S} \triangleleft Lp_\omega^{\Gamma,\Sigma_Y}(\mathcal{Q}_Y^-)$ be the least such that $\rho(\mathcal{S}) = \mathcal{Q}_Y^-$ and $\mathcal{S} \ntrianglelefteq \mathcal{Q}_Y$. Let $(\mathcal{R}, \Lambda) \in HP^\Gamma$ be such that $\lambda^\mathcal{R}$ is a limit ordinal and $L(\Gamma(\mathcal{R}, \Lambda), \mathbb{R}) \vDash$ "$\mathcal{S}$, as a Σ_Y-mouse, has an iteration strategy. Let $\alpha < \omega_1$ be such that $Code(\Lambda)$ is Suslin, co-Suslin captured by $(\mathcal{N}^*_{y_\alpha}, \delta_{y_\alpha}, \Sigma_{y_\alpha})$.

Let $\mathcal{W}^*$ be the output of $\mathcal{J}^{\vec{E},\Phi_1,\Sigma}$-construction of $\mathcal{N}^*_y$ done using extenders with critical point $> \kappa$. Let $\mathcal{W}^{**}$ be the output of $\vec{\mathcal{E}}, \oplus_\infty, \pm$-construction of $\mathcal{N}^*_{y_\alpha}$ done using extenders with critical point $> \kappa_\alpha$. Notice that it follows that $o(\mathcal{W}^*) = \delta_y$ and $o(\mathcal{W}^{**}) = \delta_{y_\alpha}$. We now compare the construction producing $\mathcal{W}^*$ and the construction producing $\mathcal{W}^{**}$. The comparison produces a tree $\mathcal{T}$ on $\mathcal{N}^*_y$ of limit length such that $\mathcal{T} \in \mathcal{N}^*_{y_\alpha}$ and if $b = \Sigma_y(\mathcal{T})$ then $\pi^\mathcal{T}_\beta(\mathcal{W}^*) = \mathcal{W}^{**}$.

Let $\mathcal{W}$ be the Γ-hod pair construction of $\mathcal{W}^{**}$ done over $\mathcal{P}$ and relative to Σ. It follows from Theorem 4.6.8 some model of the hod pair construction of $\mathcal{W}$ reaches a Λ-iterate of $\mathcal{R}$. It then follows that for some $\alpha < \lambda^\mathcal{W}$, if Φ^* is the strategy of $\mathcal{W}(\alpha)$ induced by Σ_{y_α} then $\mathcal{S}$, as a Σ_Y-mouse, has an iteration strategy in $L(\Gamma(\mathcal{W}(\alpha), \Phi^*))$. $\mathcal{W}(\alpha)$ is our universal model but we cannot yet apply $\pi_{0,1}$ to it. To do this, let $\mathcal{U} = \pi_{0,1}\mathcal{T}$. Let $b = \Sigma_y(\mathcal{T})$. The copying construction produces $\sigma : \mathcal{M}^\mathcal{T}_b \to \mathcal{M}^\mathcal{U}_b$ such that $\pi^\mathcal{U}_b \circ \pi_{0,1} = \sigma \circ \pi^\mathcal{T}_b$. Moreover, $\text{crit}(\sigma) > \kappa$.

It then follows that $\sigma(\mathcal{W}(\alpha))$ is a hod premouse over $\mathcal{P}_1$ relative to $\pi_{0,1}(\Sigma)$. Moreover, Φ^* is the σ-pullback of the strategy of $\sigma(\mathcal{W}(\alpha))$ induced by $(\Sigma_{y_1})_{\mathcal{M}^\mathcal{U}_b}$. It now follows that we can lift $\pi_{X,Y}$ to $\mathcal{W}_\alpha$ and obtain $\pi^+_{X,Y} : \mathcal{W}(\alpha) \to \mathcal{W}^{***}$ and $\tau^+_Y : \mathcal{W}^{***} \to \sigma(\mathcal{W}(\alpha))$. The rest of the proof follows very closely to the proof of Lemma 9.2.6.

Next, we need to show that Y is such that $\pi_{X,Y} \restriction \mathcal{P}^-$ is the iteration embedding according to Σ then $\pi_{0,1} = \sigma^X_Y \circ \pi_{X,Y}$. It follows from the proof of (1) that there is $(\mathcal{R}, \Lambda) \in (\mathcal{F}_{\phi, A_2^{g*h}})^{\mathcal{M}_1[g*h]}$ such that $X \cup Y \subseteq \pi^\Lambda_{\mathcal{R},\mathcal{P}_1}$ and if $Z = \pi^\Lambda_{\mathcal{R},\mathcal{P}_1}[\mathcal{R}|\delta^\mathcal{R}]$ and $k = (\pi^\Lambda_{\mathcal{R},\mathcal{P}_1})^{-1} \circ \tau_Y$, then both $k \restriction \mathcal{Q}_Y^-$ and $\pi_{X,Z} \restriction \mathcal{P}^-$ are the iteration embeddings according to Σ_Y and Σ respectively. It follows from (8) that

(10) $\pi_{0,1} = \sigma^X_Z \circ \pi_{X,Z}$

Notice that $\sigma^X_Z = \pi^{\Lambda}_{\mathcal{R},\mathcal{P}_1}$. We then have that

$$\pi_{0,1} = \sigma^X_Z \circ \pi_{X,Z} = \sigma^X_Z \circ k \circ \pi_{X,Y}.$$

Notice now that $\sigma^X_Z \circ k = \sigma^X_Y$. We then have that $\pi_{0,1} = \sigma^X_Y \circ \pi_{X,Y}$. This finishes the proof of the claim. □

We now show that X^+ is a condensing set. We omit the proof that X^+ is a weakly condensing set and move directly to verifying the clauses of Definition 9.1.8. This is because both proofs are very similar.

Let $Y \in \wp_{\omega_1}(\mathcal{P}_\infty|\delta^{\mathcal{P}_\infty}) \cap W$ be an honest extension of X^+. Let $\alpha < \omega_1$ be large enough so that $Y \subseteq rng(\pi^{\Lambda_\alpha}_{\mathcal{Q}_\alpha,\infty})$ and for some $\beta \leq \alpha$, z_β codes some $Y^* \subseteq \mathcal{Q}_\alpha$ such that $Y = \pi^{\Lambda_\alpha}_{\mathcal{Q}_\alpha,\infty}[Y^*]$. It now follows that $\bar{Y} = \pi^{-1}_{\alpha,\omega_1}[Y] \in \mathcal{N}^*_{y_\alpha}$. Indeed, $\bar{Y} = \pi^{\Lambda_\alpha}_{\mathcal{Q}_\alpha,\mathcal{P}_\alpha}[Y^*]$ (this follows from (6)).

Let $\tau_Y : \mathcal{Q}_Y \rightarrow \mathcal{P}_\infty$ be the inverse of the collapse of $Hull^{\mathcal{P}_\infty}(X^+ \cup Y)$ and let $\tau_{\bar{Y}} : \mathcal{Q}_{\bar{Y}} \rightarrow \mathcal{P}_\alpha$ be the inverse of the collapse of $Hull^{\mathcal{P}_\alpha}(\pi_{0,\alpha}[X] \cup \bar{Y})$. Let $\pi_{X^+,Y} = (\tau_Y)^{-1} \circ \tau_{X^+}$ and let $\pi_{X,\bar{Y}} = (\tau_{\bar{Y}})^{-1} \circ (\tau_{\pi_{1,\alpha}[X]})^{\mathcal{N}^*_{y_\alpha}}$. We then have that

(11) $\mathcal{Q}_Y = \mathcal{Q}_{\bar{Y}}$, $\pi_{X^+,Y} = \pi_{X,\bar{Y}}$, $\tau_Y = \pi_{\alpha,\omega_1} \circ \tau_{\bar{Y}}$ and $\pi_{0,\alpha} = \tau_{\bar{Y}} \circ \pi_{X,\bar{Y}}$.

Suppose then $\pi_{X,Y} \restriction \mathcal{P}|\delta^{\mathcal{P}}$ is the iteration embedding according to Σ. Let Σ_Y be the π_Y-pullback of $\Sigma_{\phi,\Gamma}$. Notice that it follows from (6) that

(12) $\pi_{0,\omega_1}(\Sigma) = \Sigma_{\phi,\Gamma}$.

Let $\sigma^{X^+}_Y : \mathcal{Q}_Y \rightarrow \mathcal{P}_\infty$ be given by $\sigma^{X^+}_Y(u) = \pi_X(f)(\pi^{\Sigma_Y}_{\mathcal{Q}_Y(\beta),\infty}(a))$ where $u \in \mathcal{Q}_Y$, $f \in \mathcal{P}$, $\beta < \lambda^{\mathcal{Q}_Y}$ and $a \in \mathcal{Q}_Y(\beta)^{<\omega}$ are such that $u = \pi_{X,Y}(f)(a)$. We want to show that

(13) $\tau_{X^+} =_{def} \pi_{0,\omega_1} \restriction \mathcal{P} = \sigma^{X^+}_Y \circ \pi_{X^+,Y}$.

We define $\sigma^X_{\bar{Y}}$ and $\Sigma_{\bar{Y}}$ similarly. It follows from(11) and (12) that

(14) $\Sigma_Y = \Sigma_{\bar{Y}}$ and $\sigma^{X^+}_Y = \pi_{\alpha,\omega_1} \circ \sigma^X_{\bar{Y}}$.

It follows from Claim 8 that

(15) $\pi_{0,\alpha} \restriction \mathcal{P} = \sigma^X_{\bar{Y}} \circ \pi_{X,\bar{Y}}$.

Combining (14) and (15) we get that $\pi_{0,\omega_1} \restriction \mathcal{P} = \sigma^X_Y \circ \tau_Y$. This finishes the proof of Theorem 9.3.1. $\square$

Chapter 10

Applications

10.1 The generation of the mouse full pointclasses

In this section, our goal is to show that under Strong Mouse Capturing (SMC) if Γ is a mouse full pointclass (see Definition 5.3.2) such that $\Gamma \neq \wp(\mathbb{R})$ and there is a good pointclass Γ^* with the property that $\Gamma \subset \Gamma^*$ then there is a hod pair or an sts pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma$. Recall that SMC states that for any hod pair or sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ is strongly fullness preserving and has strong branch condensation then for any $x, y \in \mathbb{R}$, $x \in OD_{y,\Sigma}$ if and only if $x \in Lp^{\Sigma}(y)$.

We work under the following minimality assumption.

$\#_{lsa}$**:** There is a pointclass $\Gamma \subset \wp(\mathbb{R})$ such that there is a Suslin cardinal bigger than $w(\Gamma)$ and $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$.

As in [10, Section 6.1], we will construct $(\mathcal{P}, \Sigma)$ as above via a hod pair construction of some sufficiently strong background universe. Here is our theorem on generation of pointclasses.

Theorem 10.1.1 (The generation of the mouse full pointclasses) *Assume* AD^+ *and* $\neg\#_{lsa}$. *Suppose* $\Gamma \neq \wp(\mathbb{R})$ *is a mouse full pointclass such that* $\Gamma \vDash \mathsf{SMC}$. *Then the following holds:*

1. *Suppose* Γ *is completely mouse full and let* $A \subseteq \mathbb{R}$ *witness it. Then the following holds:*

   (a) *Suppose* $L(A, \mathbb{R}) \vDash \neg\mathsf{LSA}$. *Then there is a hod pair* $(\mathcal{P}, \Sigma) \in L(A, \mathbb{R})$ *such that* $L(A, \mathbb{R}) \vDash$ *"Σ has strong branch condensation and is super fullness preserving" and* $\Gamma(\mathcal{P}, \Sigma) = \Gamma$.

(b) *Suppose $L(A,\mathbb{R}) \vDash \mathsf{LSA}$. Then there is an sts hod pair $(\mathcal{P},\Sigma) \in L(A,\mathbb{R})$ such that $L(A,\mathbb{R}) \vDash$ "Σ has branch condensation and is fullness preserving", $\mathcal{P}$ is of lsa type and $\Gamma^b(\mathcal{P},\Sigma) = \Gamma$. If in addition there are good pointclasses beyond $L(A,\mathbb{R})$, then there is a a hod pair $(\mathcal{P},\Sigma)$ such that $(\mathcal{P},\Sigma^{stc}) \in L(A,\mathbb{R})$ and $(\mathcal{P},\Sigma^{stc})$ satisfies the above conditions.*

2. *Suppose Γ is mouse full but not completely mouse full. Then there is a hod pair or an anomalous hod pair $(\mathcal{P},\Sigma)$ such that Σ has strong branch condensation, $\mathcal{P}$ is of limit type and either*

   (a) *$\mathcal{P}$ is of lsa type and $\Gamma^b(\mathcal{P},\Sigma^{stc}) = \Gamma$ or*

   (b) *$\mathcal{P}$ is not of lsa type and $\Gamma(\mathcal{P},\Sigma) = \Gamma$.*

Proof. Our proof has the same structure as the proof of [10, Theorem 6.1]. However, unlike that proof, we will make an important use of Theorem 9.3.1. The proof is again by induction. Suppose $\Gamma \neq \wp(\mathbb{R})$ is a mouse full pointclass such that whenever Γ^* is properly contained in Γ and is a mouse full pointclass then there is a hod pair $(\mathcal{P},\Sigma)$ as in 1 or 2. We want to show that the claim holds for Γ. Suppose not. We examine several cases.

Case 1. There is a sequence of mouse full pointclass $(\Gamma_\alpha : \alpha < \Omega)$ such that $\Gamma_\alpha \subseteq \Gamma$, $\Gamma = \bigcup_{\alpha<\Omega} \Gamma_\alpha$ and for $\alpha < \beta < \Omega$, $\Gamma_\alpha \trianglelefteq_{mouse} \Gamma_\beta$.

We will use the terminology of Section 9.3. Let $\phi(U,V)$ be the formula that expresses the fact that U is a mouse full pointclass having the properties that Γ has and V is a hod pair $(\mathcal{Q},\Lambda)$ such that $Code(\Lambda) \in U$ and Λ has strong branch condensation and is strongly U-fullness preserving.

Let $\mathcal{M}^- = \mathcal{P}^-_{\phi,\Gamma}$ and $\mathcal{M} = \mathcal{P}_{\phi,\Gamma}$. Because we are assuming that Γ is not generated by a hod pair, it follows from clause 2 of Theorem 9.3.1 that $\rho(\mathcal{M}) > o(\mathcal{M}^-)$ and that there is a condensing set $X \in \wp_{\omega_1}(\mathcal{M})$. In what follows we will use the notation introduced in Section 9.1. In particular, recall the definition of τ_Y and σ^X_Y.

Following the proof of Theorem 9.3.1, let $(A_i : i < 4)$, $\Gamma_0 <_w \Gamma_1$, F_0, and F_1 be as in that proof. We introduce two more kinds of set of reals that we need to be captured.

Let $(\alpha_i : i < \omega)$ be an enumeration of X and let $x_i = (\alpha_k : k \leq i)$. Let $(\phi_i : i < \omega)$ be an enumeration of formulas in the language of hod mice. Let $B_{i,k}$ be the set of pairs $((\mathcal{Q},\Lambda,\beta),(\mathcal{R},\Psi,\gamma))$ such that $(\mathcal{Q},\Lambda),(\mathcal{R},\Psi) \in HP^\Gamma$, $\beta < \delta^{\mathcal{Q}}$, $\gamma < \delta^{\mathcal{R}}$ and $\pi^\Psi_{\mathcal{R},\infty}(\gamma)$ is the unique ordinal ξ such that $\mathcal{M} \vDash \phi_k[x_i, \pi^\Lambda_{\mathcal{Q},\infty}(\beta), \xi]$. We then let $A_{i,k}$

be the set of reals σ such that σ codes a pair of continuous functions (σ_l, σ_r) such that $(\sigma_l^{-1}[A_0], \sigma_r^{-1}[A_0])$ is a code for a pair in $B_{i,k}$.

Next, let B be the set of $(\mathcal{Q}, \Lambda)$ such that $X \cap \delta^{\mathcal{M}} \subseteq \pi^{\Lambda}_{\mathcal{Q},\infty}[\mathcal{Q}|\delta^{\mathcal{Q}}]$ and the transitive collapse of $Hull^{\mathcal{M}}(X \cup \pi^{\Lambda}_{\mathcal{Q},\infty}[\mathcal{Q}|\delta^{\mathcal{Q}}])$ is $\mathcal{Q}$. Given $(\mathcal{Q}, \Lambda) \in B$, let $Y_{\mathcal{Q},\Lambda} = \pi^{\Lambda}_{\mathcal{Q},\infty}[\mathcal{Q}|\delta^{\mathcal{Q}}]$. Let A_4 be the set of reals σ such that σ codes two reals (σ_l, σ_r) such that σ_l is a continuous functions with the property that $\sigma_l^{-1}[A_0]$ is a code for a pair $(\mathcal{Q}, \Lambda)$ in B and σ_r is a real coding a countable sequence $X_{\mathcal{Q},\Lambda} \subseteq \mathcal{Q}$ such that $\tau_{Y_{\mathcal{Q},\Lambda}}[X_{\mathcal{Q},\Lambda}] = X$.

We now define our final set A_5. Given $x \in \mathbb{R}$, let $A_x = \{u \in \mathbb{R} : \{u\}$ is $OD^{\Gamma}_{x,X}\}$. We let $A_5 = \{(x, y) \in \mathbb{R}^2 : y$ codes $A_x\}$. Let now $x \in \text{dom}(F_0)$ be such that if $F_0(x) = (\mathcal{N}, \mathcal{M}, \delta, \Psi)$ then $(\mathcal{N}, \delta, \Psi)$ Suslin, co-Suslin captures $\oplus_{i<6} A_i$ and $\oplus_{(i,k)\in\omega^2} A_{i,k}$. Let Ψ^* be the iteration strategy of $\mathcal{M}_3^{\#,\Psi}$ and let $y \in dom(F_1)$ be such that if $F_1(y) = (\mathcal{N}^*_y, \mathcal{M}_y, \delta_y, \Sigma_y)$ then $(\mathcal{N}^*_y, \delta_y, \Sigma_y)$ Suslin, co-Suslin captures Ψ^*.

We claim that some hod pair appearing on the Γ-hod pair construction of $\mathcal{N}^*_y|\delta_y$ generates Γ. Here the proof is somewhat different than the proof of Theorem 6.1 of [10]. There the contradictory assumption that such constructions do not reach Γ led to a construction of a hod pair $(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{P}} = \delta^{\mathcal{P}}$ and $\mathcal{P} \vDash$ "$\delta^{\mathcal{P}}$ is regular". This meant that a pointclass satisfying $\mathsf{AD}_{\mathbb{R}}+$ "Θ is regular" had been reached giving the desired contradiction. In our current situation, if the constructions never stops then we will reach an lsa type hod premouse $\mathcal{P}$ of height δ_y. We need techniques to argue that this cannot happen.

We proceed by assuming that the Γ-hod pair construction of $\mathcal{N}^*_y|\delta_y$ does not reach a pair generating Γ. Let $\mathcal{P}^*$ be the final model of the Γ-hod pair construction of $\mathcal{N}^*_y|\delta_y$. Let $\mathcal{P} = \mathcal{M}^+(\mathcal{P}^*)$ and let Σ be the strategy of $\mathcal{P}$ induced by Σ_y.

Claim 1. $o(\mathcal{P}^*) = \delta_y$.

Proof. Suppose not. It follows from Theorem 4.7.6 and Theorem 8.3.1 that the only way our construction could break down before reaching δ_y is if $\mathcal{P}^*$ is of lsa type. Let $\Lambda = \Sigma_{\mathcal{P}^*}$. We then have that $(\mathcal{P}^*, \Lambda)$ is a hod pair such that $\mathcal{P}^*$ is of lsa type and Λ has strong branch condensation and is strongly Γ-fullness preserving. Because $\Gamma^b(\mathcal{P}^*, \Lambda^{stc}) \subseteq \Gamma$ and $\Gamma^b(\mathcal{P}^*, \Lambda^{stc}) \neq \Gamma$, we can fix $(\mathcal{R}, \Phi) \in HP^{\Gamma}$ such that $\lambda^{\mathcal{R}}$ is limit and $(\mathcal{P}^*, \Lambda) \in L(\Gamma(\mathcal{R}, \Phi))$. We have that in $L(\Gamma(\mathcal{R}, \Phi))$, Λ has strong branch condensation and is strongly fullness preserving. It now follows from Theorem 8.1.13 applied in $L(\Gamma(\mathcal{R}, \Phi))$ that for some $\mathcal{S} \in pI(\mathcal{P}^*, \Lambda)$, $L(\Gamma(\mathcal{S}, \Lambda_{\mathcal{S}})) \vDash \mathsf{LSA}$, contradicting our assumption that $\neg\#_{lsa}$ holds. $\square$

Let $\kappa = \delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}-1}$. Notice that in $\mathcal{P}$, κ is $< \delta^{\mathcal{P}}$-strong.

Claim 2. $(\mathcal{P}^b, \Sigma_{\mathcal{P}^b}) \in B$.

Proof. Let $g \subseteq Coll(\omega, <\kappa)$ be $\mathcal{N}_y^*$-generic. We let $\psi(u, v)$ be as in the proof of Theorem 9.3.1. Following the notation used in the proof of Theorem 9.3.1, let $\mathcal{S} = \mathcal{P}_{\psi, A_2^g}$ and $\mathcal{S}^- = \mathcal{P}^-_{\psi, A_2^g}$. It follows from the proof of Theorem 9.3.1 that $\rho(\mathcal{S}) > o(\mathcal{S})$.

We claim that $\mathcal{S}$ is an iterate of $\mathcal{P}^b$. Clearly $\mathcal{M}_\infty(\mathcal{P}^b, \Sigma_{\mathcal{P}^b}) \trianglelefteq \mathcal{S}$. This is simply because for every $\mathcal{Q} \trianglelefteq_{hod} \mathcal{P}^b$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \in HP^\Gamma$. Suppose then that $\mathcal{M}_\infty(\mathcal{P}^b, \Sigma_{\mathcal{P}^b}) \triangleleft \mathcal{S}$. Let $(\mathcal{R}, \Lambda) \in HP^\Gamma \cap \mathcal{N}_y^*[g]$[1] be such that $\mathcal{M}_\infty(\mathcal{P}^b, \Sigma_{\mathcal{P}^b}) \triangleleft \mathcal{M}_\infty(\mathcal{R}, \Lambda)$. Let $\eta < \kappa$ be such that $(\mathcal{R}, \Lambda) \in \mathcal{N}_y^*[g \cap Coll(\omega, \eta)]$.

Let $\mathcal{Q}$ be the output of the hod pair construction of $\mathcal{P}$ in which extenders used have critical points $> \eta$. It follows from universality that for some $\alpha < \lambda^{\mathcal{Q}}$, $\mathcal{Q}(\alpha)$ is a Λ-iterate of $\mathcal{R}$. Let $E \in \vec{E}^{\mathcal{P}}$ be an extender with critical point κ such that $\nu_E > \alpha$. Let E^* be the resurrection of E. It follows that in $Ult(\mathcal{N}_y^*, E^*)$, some hod pair appearing on the hod pair construction of $\pi(\mathcal{P}^b)$ in which extenders used are bigger than η is a Λ-iterate of $\mathcal{R}$. It then follows that some hod pair appearing on the hod pair construction of $\mathcal{P}^b$ in which extenders used are bigger than η is a Λ-iterate of $\mathcal{R}$. It follows that $Code(\Lambda) <_w Code(\Sigma_{\mathcal{P}^b})$ implying that $\mathcal{M}_\infty(\mathcal{R}, \Lambda) \triangleleft \mathcal{M}_\infty(\mathcal{P}^b, \Sigma_{\mathcal{P}^b})$, contradiction. This contradiction proves the claim. $\square$

It is not hard to see, by a simple Skolem hull argument using the fact that $\mathcal{P} \in \mathcal{N}_y^*$, that

(2) for a club of $\eta < \delta_y$, $\mathcal{M}^+(\mathcal{P}|\eta) \vDash$ "η is a Woodin cardinal".

Let C be the club in (2). For $\eta \in C$, let $\mathcal{R}_\eta = \mathcal{M}^+(\mathcal{P}|\eta)$, $\Sigma_\eta = \Sigma^{stc}_{\mathcal{R}_\eta}$ and $\mathcal{Q}_\eta \trianglelefteq \mathcal{P}$ be the largest Σ_η-sts mouse such that $\mathcal{Q}_\eta \vDash$ "η is a Woodin cardinal". Using Lemma 6.4.6, we can translate $\mathcal{Q}_\eta$ onto Σ_η-sts mouse $\bar{\mathcal{Q}}_\eta$ over $\mathcal{M}^+(\mathcal{N}_y^*|\eta)$. Notice that

(3) for every η, $\mathcal{Q}_\eta$ has an iteration strategy Δ witnessing that $\mathcal{Q}_\eta$ is a Σ_η-sts mouse over $\mathcal{M}^+(\mathcal{N}_y^*|\eta)$.

(3) is a consequence of the fact that $\mathcal{Q}_\eta$ appears on a Γ-hod pair construction of $\mathcal{N}_y^*$. Moreover,

(4) for every η and for every real x coding $\mathcal{N}_y^*|\eta$, $\bar{\mathcal{Q}}_\eta$ is $OD^\Gamma_{x,X}$.

[1]Here we are abusing the notation and use Λ for both the strategy in $\mathcal{N}_y^*[g]$ as well as its extension in V.

(4) follows from proofs that have already appeared in the book. For instance, see the notion of goodness that appeared in the proof of Lemma 8.1.12. We now claim that

Claim 2. for a club of $\eta \in C - (\kappa+1)$, $\mathcal{Q}_\eta \in \mathcal{J}^{\Psi}_{\nu_\eta}(\mathcal{N}^*_y|\eta)$ where ν_η is the least ordinal such that $\mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta) \vDash \mathsf{ZFC}$.

Proof. Towards a contradiction, suppose not. Let λ be least such that $\mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y) \vDash \mathsf{ZFC}$. Let $\eta \in C$ be such that $\mathcal{Q}_\eta \notin \mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta)$ and there is a map $\pi : \mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta) \to \mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y)$. Thus

(5) $\mathcal{J}^{\Psi^*}_{\nu_\eta} \vDash$ "η is a Woodin cardinal".

Using genericity iterations, we can find $\mathcal{N} \in \mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta)$ such that $\mathcal{N}$ is a Ψ^*-iterate of $\mathcal{M}_3^{\#,\Psi}$ such that $\mathcal{M}^+(\mathcal{N}^*_y|\eta)$ is generic over the extender algebra $\mathbb{B}^{\mathcal{N}}_{\delta_0}$ where δ_0 is the least Woodin cardinal of $\mathcal{N}$. Let $g \subseteq Coll(\omega,\eta)$ be $\mathcal{N}^*_y$-generic. Fix a real $x \in \mathcal{N}[\mathcal{N}^*_y|\eta][g]$ coding $\mathcal{N}^*_y|\eta$. It follows that there is $y \in \mathbb{R}$ such that $(x,y) \in A_5 \cap \mathcal{N}[\mathcal{N}^*_y|\eta][g]$. Therefore $\mathcal{Q}_\eta \in \mathcal{N}[\mathcal{N}^*_y|\eta][g]$, implying that $\mathcal{Q}_\eta \in \mathcal{J}^{\Psi^*}_{\nu_\eta}$. It follows that $\mathcal{J}^{\Psi^*}_{\nu_\eta} \vDash$ "η is not a Woodin cardinal", contradicting (5). $\square$

The rest of the proof is easy. It follows from Claim 2 that we can find an η such that $\mathcal{Q}_\eta \in \mathcal{J}^{\Psi}_{\nu_\eta}(\mathcal{N}^*_y|\eta)$ and there is an elementary embedding $\pi : \mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta) \to \mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y)$ where λ is the least such that $\mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y) \vDash \mathsf{ZFC}$. Because $\mathcal{Q}_\eta \in \mathcal{J}^{\Psi}_{\nu_\eta}(\mathcal{N}^*_y|\eta)$, we have that $\mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta) \vDash$ "η is not a Woodin cardinal", and because $\pi : \mathcal{J}^{\Psi^*}_{\nu_\eta}(\mathcal{N}^*_y|\eta) \to \mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y)$, we have that $\mathcal{J}^{\Psi^*}_{\lambda}(\mathcal{N}^*_y|\delta_y) \vDash$ "δ_y is not a Woodin cardinal". This is an obvious contradiction! Thus, we must have that the Γ hod pair construction of $\mathcal{N}^*_y$ reaches a generator for Γ. We now move to case 2.

Case 2. Γ is a completely mouse full pointclass such that for some α, $L(\Gamma,\mathbb{R}) \vDash \theta_{\alpha+1} = \Theta$.

Because we are assuming $\neg\#_{lsa}$, we must have that $L(\Gamma,\mathbb{R}) \vDash \neg\mathsf{LSA}$. The rest of the proof is very much like the proof of [10, Theorem 6.1]. To complete it, we need to use Theorem 7.2.2 instead of [10, Theorem 4.24]. We leave the details to the reader. $\square$

Theorem 10.1.1 has one shortcoming. It cannot be used to compute HOD of the minimal model of LSA as it only generates pointclasses whose Wadge ordinal is strictly smaller than the largest Suslin cardinal. To compute HOD of the minimal model of LSA we will need the following theorem.

Theorem 10.1.2 *Assume $\mathsf{AD}^+ + \mathsf{LSA}$ and suppose $\neg\#_{lsa}$. Let α be such that $\theta_{\alpha+1} = \Theta$, and suppose that there is a hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ is strongly fullness preserving and has strong branch condensation and $\Gamma^b(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Then $(\mathcal{P}, \Sigma)$ is an sts hod pair and for any $B \in \mathbb{B}[\mathcal{P}, \Sigma]$ there is $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ such that $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ is strongly B-iterable.*

Proof. Towards a contradiction, assume not. We reflect the failure of our claim to $\utilde{\Delta}^2_1$. Let (β, γ) be lexicographically least such that letting $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \gamma\}$,

1. $\Gamma = \wp(\mathbb{R}) \cap \mathcal{J}_\beta(\Gamma, \mathbb{R})$ and $L_\beta(\Gamma, \mathbb{R}) \vDash \mathsf{LSA} + \mathsf{ZF}-$*Replacement*,
2. letting α be such that $L_\beta(\Gamma, \mathbb{R}) \vDash$ "$\theta_{\alpha+1} = \Theta$", $L_\beta(\Gamma, \mathbb{R}) \vDash$ "there is a hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ is strongly fullness preserving and has strong branch condensation and $\Gamma^b(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$ but either
   (a) $(\mathcal{P}, \Sigma)$ is not an sts hod pair or
   (b) there is a $B \in \mathbb{B}[\mathcal{P}, \Sigma]$ such that whenever $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ is not strongly B-iterable".

Because (β, γ) is minimized, we have that $\Gamma \subset \utilde{\Delta}^2_1$. Fix $(\mathcal{P}, \Sigma)$ as above. First we claim that

Claim. Σ is not an iteration strategy.

Proof. Suppose not. Let Γ_0 be a good pointclass beyond Γ and let F be as in Theorem 4.1.6 for Γ_0. Let $x \in dom(F)$ be such that letting $F(x) = (\mathcal{N}^*_x, \mathcal{M}_x, \delta_x, \Sigma_x)$, $(\mathcal{N}^*_x, \delta_x, \Sigma_x)$ Suslin, co-Suslin captures $Code(\Sigma)$ and Γ. It follows that $(\mathcal{J}^{\vec{E},\Sigma})^{\mathcal{N}^*_x|\delta_x}$ reaches $\mathcal{M}_2^{\#,\Sigma}$. Let Ψ be the iteration strategy of $\mathcal{M}_2^{\#,\Sigma}$. Notice that

(1) $\Psi \in L_\beta(\Gamma, \mathbb{R})$.

Because Σ is an iteration strategy, it follows from clause 1 of Theorem 6.1.5 that there are trees $(T, S) \in \mathcal{M}_2^{\#,\Sigma}$ such that letting $\delta_0 < \delta_1$ be the Woodin cardinals of $\mathcal{M}_1^{\#,\Sigma}$

1. $\mathcal{M}_2^{\#,\Sigma} \vDash$ "(T, S) are δ_1-complementing",

2. whenever $\pi : \mathcal{M}_2^{\#,\Sigma} \to \mathcal{N}$ is an iteration according to Ψ and $g \subseteq Coll(\omega, \pi(\delta_0))$ is $\mathcal{N}$-generic then $Code(\Sigma) \cap \mathbb{R}^{\mathcal{N}|\delta_1[g]} = p[\pi(T)]$ and $(Code(\Sigma))^c \cap \mathbb{R}^{\mathcal{N}|\delta_1[g]} = p[\pi(S)]$.

Let $\mathcal{M}_\infty$ be the direct limit of all Ψ-iterates of $\mathcal{M}_2^{\#,\Sigma}$ and let $\pi : \mathcal{M}_2^{\#,\Sigma} \to \mathcal{M}_\infty$ be the direct limit embedding. It then follows that $Code(\Sigma) = p[\pi(T)]$ and $(Code(\Sigma))^c = p[\pi[S]]$. It follows from (6) that $T, S \in L(\Gamma, \mathbb{R})$, implying that $L(\Gamma, \mathbb{R}) \vDash$ "$Code(\Sigma)$ is Suslin, co-Suslin". It follows that $Code(\Sigma) \in \Gamma(\mathcal{P}, \Sigma)$, contradiction! □

It follows from Claim 1 that $(\mathcal{P}, \Sigma)$ is an sts hod pair. Hence, we must have that

(2) there is $B \in \mathbb{B}[\mathcal{P}, \Sigma]$ such that whenever $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ is not strongly B-iterable.

We can now finish by appealing to Theorem 8.1.14.

□

10.2 A proof of the Mouse Set Conjecture below LSA

Throughout we will assume $\mathsf{AD}^{++} =_{\mathsf{def}} \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$. Let

$\#_{lsa}$: There is a pointclass $\Gamma \subset \wp(\mathbb{R})$ such that there is a Suslin cardinal bigger than $w(\Gamma)$ and $L(\Gamma, \mathbb{R}) \vDash \mathsf{LSA}$.

The following is the main theorem of this section.

Theorem 10.2.1 *Assume $\mathsf{AD}^{++} + \neg\#_{\mathsf{lsa}}$. Then the Strong Mouse Capturing holds.*

The rest of this section is devoted to the proof of Theorem 10.2.1. Recall that Strong Mouse Capturing (SMC) is the statement that for any hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that Σ has strong branch condensation and is strongly fullness preserving, and for any reals x, y, x is ordinal definable from Σ and y if and only if x is in some Σ-mouse over y. We assume familiarity with the proof of [10, Theorem 6.19] and build directly on it. We start by stating the main steps of [10, Theorem 6.19]. We will follow these steps and provide proofs only for the new cases.

Towards a contradiction assume that the Strong Mouse Capturing (SMC) is false. Our first step is to locate the minimal level of Wadge hierarchy over which SMC becomes false. For simplicity we assume that the Mouse Capturing, instead of the Strong Mouse Capturing, is false. Mouse Capturing is the same as SMC when the pair $(\mathcal{P}, \Sigma) = \emptyset$. The general case is only different in one aspect, it needs to be relativized to some strategy or a short tree strategy Σ. Let Γ be the least Wadge initial segment such that for some α

1. $\Gamma = \wp(\mathbb{R}) \cap L_\alpha(\Gamma, \mathbb{R})$,
2. $L_\alpha(\Gamma, \mathbb{R}) \vDash \mathsf{SMC}$,
3. there are reals x and y such that $L_{\alpha+1}(\Gamma, \mathbb{R}) \vDash$ "y is $OD(x)$" yet no x-mouse has y as a member.

For the purposes of this section we make the following definition.

Definition 10.2.2 *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and Γ^* is a pointclass. We say $(\mathcal{P}, \Sigma)$ is Γ^*-perfect if the following conditions are met.*

1. *Σ is Γ^*-super fullness preserving and has strong branch condensation.*
2. *For every $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$ such that $\lambda^{\mathcal{Q}}$ is a successor ordinal and $\mathcal{Q}$ is meek there is $\vec{B} = (B_i : i \leq \omega) \subseteq \mathbb{B}[\mathcal{Q}(\lambda^{\mathcal{Q}} - 1), \Sigma_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1)})$ such that $\vec{B}$ strongly guides $\Sigma_{\mathcal{Q}}$.*

If $\Gamma^* = \wp(\mathbb{R})$ then we omit Γ^* from our notation.

The following theorem was heavily used in [10]. It is essentially due to Steel and Woodin (see [23]).

Theorem 10.2.3 *Assume AD^+ and suppose $(\mathcal{P}, \Sigma)$ is a hod pair or an sts hod pair such that $L(\Sigma, \mathbb{R}) \vDash$ "$(\mathcal{P}, \Sigma)$ is perfect". Then $L(\Sigma, \mathbb{R}) \vDash \mathsf{MC}(\Sigma)$.*

A key theorem used in the proof of Theorem 10.2.1 is the following capturing theorem. Its precursor is stated as [10, Theorem 6.5].

Theorem 10.2.4 *Suppose $(\mathcal{P}, \Sigma)$ is a perfect hod pair and Γ_1 is a good pointclass such that $Code(\Sigma) \in \underset{\sim}{\Delta}_{\Gamma_1}$. Suppose F is as in Theorem 4.1.6 for Γ_1 and $z \in dom(F)$ is such that if $F(z) = (\mathcal{N}^*_z, \mathcal{M}_z, \delta_z, \Sigma_z)$ then $(\mathcal{N}^*_z, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures $Code(\Sigma)$. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}})^{\mathcal{N}^*_z|\delta_z}$. Then there is $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.*

The next key lemma that is used in the proof of Theorem 10.2.1 is the following generation lemma that can be traced to [10, Lemma 6.23].

Lemma 10.2.5 *There is a Γ-perfect pair $(\mathcal{P}, \Sigma)$ such that*

$$\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R}).$$

Our goal now is to outline how to use Theorem 10.2.4 and Lemma 10.2.5 to prove Theorem 10.2.1.

10.2.1 The structure of the proof of the Mouse Set Conjecture

We make the following convention. If $\mathcal{P}$ is a hod pair then $\mathcal{P}(-1) = \emptyset$. If Σ is a strategy for $\mathcal{P}$ then $\Sigma_{\mathcal{P}(-1)} = \emptyset$. First we outline the proof of the following general theorem.

Theorem 10.2.6 *Suppose $(\mathcal{P}, \Sigma)$ is a perfect pair. Then $L(\Sigma, \mathbb{R}) \vDash$ "for every $\beta \in [-1, \lambda^{\mathcal{P}})$, Mouse Capturing holds for $\Sigma_{\mathcal{P}(\beta)}$".*

Proof. We only outline the proof as the full proof is presented in [10, Section 6.4]. Fix $\beta < \lambda^{\mathcal{P}}$. We want to show that

(1) $L(\Sigma, \mathbb{R}) \vDash$ "Mouse Capturing holds for $\Sigma_{\mathcal{P}(\beta)}$".

For simplicity we assume $\beta = -1$. The general case is only notationally more complex. Suppose $x, y \in \mathbb{R}$ are such that $L(\Sigma, \mathbb{R}) \vDash$ "$y \in OD(x)$". It follows from Theorem 10.2.3 that there is a Σ-mouse $\mathcal{M}$ over $(\mathcal{P}, x)$ containing y such that $\mathcal{M}$ has an iteration strategy in $L(\Sigma, \mathbb{R})$. In fact, it follows from Theorem 10.2.3 that

(2) for every $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ there is a $\Sigma_{\mathcal{Q}}$-mouse $\mathcal{M}$ over $(\mathcal{Q}, x)$ such that $y \in \mathcal{M}$ and $\mathcal{M}$ has an iteration strategy in $L(\Sigma, \mathbb{R})$.

Let $\mathcal{M}_{\mathcal{Q}}$ be the least $\Sigma_{\mathcal{Q}}$-mouse over $(\mathcal{Q}, x)$ such that y is definable over $\mathcal{M}_{\mathcal{Q}}$. Let $\Lambda_{\mathcal{Q}}$ be the iteration strategy of $\mathcal{M}_{\mathcal{Q}}$ (witnessing that $\mathcal{M}_{\mathcal{Q}}$ is a $\Sigma_{\mathcal{Q}}$-mouse). Let $\Gamma^* \in L(\Sigma, \mathbb{R})$ be a good pointclass such that the set

$$A = \{(z, u) \in \mathbb{R}^2 : z \text{ codes some } \mathcal{M}_{\mathcal{Q}} \text{ and } u \text{ is an iteration according to } \Sigma_{\mathcal{Q}}\}$$

is in $\utilde{\Delta}_{\Gamma^*}$. Let F be as in Theorem 4.1.6 for Γ^* and let $z \in dom(F)$ be such that if $F(z) = (\mathcal{N}_z^*, \mathcal{M}_z, \delta_z, \Sigma_z)$ then $(\mathcal{N}_z^*, \delta_z, \Sigma_z)$ Suslin, co-Suslin captures Σ and the set A. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}}(x))^{\mathcal{N}_z^*|\delta_z}$. It follows from Theorem 10.2.4 that

(3) there is a $\mathcal{Q} \in \mathcal{N}$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.

It follows from the universality of $\mathcal{N}$ that $\mathcal{M}_{\mathcal{Q}} \in \mathcal{N}$ (this is because $(\mathcal{J}^{\vec{E},\Sigma_{\mathcal{Q}}})^{\mathcal{N}}$ is universal in $\mathcal{N}_z^*$ and the strategy of $\mathcal{M}_{\mathcal{Q}}$ is captured by $\mathcal{N}_z^*$). It then follows that $y \in \mathcal{N}$. As $\mathcal{N}$ is an x-mouse, this completes the proof. □

Suppose now that $(\mathcal{P}, \Sigma)$ is a Γ-perfect pair such that $\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R})$. Such a pair is given to us by Lemma 10.2.5.

We now apply Theorem 10.2.3. For each $\mathcal{Q} \in pI(\mathcal{P}, \Sigma)$ there is a $\Sigma_{\mathcal{Q}}$-mouse $\mathcal{M}_{\mathcal{Q}}$ over $(\mathcal{Q}, x)$ such that y is definable over $\mathcal{M}_{\mathcal{Q}}$. We then again can find an x-mouse $\mathcal{N}$ such that for some $\mathcal{Q} \in \mathcal{N} \cap pI(\mathcal{P}, \Sigma)$, $\mathcal{M}_{\mathcal{Q}} \in \mathcal{N}$. It follows that $y \in \mathcal{N}$. Thus, to finish the proof of Theorem 10.2.1, it is enough to establish Theorem 10.2.4 and Lemma 10.2.5.

10.2.2 Review of basic notions

In this subsection we review basic notions introduced in [10, Theorem 6.5] for proving a version of Theorem 10.2.4. We are in fact working towards the proof of Theorem 10.2.4, and the notation and the terminology of this subsection will be used in the later subsections.

Fix $(\mathcal{P}, \Sigma)$, Γ_1, F and z as in the statement of Theorem 10.2.4. Let $\mathcal{N} = (\mathcal{J}^{\vec{E}})^{\mathcal{N}_z^*}$. We are looking for $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{Q}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$. We start working in $\mathcal{N}_z^*$. Without loss of generality we can assume that

(1) whenever $\mathcal{R} \in pB(\mathcal{P}, \Sigma) \cap (\mathcal{N}_z^* | \delta_z)$ there is $\mathcal{S} \in pI(\mathcal{R}, \Sigma_{\mathcal{R}}) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{S}} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$.

As in [10], there are several cases.

1. $\lambda^{\mathcal{P}}$ is a successor and $\mathcal{P}$ is meek.
2. $\lambda^{\mathcal{P}}$ is a limit.
3. $\lambda^{\mathcal{P}}$ is a successor, $\mathcal{P}$ is non-meek but $\mathcal{P}$ is not of lsa type.
4. $(\mathcal{P}, \Sigma)$ is an sts hod pair.

The first two cases are just like the cases considered in [10, Theorem 6.5], we leave those to the reader. Here we analyze the remaining two cases. To start, we need to import some notions from [10, Section 6.3].

Suppose for a moment that we are working in some model of ZFC. Suppose κ is an inaccessible cardinal. We say that $(\mathcal{Q},\Lambda)$ is a *hod pair at κ* if

1. $(\mathcal{Q},\Lambda)$ is a hod pair,
2. $\mathcal{Q}\in HC$,
3. Λ is a (κ,κ)-iteration strategy,
4. $Code(\Lambda)$ is a κ-universally Baire set of reals.

Suppose $(\mathcal{Q},\Lambda)$ is a hod pair at κ. Then we let

$$Lp^{\Lambda,\kappa}(a)=\bigcup\{\mathcal{M}:\mathcal{M}\text{ is a sound }\Lambda\text{-mouse over }a\text{ such that }\rho_\omega(\mathcal{M})=a\text{ and }\mathcal{M}\trianglelefteq(\mathcal{J}^{\vec{E},\Lambda}(a))^{V_\kappa}\}.$$

As is customary, we let $Lp^{\Lambda,\kappa}_\alpha(a)$ be the αth iterate of $Lp^{\Lambda,\kappa}(a)$. Below $\mathcal{S}^*(\mathcal{R})$ is the $*$-transform of $\mathcal{S}$ into a hybrid mouse over $\mathcal{R}$, it is defined when $\mathcal{R}$ is a cutpoint of $\mathcal{S}$ (cf. [18]).

Definition 10.2.7 (Fullness preservation in models of ZFC) *Suppose now that $(\mathcal{Q},\Lambda)$ is a hod pair at κ. We then say Λ is κ-fullness preserving if whenever $(\vec{\mathcal{T}},\mathcal{R})\in I(\mathcal{Q},\Lambda)\cap V_\kappa$,*

1. *For all limit type $\mathcal{S}\in Y^{\mathcal{S}}$, $\mathcal{S}^b=Lp_\omega^{\oplus_{\mathcal{W}\in Y^{\mathcal{S}^b}}\Lambda_{\mathcal{W},\vec{\mathcal{T}},\kappa}}(\mathcal{S}|\delta^{\mathcal{S}})$.*
2. *For all successor type $\mathcal{S}\in Y^{\mathcal{R}}$,*
$$\mathcal{S}=Lp_\omega^{\oplus_{\mathcal{W}\in Y^{\mathcal{S}^b}}\Lambda_{\mathcal{W},\vec{\mathcal{T}},\kappa}}(\mathcal{S}|\delta^{\mathcal{S}}).$$
3. *If $\mathcal{R}$ is of lsa type then $\mathcal{R}=Lp_\omega^{\Lambda^{stc}_{\mathcal{M}^+(\mathcal{R}|\delta^{\mathcal{R}}),\vec{\mathcal{T}},\kappa}}(\mathcal{R}|\delta^{\mathcal{R}})$[2],*
4. *If η is a cardinal cutpoint of $\mathcal{R}$ such that for some $\mathcal{R}_1,\mathcal{R}_2\in Y^{\mathcal{R}}$ such that $\mathcal{R}_2$ is the $\mathcal{R}$-successor of $\mathcal{R}_1$ (see Definition 3.9.2), $\mathcal{R}_1$ is a cutpoint of $\mathcal{R}$ and $\eta\in(\delta^{\mathcal{R}_1},\delta^{\mathcal{R}_2})$ then*
$$(\mathcal{R}|(\eta^+)^{\mathcal{R}})^*=Lp^{\Lambda_{\mathcal{R}_1,\vec{\mathcal{T}},\kappa}}(\mathcal{R}|\eta).$$

[2]Here, if Λ is a short tree strategy then $\Lambda^{sts}=\Lambda$.

Continuing our work inside some model of ZFC, suppose $(\mathcal{Q}, \Lambda)$ is a hod pair at κ such that Λ has branch condensation and is κ-fullness preserving. Suppose $\lambda < \kappa$ is an inaccessible cardinal. Then we say

Definition 10.2.8 (Universal tail) *$(\mathcal{Q}^*, \Lambda^*)$ is a λ-universal tail of $(\mathcal{Q}, \Lambda)$ if there is a stack $\vec{\mathcal{T}}$ according to Λ on $\mathcal{Q}$ with last model $\mathcal{Q}^*$ such that*

1. *$lh(\vec{\mathcal{T}}) = \lambda$ and for all $\beta < lh(\vec{\mathcal{T}})$, $\vec{\mathcal{T}} \restriction \beta \in V_\lambda$;*
2. *for any $(\vec{\mathcal{S}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda) \cap V_\lambda$ there is a stack $\vec{\mathcal{U}}$ on $\mathcal{R}$ according to $\Lambda_{\mathcal{R}}$ with its last model on the main branch of $\vec{\mathcal{T}}$.*

If $\vec{\mathcal{T}}$ is as above then we say $\vec{\mathcal{T}}$ is a λ-universal stack on $\mathcal{Q}$ according to Λ.

We now resume the proof of Theorem 10.2.4 and start working in $\mathcal{N}_z^*$. Observe that because of our assumption on $(\mathcal{P}, \Sigma)$, whenever $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma)$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{R}, \Sigma_{\mathcal{R}})$ have a common tail in $\mathcal{N}_z^*|\delta_z$. In fact more is true. Suppose κ is a strong cardinal of $\mathcal{N}_z^*$. Then it follows from Corollary 4.6.10 that if $\mathcal{Q}, \mathcal{R} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}_z^*|\kappa$ then $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ and $(\mathcal{R}, \Sigma_{\mathcal{R}})$ have a common tail in $\mathcal{N}_z^*|\kappa$. This means that whenever $\kappa < \delta_z$ is a cardinal of $\mathcal{N}_z^*$ and $\mathcal{Q} \in (pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)) \cap \mathcal{N}_z^*|\kappa$, we can form the direct limit of all $\Sigma_{\mathcal{Q}}$ iterates of $\mathcal{Q}$ that are in $\mathcal{N}_z^*|\kappa$. Let $\mathcal{R}_\kappa^{\mathcal{Q}, \Sigma_{\mathcal{Q}}}$ be this direct limit. The next lemma shows that the universal tails are unique. It appeared as [10, Lemma 6.8].

Lemma 10.2.9 (Uniqueness of universal tails) *Suppose $\mathcal{Q} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}_z^*|\delta_z$. Then for each $\mathcal{N}$-strong $\kappa < \delta_z$ such that $\mathcal{Q} \in \mathcal{N}_z^*|\kappa$ and $\alpha \leq \lambda^{\mathcal{Q}}$, there is a unique κ-universal tail of $(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)})$. In fact, letting $\mathcal{R} = \mathcal{R}_\kappa^{\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)}}$, $(\mathcal{R}, \Sigma_{\mathcal{R}})$ is the unique κ-universal tail of $(\mathcal{Q}(\alpha), \Sigma_{\mathcal{Q}(\alpha)})$*

Suppose $\mathcal{Q} \in (pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)) \cap \mathcal{N}_z^*|\delta_z$ and κ is an $\mathcal{N}$-strong cardinal such that $\mathcal{Q} \in \mathcal{N}_z^*|\kappa$.

Definition 10.2.10 *Then we say $\mathcal{N}$ captures a tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ below κ if there is a hod pair $(\mathcal{R}, \Lambda) \in \mathcal{N}$ such that Λ is a (κ, κ)-iteration strategy and there is a term relation $\tau \in \mathcal{N}^{Coll(\omega, <\kappa)}$ such that whenever $g \subseteq Coll(\omega, |\mathcal{R}|^+)$ is $\mathcal{N}$-generic,*

1. *$\mathcal{N}[g] \vDash$ "$(\mathcal{R}, \tau_g)$ is a hod pair at κ such that τ_g is κ-fullness preserving" and $\tau_g \restriction \mathcal{N} = \Lambda$,*
2. *for some $\lambda < \kappa$, $\mathcal{R} = \mathcal{R}_\lambda^{\mathcal{Q}}$ and letting $T, U \in \mathcal{N}[g]$ witness that τ_g is κ-uB, whenever $h \subseteq Coll(\omega, < \kappa)$ is $\mathcal{N}[g]$-generic, $(p[T])^{\mathcal{N}[g][h]} = Code(\Sigma_{\mathcal{R}}) \cap \mathcal{N}[g][h]$.*

We say $\mathcal{N}$ captures $B(\mathcal{Q}, \Sigma_{\mathcal{Q}})$ below κ if whenever $\mathcal{R} \in pB(\mathcal{Q}, \Sigma_{\mathcal{Q}}) \cap \mathcal{N}_z^*|\kappa$, $\mathcal{N}$ captures $(\mathcal{R}, \Sigma_{\mathcal{R}})$ below κ.

Towards a contradiction, we assume that $\mathcal{N}$ does not capture a tail of $(\mathcal{P}, \Sigma)$ and that either

1. $\lambda^{\mathcal{P}}$ is a successor, $\mathcal{P}$ is not of lsa type and $\mathcal{P}$ is non-meek or
2. $(\mathcal{P}, \Sigma)$ is an sts hod pair.

Notation 10.2.11 *For each $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, we let $\lambda_{\mathcal{Q}}$ be the least $\mathcal{N}$-strong cardinal ν such that $\mathcal{N}$ captures the ν-universal tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. We let $(\mathcal{R}^{\mathcal{Q}}, \Psi^{\mathcal{Q}})$ be the $\lambda_{\mathcal{Q}}$-universal tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$. For each inaccessible cardinal ν such that $\mathcal{Q} \in \mathcal{N}|\nu$, we let $(\mathcal{R}^{\mathcal{Q}}_{\nu}, \Psi^{\mathcal{Q}}_{\nu})$ be the ν-universal tail of $(\mathcal{Q}, \Sigma_{\mathcal{Q}})$.*

10.2.3 The ideas behind the proof

The notation and the terminology introduced in this subsection will be used in the next few subsections. Suppose now κ is an $\mathcal{N}$-strong cardinal that reflects the set of $\mathcal{N}$-strong cardinals. Let

$$\mathcal{E} = \{E \in \vec{E}^{\mathcal{N}} : \mathcal{N} \vDash \text{“}\nu(E) \text{ is inaccessible” and for all } \eta \in (\kappa, \nu(E)), \mathcal{N} \vDash \text{“}\eta \text{ is a strong cardinal” if and only if } Ult(\mathcal{N}, E) \vDash \text{“}\eta \text{ is a strong cardinal”}\}.$$

Notation 10.2.12 *Working in $\mathcal{N}$, let*

$$\mathcal{F} = \{(\mathcal{Q}, \Lambda) : \mathcal{Q} \in \mathcal{N}|\delta \wedge \mathcal{J}[\mathcal{N}] \vDash \text{“}(\mathcal{Q}, \Lambda) \textit{ is a hod pair at } \delta_z \textit{ and } \Lambda \textit{ has branch condensation and is } \delta_z\textit{-fullness preserving”}\}.$$

We have that $\mathcal{F}$ is a directed system. Let for $\lambda \leq \delta_z$,

$$\mathcal{F} \restriction \lambda = \{(\mathcal{Q}, \Lambda) \in \mathcal{F} : \mathcal{Q} \in \mathcal{N}|\lambda\}.$$

We let $\mathcal{R}^*$ be the direct limit of $\mathcal{F} \restriction \kappa$ under the iteration maps. Let

$$\mathcal{R} = (\mathcal{R}^{\mathcal{P}}_{\kappa})^b.$$

The next lemma summarizes what was proved in [10].

Lemma 10.2.13 *The following holds.*

1. *Suppose $\mathcal{Q} \in pB(\mathcal{P}, \Sigma) \cap \mathcal{N}_x^*|\kappa$. Then $\lambda_{\mathcal{Q}} < \kappa$. Thus, $\mathcal{R}^{\mathcal{Q}} \in \mathcal{N}|\kappa$.*

2. *Suppose* $\mathcal{Q} \in pB(\mathcal{P}, \Sigma)$, $\lambda > \kappa$ *is a strong cardinal such that* $\lambda_{\mathcal{Q}} < \lambda$, *and* $E \in \mathcal{E}$ *is an extender with critical point* κ *such that* $\nu(E) > (\lambda^+)^{\mathcal{N}^*_x}$. *Then* $\Psi^{\mathcal{Q}} \restriction (Ult(\mathcal{N}, E)|\delta) \in Ult(\mathcal{N}, E)$.

3. *Either* $\mathcal{R} \trianglelefteq_{hod} \mathcal{R}^*$ *or* $\mathcal{R}|\delta^{\mathcal{R}} = \mathcal{R}^*$. *Moreover,* $\mathcal{R} \in \mathcal{N}$ *and* $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in L[\mathcal{N}]$.

Clause 1 is just [10, Lemma 6.11], clause 2 is [10, Lemma 6.12] and clause 3 is [10, Lemma 6.13].

In the sequel, we will develop a technology for recovering a full iterate of $\mathcal{P}$. Let $\mathcal{R}^+ = \mathcal{R}^{\mathcal{P}}_\kappa$ be the iterate of $\mathcal{P}$ extending $\mathcal{R}$ and let $i : \mathcal{P} \to \mathcal{R}^+$ be the iteration embedding. We will recover an iterate of $\mathcal{R}^+$ inside $\mathcal{N}$ as an output of a backgrounded construction that is done over $\mathcal{R}$. Such constructions are called mixed hod pair constructions. The details of this construction appear in Section 10.2.9.

There are two kind of extenders that we will use in this construction. The extenders with critical point $> \delta^{\mathcal{R}}$ will have traditional background certificates. We will use the total extenders on the sequence of $\mathcal{N}$ to certify such extenders. The extenders with critical point $\delta^{\mathcal{R}}$ will come from a different source. The following key lemma illustrates the idea. Let $\delta = \delta^{\mathcal{R}}$.

Lemma 10.2.14 *Suppose* $\mathcal{S} \in pI(\mathcal{R}^+, \Sigma_{\mathcal{R}^+})$ *is a normal iterate of* $\mathcal{R}^+$ *that is obtained by iterating entirely above* $\delta^{\mathcal{R}}$. *Suppose that* $E_\alpha \in \vec{E}^{\mathcal{S}}$ *is such that* $\mathrm{crit}(E_\alpha) = \delta^{\mathcal{R}^b}$, $\mathcal{S}|\alpha \in \mathcal{N}$ *and* $\Sigma_{\mathcal{S}|\alpha} \restriction \mathcal{N} \in L[\mathcal{N}]$. *Then* $E_\alpha \in \mathcal{N}$. *Moreover,* $(a, A) \in E_\alpha$ *if and only if* $a \in \nu^{<\omega}_{E_\alpha}$, $A \in [\delta]^{|a|}$ *and whenever* $F \in \mathcal{E}$ *is such that* $\mathrm{crit}(F) = \kappa$ *and*

$$\mathcal{N} \vDash \text{“}\textit{there is a strong cardinal } \nu \textit{ in the interval } (\kappa, lh(F)) \textit{ such that } \mathcal{S} \in \mathcal{N}|\nu\text{”},$$

$\pi^{\Sigma_{\mathcal{S}|\alpha}}_{\mathcal{S}|\alpha, \pi_F(\mathcal{R})}(a) \in \pi_F(A)$.

Proof. Set $\mathcal{M}^+ = Ult(\mathcal{R}^+, E_\alpha)$ and $\mathcal{M} = Ult(\mathcal{R}, E_\alpha)$. Let F^* be the resurrection of F and let $\sigma : Ult(\mathcal{N}, F) \to Ult(\mathcal{N}^*_z, F^*)$ be the canonical factor map. We have that $\sigma \restriction \nu_F = id$. Thus, $\pi_{F^*} \restriction \mathcal{N} = \sigma \circ \pi_F$. It follows that $\pi_{F^*} \restriction \mathcal{R}^+$ is the iteration embedding implying

(1) $\pi_{F^*} \restriction \mathcal{R}^+ = \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}^+)} \circ \pi^{\mathcal{R}^+}_{E_\alpha}$.

We now have that

$$\begin{aligned}(a, A) \in E_\alpha &\leftrightarrow a \in \pi^{\mathcal{R}^+}_{E_\alpha}(A)\\ &\leftrightarrow \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}^+)}(a) \in \pi^{\Sigma_{\mathcal{M}^+}}_{\mathcal{M}^+, \pi_{F^*}(\mathcal{R}^+)}(\pi^{\mathcal{R}^+}_{E_\alpha}(A))\\ &\leftrightarrow \sigma(\pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R})}(a)) \in \pi_{F^*}(A)\\ &\leftrightarrow \sigma(\pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R})}(a)) \in \sigma(\pi_F(A))\\ &\leftrightarrow \pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R})}(a) \in \pi_F(A)\end{aligned}$$

Therefore,

$$(a, A) \in E_\alpha \leftrightarrow \pi^{\Sigma_{\mathcal{M}}}_{\mathcal{M}, \pi_F(\mathcal{R})}(a) \in \pi_F(A).$$

By our assumption, the right hand side of the equivalence can be computed in $\mathcal{N}$. Hence $E_\alpha \in \mathcal{N}$. □

Thus, the extenders with critical point $\delta^{\mathcal{R}}$ that we will use in our mixed hod pair construction have the following property. If $\mathcal{Q}$ is the current level of the construction and Λ is its strategy then let E be the set of pairs (a, A) such that $(a \in \delta^{\mathcal{R}})^{<\omega}$ and for every $F \in \mathcal{E}$ such that $\mathrm{crit}(F) = \kappa$ and

$$\mathcal{N} \vDash \text{"there is a strong cardinal } \nu \text{ in the interval } (\kappa, lh(F)) \text{ such that } \mathcal{Q} \in \mathcal{N}|\nu\text{"},$$

$\pi^{\Lambda}_{\mathcal{Q}, \pi_F(\mathcal{R})}(a) \in \pi_F(A)$.

There is one problem with this approach. We need to know the strategy Λ of $\mathcal{Q}$ before we can find the relevant extender. To resolve this problem, we will first define the strategy Λ. Essentially Λ will pick branches that, for some η, are π_E-realizable for all $E \in \mathcal{E}$ such that $lh(E) > \eta$. We will call such strategies $\mathcal{E}$-certified.

In the sequel, we will first introduce the $\mathcal{E}$-certified strategies. Then we will prove basic fact about them. Then we will introduce the mixed hod pair constructions and show that some model appearing on this construction is an iterate of $\mathcal{R}^+$ via an iteration that is entirely above $\delta^{\mathcal{R}}$.

10.2.4 $\mathcal{E}$-certified iteration strategies

The following is a modification of [10, Definition 6.14].

Definition 10.2.15 (π_E-realizable iterations) *Suppose*

1. $\mathcal{M} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{M}^b$,

2. $\vec{\mathcal{T}} \in \mathcal{N}$ is a stack on $\mathcal{M}$ played either according to the usual rules of the iteration game or (in the case $\mathcal{M}$ is of lsa type) according to the rules of the short tree game,

3. $E \in \mathcal{E}$.

Set $B_{\vec{\mathcal{T}}} = \{\mathcal{Q} \in tn(\vec{\mathcal{T}}) : \mathcal{Q} \in tn(\vec{\mathcal{T}})$ *and* $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{Q}},b}$ *exists*$\}$.

We say $\vec{\mathcal{T}}$ *is* π_E*-realizable if there is a strong cardinal* $\lambda < \nu(E)$ *such that* $\vec{\mathcal{T}} \in \mathcal{N}|\lambda$*, a sequence* $(\sigma_{\mathcal{Q}} : \mathcal{Q} \in tn(\vec{\mathcal{T}})) \in \mathcal{N}|\lambda$ *and a sequence* $((\mathcal{W}_{\mathcal{Q}}, \Psi_{\mathcal{Q}}) \in \mathcal{F} \restriction \lambda : \mathcal{Q} \in B_{\vec{\mathcal{T}}}) \in \mathcal{J}[\mathcal{N}]$ *such that the following holds:*

1. $\sigma_{\mathcal{R}} = \pi_E \restriction \mathcal{R}$.

2. For all terminal nodes $\mathcal{Q}$ of $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}}_{\leq \mathcal{Q}},b}$ exists, $\sigma_{\mathcal{Q}} : \mathcal{Q}^b \to \pi_E(\mathcal{R})$.

3. For all $\mathcal{Q}, \mathcal{S} \in B_{\vec{\mathcal{T}}}$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{S}$, $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{S}} \circ \pi^{\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{S}},b}_{\mathcal{Q},\mathcal{S}}$.

4. For all $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, letting $\mathcal{S}_{\mathcal{Q}} \trianglelefteq \pi_E(\mathcal{R})$ be the $\Psi_{\mathcal{Q}}$-iterate of $\mathcal{W}_{\mathcal{Q}}$, $\delta^{\mathcal{S}_{\mathcal{Q}}} = \sigma_{\mathcal{Q}}(\delta^{\mathcal{Q}^b})$ and $\sigma_{\mathcal{Q}}[\mathcal{Q}^b] \subseteq rng(\pi^{\Psi_{\mathcal{Q}}}_{\mathcal{W}_{\mathcal{Q}},\mathcal{S}_{\mathcal{Q}}})$.

5. For all $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, letting $k_{\mathcal{Q}} : \mathcal{Q}^b \to \mathcal{W}_{\mathcal{Q}}$ be given by $k_{\mathcal{Q}}(x) = y$ if and only if $\sigma_{\mathcal{Q}}(x) = \pi^{\Psi_{\mathcal{Q}}}_{\mathcal{W}_{\mathcal{Q}},\mathcal{S}_{\mathcal{Q}}}(y)$, $k_{\mathcal{Q}}\vec{\mathcal{T}}_{\mathcal{Q}}$ is according to $\Psi_{\mathcal{Q}}$.

6. For all $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, $(\mathcal{Q}^b, \Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}}) \in \pi_E(\mathcal{F})$ and $\sigma_{\mathcal{Q}} \restriction (\mathcal{Q}^b|\delta^{\mathcal{Q}^b})$ is the iteration embedding according to $\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}}$.

7. For all $\mathcal{Q}, \mathcal{K} \in B_{\vec{\mathcal{T}}}$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{K}$, letting β be such that $\mathcal{K}(\beta) = \pi^{\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{K}},b}_{\mathcal{Q},\mathcal{K}}(\mathcal{Q}^b)$,
$$(\Psi^{k_{\mathcal{K}}}_{\mathcal{K}})_{\mathcal{K}(\beta)} = (\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}})_{\mathcal{K}(\beta)}$$
and $\sigma_{\mathcal{K}} \restriction (\mathcal{K}(\beta)|\delta^{\mathcal{K}(\beta)})$ is the iteration embedding according to $(\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}})_{\mathcal{K}(\beta)}$.

8. Suppose there is a main drop at $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$. Then $\vec{\mathcal{T}}_{\geq \mathcal{Q}}$ is according to $\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}}$.

We say that $(\sigma_{\mathcal{Q}} : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ *are the* π_E*-realizable embeddings of* $\vec{\mathcal{T}}$ *and* $((\mathcal{W}_{\mathcal{Q}}, \Psi_{\mathcal{Q}}) : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ *are the* π_E*-realizable pairs of* $\vec{\mathcal{T}}$*. We say* $\vec{\mathcal{T}}$ *is* $\mathcal{E}$*-realizable if for some* η*,* $\vec{\mathcal{T}}$ *is* E*-realizable for every* $E \in \mathcal{E}$ *with the property that* $lh(E) > \eta$*.*

We now introduce a kind of backgrounded constructions reminiscent to the backgrounded construction introduced in Definition 4.2.1. We will use them to find the $\mathcal{Q}$-structures of various iterations.

Definition 10.2.16 ($\mathcal{E}$-realizable backgrounded constructions) *Suppose $\mathcal{M}, \vec{\mathcal{T}}, \mathcal{S}, \mathcal{Q}, \eta$ are such that*

1. *$\mathcal{M} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{M}^b = \mathcal{R}$,*
2. *$\vec{\mathcal{T}}$ is a $\mathcal{E}$-realizable stack on $\mathcal{M}$ (played either according to the usual rules of the iteration game or according to the rules of the short tree game) such that $\pi^{\vec{\mathcal{T}},b}$ exists,*
3. *$\mathcal{S} \in ntn(\vec{\mathcal{T}})$ and $\mathcal{U}$ is the largest normal initial segment of $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ that is based on $\mathcal{S}$ and is above $\delta^{\mathcal{S}^b}$,*
4. *if $\mathcal{U}$ is of limit length then $\mathcal{Q} = \mathcal{M}(\mathcal{U})$ and otherwise for some $\alpha < lh(\mathcal{U})$, $\mathcal{Q} = \mathcal{M}_\alpha$,*
5. *$\mathcal{M}^+(\mathcal{Q}|\eta) \vDash$ "η is a Woodin cardinal".*

Then for $\xi \leq \delta$, $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \xi), (F_\gamma : \gamma < \xi)) \in \mathcal{J}[\mathcal{N}]$ is the ξth initial segment of the output of the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{Q}|\eta)$ done in $\mathcal{N}$ if the following is true.

1. *$\mathcal{M}_0 = \mathcal{J}_1(X)$, and for all $\xi < \eta$, $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ are sts premice such that if $\vec{\mathcal{W}}$ is a stack indexed either in $\mathcal{M}_\xi$ or $\mathcal{N}_\xi$ then $\vec{\mathcal{T}}_{\leq \mathcal{Q}} {}^\frown \vec{\mathcal{W}}$ is $\mathcal{E}$-realized.*
2. *Suppose $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \beta), (F_\gamma : \gamma < \beta))$ has been defined for $\beta < \xi$. Then we define $\mathcal{M}_{\beta+1}$, $\mathcal{N}_{\beta+1}$ and F_β as follows.*
   (a) *Suppose $\mathcal{M}_\beta = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, i.e., with no last predicate, and there is a total extender $F^* \in \vec{E}^{\mathcal{N}}$ such that F^* coheres $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma \leq \beta), (F_\gamma : \gamma < \beta))$, an extender F over $\mathcal{M}_\beta$, and an ordinal $\nu < \alpha$ such that $\mathcal{N}|\nu + \omega \subseteq Ult(\mathcal{N}, F^*)$ and*

   $$F \upharpoonright \nu = F^* \cap ([\nu]^\omega \times \mathcal{J}_\alpha^{\vec{E},f}).$$

   Then

   $$\mathcal{N}_{\beta+1} = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f, \tilde{F})$$

and $\nu = \nu^{\mathcal{N}_{\beta+1}}$ where $\tilde{F}$ is the amenable code of F[3]*. Also, if $\mathcal{N}_{\beta+1}$ is reliable then $\mathcal{M}_{\beta+1} = \mathcal{C}(\mathcal{N}_{\beta+1})$*[4] *and $F_\beta = F$. If $\mathcal{N}_{\beta+1}$ is not reliable then we stop the construction.*

(b) *Suppose $\mathcal{M}_\beta = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, the hypothesis of item 2.a above doesn't hold, $\mathcal{M}_\beta \vDash$ ZFC-Replacement, and $\mathcal{M}_\beta$ is ambiguous. Let $t = (\mathcal{Q}, \mathcal{T}, \mathcal{Q}_1, \vec{\mathcal{U}}) \in \mathcal{J}_\alpha^{\vec{E},f} \cap dom(\Lambda)$ on $\mathcal{Q}$ be the $\mathcal{M}_\beta$-least stack of length 2 witnessing that $\mathcal{M}_\beta$ is ambiguous and such that $lh(\mathcal{T})$ is not of measurable cofinality in $\mathcal{M}_\beta$ and $lh(\vec{\mathcal{U}})$ is not of measurable cofinality in $\mathcal{M}_\beta$. Suppose there is a branch $b \in \mathcal{N}$ of $\vec{\mathcal{U}}$ such that $\vec{\mathcal{T}}_{\leq \mathcal{Q}} {}^\frown t {}^\frown \{\mathcal{M}_b^{\vec{\mathcal{U}}}\}$ is $\mathcal{E}$-realizable. Let b be the $\mathcal{N}$-least such branch*[5] *and set $\nu = \sup b$ and $\mathcal{W} = \mathcal{J}_\nu(\mathcal{M}_\beta)$. If $\rho(\mathcal{W}) \geq \alpha$ then*

$$\mathcal{N}_\beta = (\mathcal{J}_\beta^{\vec{E},f^+}, \in, \vec{E}, f^+)$$

where $f^+ = f \cup (\mathcal{J}_\omega(t), \tilde{b})$ where $\tilde{b} \subseteq \alpha+\nu$ is defined by $\alpha+\nu^* \in \tilde{b} \leftrightarrow \nu^* \in b$. If $\rho(\mathcal{W}) < \alpha$ then let $\gamma \in (\alpha, \nu]$ be least such that $\rho(\mathcal{J}_\gamma(\mathcal{M}_\beta)) < \alpha$ and let $\mathcal{N}_{\beta+1} = \mathcal{C}(\mathcal{J}_\gamma(\mathcal{M}_\beta))$. Also, if $\mathcal{N}_{\beta+1}$ is reliable then $\mathcal{M}_{\beta+1} = \mathcal{C}(\mathcal{N}_{\beta+1})$ and $F_\beta = \emptyset$. If $\mathcal{N}_{\beta+1}$ is not reliable then we stop the construction. If there is no such branch b then stop the construction.

(c) *Suppose $\mathcal{M}_\beta = (\mathcal{J}_\alpha^{\vec{E},f}, \in, \vec{E}, f)$ is a passive hp, the hypothesis of item 2.a and 2.b above don't hold, $\mathcal{M}_\beta \vDash$ ZFC, $\mathcal{M}_\beta$ is unambiguous and there is a normal terminal $\mathcal{T} \in \mathcal{J}_\alpha^{\vec{E},f} \cap dom(\Lambda)$ such that $\mathcal{M}_\beta \vDash$ "$\mathcal{T}$ is ambiguous and $lh(\mathcal{T})$ is not of measurable cofinality", $f^{\mathcal{M}_\beta}(\mathcal{T})$ isn't defined and there is an $\mathcal{M}_\beta$-minimal shortness witness for $\mathcal{T}$. Let $\mathcal{U}$ be the $\mathcal{M}_\beta$-least such tree, (ϕ, ζ, b) be a shortness witness for $\mathcal{U}$, $\nu = \sup b$ and $\mathcal{W} = \mathcal{J}_\nu(\mathcal{M}_\beta)$. If $\rho(\mathcal{W}) \geq \alpha$ then*

$$\mathcal{N}_\beta = (\mathcal{J}_\nu^{\vec{E},f^+}, \in, \vec{E}, f^+)$$

where $f^+ = f \cup \{(\mathcal{J}_\omega(\mathcal{U}), \tilde{b})\}$ where $\tilde{b} \subseteq \alpha + \nu$ is defined by $\alpha + \nu^* \in \tilde{b} \leftrightarrow \nu^* \in b$. If $\rho(\mathcal{W}) < \alpha$ then let $\gamma \in (\alpha, \nu]$ be least such that $\rho(\mathcal{J}_\gamma(\mathcal{M}_\beta)) < \alpha$ and let $\mathcal{N}_{\beta+1} = \mathcal{C}(\mathcal{J}_\gamma(\mathcal{M}_\beta))$. Also, if $\mathcal{N}_{\beta+1}$ is reliable then $\mathcal{M}_{\beta+1} = \mathcal{C}(\mathcal{N}_{\beta+1})$ and $F_\beta = \emptyset$. If $\mathcal{N}_{\beta+1}$ is not reliable then we stop the construction.

3. *Suppose $\beta \leq \eta$ is a limit ordinal and $((\mathcal{M}_\gamma, \mathcal{N}_\gamma : \gamma < \beta), (F_\gamma : \gamma < \beta))$ has been defined. Then we define $\mathcal{M}_\beta$ and $\mathcal{N}_\beta$ as follows*[6]*. Let $\nu = limsup_{\lambda \to \beta} (\rho^+)^{\mathcal{M}_\beta}$.*

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[3] For the definition of the "amenable code" see the last paragraph on page 14 of [28].

[4] Recall that $\mathcal{C}(\mathcal{M})$ is the core of $\mathcal{M}$.

[5] Such branches are essentially unique, see Lemma 10.2.19.

[6] F_β will be defined at the next stage of the induction as in clause 2.

Then we let $\mathcal{N}_\beta$ be the passive lhp $\mathcal{W} = \mathcal{J}_\nu^{\mathcal{W}}$, where for all $\beta < \nu$ we set $\mathcal{J}_\beta^{\mathcal{W}}$ be the eventual value of $\mathcal{J}_\beta^{\mathcal{M}_\lambda}$ as $\lambda \to \beta$. Also if $\mathcal{N}_\beta$ is reliable then $\mathcal{M}_\beta = \mathcal{C}(\mathcal{N}_\beta)$. If $\mathcal{N}_\beta$ is not reliable then we stop the construction.

We can now define the $\mathcal{E}$-certified iterations.

Definition 10.2.17 *Suppose $\mathcal{M} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{M}^b$. Suppose $\vec{\mathcal{T}} \in \mathcal{N}$ is a stack on $\mathcal{M}$ (played either according to the usual rules of the iteration game or according to the rules of the short tree game) and $E \in \mathcal{E}$. We say $\vec{\mathcal{T}}$ is E-certified if the following conditions are satisfied.*

1. *$\vec{\mathcal{T}}$ is π_E-realizable.*
2. *Suppose $\mathcal{S} \in B_{\vec{\mathcal{T}}}$ and let $\mathcal{U}$ be the largest normal initial segment of $\vec{\mathcal{T}}_{\geq \mathcal{S}}$ that is based on $\mathcal{S}$ and is above $\mathcal{S}^b$. Let $\alpha < lh(\mathcal{U})$ be a limit ordinal and let c be the branch of $\mathcal{U} \restriction \alpha$ chosen by $\mathcal{U}$ if there is such a branch. Then the following conditions hold.*
   - (a) *$\mathcal{M}^+(\mathcal{M}(\mathcal{U} \restriction \alpha)) \vDash$ "$\delta(\mathcal{U} \restriction \alpha)$ is not a Woodin cardinal". Then $\mathcal{Q}(b, \mathcal{U} \restriction \alpha)$ exists and $\mathcal{Q}(b, \mathcal{U} \restriction \alpha) \trianglelefteq \mathcal{M}^+(\mathcal{M}(\mathcal{U} \restriction \alpha))$.*
   - (b) *$\mathcal{M}^+(\mathcal{M}(\mathcal{U} \restriction \alpha)) \vDash$ "$\delta(\mathcal{U} \restriction \alpha)$ is a Woodin cardinal" and there is $\mathcal{W}$ that is an initial segment of the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{M}(\mathcal{U}| \restriction \alpha))$ and is such that $\mathcal{J}[\mathcal{W}] \vDash$ "$\delta(\mathcal{U} \restriction \alpha)$ is not a Woodin cardinal". Then $\mathcal{Q}(b, \mathcal{U} \restriction \alpha)$ exists and $\mathcal{Q}(b, \mathcal{U} \restriction \alpha) = \mathcal{W}$.*
   - (c) *The above two clauses fail. Then in $\mathcal{U}$, player II played $\mathcal{M}^+(\mathcal{M}(\mathcal{U} \restriction \alpha))$ at stage α, and $\mathcal{U} = (\mathcal{U} \restriction \alpha)^\frown \mathcal{M}^+(\mathcal{M}(\mathcal{U} \restriction \alpha))$.*

We say that $\vec{\mathcal{T}}$ is $\mathcal{E}$-certified if for some λ, $\vec{\mathcal{T}}$ is E-certified for every $E \in \mathcal{E}$ such that $lh(E) > \lambda$.

And finally we define $\mathcal{E}$-certified strategies.

Definition 10.2.18 *Suppose $\mathcal{M} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{M}^b$. We let $\Lambda_{\mathcal{M}}$ be the partial strategy of $\mathcal{M}$ with the property that*

1. *$dom(\Lambda)$ consist of $\mathcal{E}$-certified stacks $\vec{\mathcal{T}}$, and*
2. *for all $\vec{\mathcal{T}} \in dom(\Lambda)$, $\Lambda(\vec{\mathcal{T}}) = b$ if b is the unique branch of $\vec{\mathcal{T}}$ such that $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is $\mathcal{E}$-certified.*

We say $\Lambda_{\mathcal{M}}$ is the $\mathcal{E}$-certified strategy of $\mathcal{M}$.

10.2.5 Uniqueness of $\mathcal{E}$-certified strategies

In this subsection we show that $\mathcal{E}$-certified strategies are unique.

Lemma 10.2.19 *Suppose $\mathcal{M} \in \mathcal{N}$ is a hod premouse extending $\mathcal{R}$ such that $\mathcal{R} = \mathcal{M}^b$. Suppose Λ and Ψ are two $\mathcal{E}$-certified strategies for $\mathcal{M}$. Then $\Lambda = \Psi$.*

Proof. Suppose not. It follows from Lemma 4.6.3 that there is a low level disagreement between Λ and Ψ. Let $(\vec{\mathcal{T}}, \mathcal{Q})$ constitute a low level disagreement between Λ and Ψ. Let $\mathcal{Q}^+$ be the last model of $\vec{\mathcal{T}}$. Because both Λ and Ψ are $\mathcal{E}$-certified, we can find $E \in \mathcal{E}$ such that there are

1. an $\mathcal{N}$-strong cardinal $\lambda < \nu_E$,
2. $(\mathcal{W}_0, \Phi_0), (\mathcal{W}_1, \Phi_1) \in \mathcal{F} \restriction \lambda$,
3. $\sigma_0 : (\mathcal{Q}^+)^b \to \pi_E(\mathcal{R})$ and $\sigma_1 : (\mathcal{Q}^+)^b \to \pi_E(\mathcal{R})$,
4. for $i < 2$, $\sigma_i[\mathcal{Q}] \subseteq rng(\pi^{\Phi_i}_{\mathcal{W}_i, \sigma_i(\mathcal{Q})})$,
5. for $i < 2$, letting $k_i : \mathcal{Q} \to \mathcal{W}_i$ be the embedding $(\pi^{\Phi_i}_{\mathcal{W}_i, \sigma_i(\mathcal{Q})})^{-1} \circ (\sigma_i \restriction \mathcal{Q})$, $\Lambda_{\mathcal{Q}, \vec{\mathcal{T}}} = k_0$-pullback of Φ_0 and $\Psi_{\mathcal{Q}, \vec{\mathcal{T}}} = k_1$-pullback of Φ_1.

Recall the definition of low level disagreement, Definition 4.6.2. It follows that

(1) $\Lambda_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1), \vec{\mathcal{T}}} = \Psi_{\mathcal{Q}(\lambda^{\mathcal{Q}}-1), \vec{\mathcal{T}}}$ and
(2) $\delta^{\mathcal{Q}} = \sup(\{\pi^{\vec{\mathcal{T}}}(f)(a) : f \in \mathcal{R} \wedge a \in (\mathcal{Q}(\lambda^{\mathcal{Q}}-1))^{<\omega}\} \cap \delta^{\mathcal{Q}})$.

Let then $\mathcal{U}$ be a normal tree on $\mathcal{Q}$ such that if $a = \Lambda_{\mathcal{Q}, \vec{\mathcal{T}}}(\mathcal{U})$ and $c = \Psi_{\mathcal{Q}, \vec{\mathcal{T}}}(\mathcal{U})$ then

(3) $\mathcal{M}^{\mathcal{U}}_a = \mathcal{M}^{\mathcal{U}}_c =_{def} \mathcal{W}$ and $\Lambda_{\mathcal{W}, \vec{\mathcal{T}}^\frown \mathcal{U}} = \Psi_{\mathcal{W}, \vec{\mathcal{T}}^\frown \mathcal{U}}$.

We also have that for $i < 2$ there are embeddings $j_i : \mathcal{W} \to \sigma_i(\mathcal{Q})$ such that

(4) $\sigma_0 \restriction \mathcal{Q} = j_0 \circ \pi^{\mathcal{U}}_a$ and $\sigma_1 \restriction \mathcal{Q} = j_1 \circ \pi^{\mathcal{U}}_c$.

It follows that if $\mathcal{W}^+_b$ is the result of applying $\mathcal{U}$ and a to $\mathcal{Q}^+$ and $\mathcal{W}^+_c$ is the result of applying $\mathcal{U}$ and c to $\mathcal{Q}^+$ then we can extend j_0 to $j^+_0 : (\mathcal{W}^+_a)^b \to \pi_E(\mathcal{R})$ and j_1 to $j^+_1 : (\mathcal{W}^+_c)^b \to \pi_E(\mathcal{R})$ such that

(5) $\sigma_0 \restriction (\mathcal{Q}^+)^b = j_0^+ \circ \pi_a^{\mathcal{U}}$ and $\sigma_1 \restriction (\mathcal{Q}^+)^b = j_1^+ \circ \pi_c^{\mathcal{U}}$.

Combining (2), (3), and(5) we get that

(6) $\sup(rng(\pi_a^{\mathcal{U}}) \cap rng(\pi_c^{\mathcal{U}}) \cap \delta(\mathcal{U})) = \delta(\mathcal{U})$

(6) implies that $a = c$, contradiction. □

10.2.6 Canonical certification witnesses

Suppose $\mathcal{S}^*$ is a $\Sigma_{\mathcal{R}^+}$-iterate of $\mathcal{R}^+$ via an iteration that is entirely above $\delta^{\mathcal{R}^+}$. Suppose further that $\mathcal{S} \trianglelefteq \mathcal{S}^*$ is such that $\mathcal{S}^b = \mathcal{R}$ and $\mathcal{S} \in \mathcal{N}$. Let $\vec{\mathcal{T}} \in \mathcal{N}$ be a stack on $\mathcal{S}$. We will use $\mathcal{S}$ and $\vec{\mathcal{T}}$ throughout this section.

Suppose that $E \in \mathcal{E}$ and E^* is the background certificate of E. Assume that $\vec{\mathcal{T}}$ isπ_E-realizable and is according to $\Sigma_{\mathcal{S}}$. We want to show that we can choose canonical embeddings and pairs that witness that $\vec{\mathcal{T}}$ is π_E-realizable. This is shown in Corollary 10.2.23. First we prove two useful lemmas.

Lemma 10.2.20 *$\mathcal{K} \in B_{\vec{\mathcal{T}}}$. Suppose further that for every $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$ such that $\mathcal{Q} \prec^{\vec{\mathcal{T}}} \mathcal{K}$,*

1. *$\tau(\sigma_{\mathcal{Q}}) \restriction \delta^{\mathcal{Q}^b}$ is the iteration embedding according to $\Sigma_{\mathcal{Q}^b}$ and*
2. *$\Psi_{\mathcal{Q}}^{k_{\mathcal{Q}}} = \Sigma_{\mathcal{Q}^b}$.*

Then $\tau(\sigma_{\mathcal{K}}) \restriction \delta^{\mathcal{K}^b}$ is the iteration embedding according to $\Sigma_{\mathcal{K}^b}$. Moreover, if F is the $(\delta^{\mathcal{R}}, \delta^{\mathcal{K}^b})$-extender derived from $\pi^{\vec{\mathcal{T}},b}$ then $\tau(\sigma_{\mathcal{K}})$ extends to

$$\sigma_{\mathcal{K}}^+ : Ult(\mathcal{K}, F) \to \pi_{E^*}(\mathcal{R}^+)$$

and $\sigma_{\mathcal{K}}^+$ is the iteration embedding according to $\Sigma_{Ult(\mathcal{K},F)}$.

Proof. Set

$$\alpha = \sup(\{\pi_{\mathcal{Q},\mathcal{K}}^{\vec{\mathcal{T}}_{\mathcal{Q},\mathcal{K}},b}(\delta^{\mathcal{Q}^b}) : \mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{K}\}).$$

We have that

$$\mathcal{K}^b = \{\pi^{\vec{\mathcal{T}}_{\leq\mathcal{K}},b}(f)(a) : f \in \mathcal{R} \wedge a \in (\delta_\alpha^{\mathcal{K}^b})^{<\omega}\}.$$

We also have that for every $x \in \mathcal{K}^b$,

$$\pi^{\Sigma_{\mathcal{K}^b}}_{\mathcal{K}^b,\pi_{E^*}(\mathcal{R})}(x) = \pi_{E^*}(f)(\pi^{\Sigma_{\mathcal{K}(\alpha)}}_{\mathcal{K}(\alpha),\pi_{E^*}(\mathcal{R})}(a))$$

where $f \in \mathcal{R}$ and $a \in \alpha^{<\omega}$. It is then enough to see that

$$\pi^{\Sigma_{\mathcal{K}(\alpha)}}_{\mathcal{K}(\alpha),\pi_{E^*}(\mathcal{R})} \restriction \delta^{\mathcal{K}}_\alpha = \sigma_{\mathcal{K}} \restriction \delta^{\mathcal{K}}_\alpha \qquad (*).$$

Suppose first α is a limit ordinal. Then for each $\beta < \alpha$, there are $\mathcal{Q} \prec^{\vec{\mathcal{T}},s} \mathcal{K}$ and $\xi < \lambda^{\mathcal{Q}^b}$ such that $\mathcal{K}(\beta)$ is a $\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}}$-iterate of $\mathcal{Q}(\xi)$. Because we are assuming that $\Psi^{k_{\mathcal{Q}}}_{\mathcal{Q}} = \Sigma_{\mathcal{Q}^b}$, and because of clause 6 and 7 of Definition 10.2.15, we get that

$$\sigma^{\mathcal{K}} \restriction \delta^{\mathcal{K}}_\beta = \pi^{\Sigma}_{\mathcal{K}(\beta),\pi_{E^*}(\mathcal{R})} \restriction \delta^{\mathcal{K}}_\beta.$$

As the above equality holds for any $\beta < \alpha$, $(*)$ follows. The case when α is a successor ordinal is very similar. The rest follows easily because

$$Ult(\mathcal{K}, F) = \{\pi_F(f)(a) : f \in \mathcal{R}^+ \wedge a \in (\delta^{\mathcal{K}^b})^{<\omega}\}$$

implying that setting $\mathcal{W} = Ult(\mathcal{K}, F)$

$$\pi^{\Sigma_{\mathcal{W}}}_{\mathcal{W},\pi_{E^*}(\mathcal{R})}(x) = \pi_{E^*}(f)(\pi^{\Sigma_{\mathcal{K}^b}}_{\mathcal{K}^b,\pi_{E^*}(\mathcal{R})}(a))$$

where $f \in \mathcal{R}^+$ and $a \in (\delta^{\mathcal{K}^b})^{<\omega}$ are such that $x = \pi_F(f)(a)$. $\square$

Lemma 10.2.21 *Let $(\sigma_{\mathcal{Q}} : \mathcal{Q} \in B_{\vec{\mathcal{T}}}) \in \mathcal{N}|\lambda$ and $((\mathcal{W}_{\mathcal{Q}}, \Psi_{\mathcal{Q}}) \in \mathcal{F} \restriction \lambda : \mathcal{Q} \in B_{\vec{\mathcal{T}}}) \in \mathcal{J}[\mathcal{N}]$ witness that $\vec{\mathcal{T}}$ is π_E-realizable, and let $k_{\mathcal{Q}} : \mathcal{Q}^b \to \mathcal{W}_{\mathcal{Q}}$ be the embedding described in clause 2 of Definition 10.2.15. Then for any $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$,*

$$\Sigma_{\mathcal{Q}^b} \restriction \mathcal{N} = k_{\mathcal{Q}}\text{−}pullback\ of\ \Psi_{\mathcal{Q}}.$$

Proof. Let $\lambda \in (\kappa, lh(E))$ be an $\mathcal{N}$-strong cardinal witnessing the clauses of Definition 10.2.15. Let E^* be the background certificate of E and let $\tau : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be such that $\pi_{E^*} \restriction \mathcal{N} = \tau \circ \pi_E$. Because $\sigma_{\mathcal{R}} \restriction \delta^{\mathcal{R}} = \pi_{E^*} \restriction \delta^{\mathcal{R}}$ and $\pi_{E^*} \restriction \mathcal{R}$ is the iteration embedding according to Σ, Lemma 10.2.20 implies, by induction, that

(1) for every $\mathcal{K} \in B_{\vec{\mathcal{T}}}$, $\sigma_{\mathcal{K}} \restriction (\mathcal{K}^b|\delta^{\mathcal{K}^b})$ is the iteration embedding according to $\Sigma_{\mathcal{K}^b}$.

Now fix $\mathcal{K} \in B_{\vec{\mathcal{T}}}$. It follows from Clause 6 of Definition 10.2.15 that

(2) $\sigma_{\mathcal{K}} \restriction (\mathcal{K}^b|\delta^{\mathcal{K}^b})$ is the iteration embedding according to $\Psi^{k_{\mathcal{K}}}_{\mathcal{K}}$.

It follows from (1) and (2) that $\Psi^{k_{\mathcal{K}}}_{\mathcal{K}}$ and $\Sigma_{\mathcal{K}^b}$ are both $\sigma_{\mathcal{K}} \restriction (\mathcal{K}^b|\delta^{\mathcal{K}^b})$-pullbacks of $\pi_{E^*}(\Sigma_{\mathcal{R}})$, and hence,

$$\Psi_{\mathcal{K}}^{k_{\mathcal{K}}} = \Sigma_{\mathcal{K}^b} \restriction \mathcal{N},$$

finishing the proof of the lemma. □

We continue with our $\mathcal{S}$ and $\vec{\mathcal{T}}$. Let $\lambda^* = \sup\{\lambda_{\mathcal{Q}^b} : \mathcal{Q} \in B_{\vec{\mathcal{T}}}\}$ and let $\lambda_{\vec{\mathcal{T}}}$ be the least $\mathcal{N}$-strong cardinal $\geq \lambda^*$. Let $\lambda \geq \lambda_{\vec{\mathcal{T}}}$ be any $\mathcal{N}$-strong cardinal. Let

$$\mathcal{W}^* = \cup\{\mathcal{R}_\lambda^{\mathcal{Q}^b} : \mathcal{Q} \in B_{\vec{\mathcal{T}}}\},\ \Psi = \oplus_{\mathcal{Q}\in B_{\vec{\mathcal{T}}}} \Psi_\lambda^{\mathcal{Q}^b} \text{ and } \mathcal{W} = Lp_\omega^{\oplus_{\mathcal{Q}\in B_{\vec{\mathcal{T}}}} \Psi_\lambda^{\mathcal{Q}^b}}(\mathcal{W}^*).$$

Notice that $\mathrm{cf}(\lambda^{\mathcal{W}}) < \lambda$ implying that $\mathcal{W} \vDash$ "$\mathrm{cf}(\lambda^{\mathcal{W}})$ is not a measurable ordinal". It follows that Ψ is an iteration strategy for $\mathcal{W}$.

Suppose now that $E \in \mathcal{E}$ is an extender such that $\lambda < lh(E)$ and let E^* be the background certificate of E. Let $\tau : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor map. Given $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, we let $\sigma^*_{\mathcal{Q},E} : \mathcal{Q}^b \to \pi_{E^*}(\mathcal{R})$ be such that

$$\sigma^*_{\mathcal{Q},E}(x) = \pi_{E^*}(f)(\pi^{\Sigma_{\mathcal{Q}^b}}_{\mathcal{Q}^b,\pi_{E^*}(\mathcal{R})}(a))$$

where $f \in \mathcal{R}$ and $a \in (\delta^{\mathcal{Q}^b})^{<\omega}$ are such that $x = \pi^{\vec{\mathcal{T}}_{\leq\mathcal{Q}},b}(f)(a)$. The following is an easy corollary of Lemma 10.2.20.

Corollary 10.2.22 *There is a sequence $(\sigma_{\mathcal{Q},E} : \mathcal{Q} \in B_{\vec{\mathcal{T}}}) \in \mathcal{N}[g]$ such that for each $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, $\tau(\sigma_{\mathcal{Q},E}) = \sigma^*_{\mathcal{Q},E}$.*

Proof. It follows from Lemma 10.2.20 that $\tau(l_{\mathcal{Q}}) = \sigma^*_{\mathcal{Q},E}$ whenever $(l_{\mathcal{Q}} : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ witnesses that $\vec{\mathcal{T}}$ is E-certified. □

It is now routine to check that

Corollary 10.2.23 *$(\sigma_{\mathcal{Q},E} : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ and $((\mathcal{R}^{\mathcal{Q}}_\lambda, \Psi^{\mathcal{Q}}_\lambda) : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ witness that $\vec{\mathcal{T}}$ is E-certified.*

Corollary 10.2.24 *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{S}$ such that all of its initial segments are according to $\Lambda_{\mathcal{S}}$. Then $\vec{\mathcal{T}} \in dom(\Lambda_{\mathcal{S}})$.*

Proof. Let $\lambda^* = \sup\{\lambda_{\vec{\mathcal{T}}_{\leq\mathcal{Q}}} : \mathcal{Q} \in B_{\vec{\mathcal{T}}}\}$ and let λ be any $\mathcal{N}$-strong cardinal greater than λ^*. Then for any $E \in \mathcal{E}$ such that $lh(E) > \lambda$ and for any $\mathcal{K} \in B_{\vec{\mathcal{T}}}$, $(\sigma_{\mathcal{Q},E} : \mathcal{Q} \in B_{\vec{\mathcal{T}}_{\leq\mathcal{K}}})$ and $((\mathcal{R}^{\mathcal{Q}}_\lambda, \Psi^{\mathcal{Q}}_\lambda) : \mathcal{Q} \in B_{\vec{\mathcal{T}}_{\leq\mathcal{K}}})$ witness that $\vec{\mathcal{T}}_{\leq\mathcal{K}}$ is E-certified. It follows that $(\sigma_{\mathcal{Q},E} : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ and $((\mathcal{R}^{\mathcal{Q}}_\lambda, \Psi^{\mathcal{Q}}_\lambda) : \mathcal{Q} \in B_{\vec{\mathcal{T}}})$ witness that $\vec{\mathcal{T}}$ is E-certified. □

10.2.7 Correctness of $\mathcal{Q}$-structures

In this subsection, we work towards showing that $\mathcal{E}$-certified constructions produce $\mathcal{Q}$-structures that are according to Σ. Our first lemma of this subsection shows that we can always embeddings witnessing certification.

Lemma 10.2.25 *Suppose $\mathcal{Q}, \mathcal{W} \in \mathcal{N}$ are such that there are stacks $\vec{\mathcal{U}}_0$ on $\mathcal{R}^+$ with last model $\mathcal{Q}^+$ and $\vec{\mathcal{U}}_1$ on $\mathcal{Q}^+$ with last model $\mathcal{W}^+$ such that $\pi^{\vec{\mathcal{U}}_0,b}$ and $\pi^{\vec{\mathcal{U}}_1,b}$ exist, $\mathcal{Q} = (\mathcal{Q}^+)^b$ and $\mathcal{W} = (\mathcal{W}^+)^b$. Suppose further that $\mathcal{Q}, \mathcal{W} \in \mathcal{N}$. Suppose λ is a strong cardinal greater than $\max(\lambda_{\mathcal{Q}}, \lambda_{\mathcal{W}})$. Then $\pi^{\vec{\mathcal{U}}_0,b}, \pi^{\vec{\mathcal{U}}_1,b} \in \mathcal{N}$ and moreover, for any $E \in \mathcal{E}$ such that $lh(E) > \lambda$, there are $\sigma_{\mathcal{Q}} : \mathcal{Q} \to \pi_E(\mathcal{R})$ and $\sigma_{\mathcal{W}} : \mathcal{W} \to \pi_E(\mathcal{R})$ such that*

$$\pi_E \restriction \mathcal{R} = \sigma_{\mathcal{Q}} \circ \pi^{\vec{\mathcal{U}}_0,b} \text{ and } \sigma_{\mathcal{Q}} = \sigma_{\mathcal{W}} \circ \pi^{\vec{\mathcal{U}}_1,b}.$$

Proof. Let $E \in \mathcal{E}$ be such that $lh(E) > \lambda$ and let E^* be the background certificate of E. Let $Y = \pi^{\Sigma_{\mathcal{R}^{\mathcal{Q}}}}_{\mathcal{R}^{\mathcal{Q}}, \pi_{E^*}(\mathcal{R})}[\mathcal{R}^{\mathcal{Q}}]$. Then (in $\mathcal{N}^*_z$) there is $\tau^* : \mathcal{Q} \to \pi_{E^*}(\mathcal{R})$ such that

$$rng(\tau^*) \subseteq \{\pi_{E^*}(f)(a) : f \in \mathcal{R} \wedge a \in Y\}.$$

Let $k : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the factor map. Because

$$\{\pi_{E^*}(f)(a) : f \in \mathcal{R} \wedge a \in Y\} \subseteq rng(k),$$

we have (in $\mathcal{N}^*_z$) $\tau : \mathcal{Q} \to \pi_E(\mathcal{R})$ such that

$$rng(\tau) \subseteq k^{-1}(\{\pi_{E^*}(f)(a) : f \in \mathcal{R} \wedge a \in Y\}) = \{\pi_E(f)(a) : f \in \mathcal{R} \wedge a \in k^{-1}[Y]\}.$$

Therefore, there is such a $\tau \in \mathcal{N}[g]$ where $g \subseteq Coll(\omega, \lambda)$ is $\mathcal{N}$-generic. But it follows from Lemma 9.1.9 that for any such τ, τ-pullback of $\pi_E(\Sigma_{\mathcal{R}})$ (which is the same as τ-pullback of $\pi_{E^*}(\Sigma_{\mathcal{R}})$) is just $\Sigma_{\mathcal{Q}}$. It follows that $\pi^{\Sigma_{\mathcal{Q}}}_{\mathcal{Q}, \pi_E(\mathcal{R})}[\mathcal{Q}|\delta^{\mathcal{Q}}] \in Ult(\mathcal{N}, E)[g]$ for any such generic g. Set

$$\sigma_{\mathcal{Q}}(x) = \pi_E(f)(a)$$

where $f \in \mathcal{R}$ and $a \in (\delta^{\mathcal{Q}})^{<\omega}$ are such that $\pi^{\vec{\mathcal{U}}_0,b}(f)(a) = x$. It follows that $\sigma_{\mathcal{Q}} \in \mathcal{N}[g]$ for any generic g. Therefore, $\sigma_{\mathcal{Q}} \in \mathcal{N}$. We then have that

$$\pi^{\vec{\mathcal{U}}_0,b}(x) = \sigma_{\mathcal{Q}}^{-1}(\pi_E(x)).$$

The rest of the argument is very similar. □

Suppose now that $\mathcal{S}^*$ is a $\Sigma_{\mathcal{R}^+}$-iterate of $\mathcal{R}^+$ via an iteration that is entirely above $\delta^{\mathcal{R}^+}$. Suppose further that $\mathcal{S} \trianglelefteq \mathcal{S}^*$ is such that $\mathcal{S}^b = \mathcal{R}$ and $\mathcal{S} \in \mathcal{N}$. Let $\vec{\mathcal{T}} \in \mathcal{N}$ be a stack on $\mathcal{S}$. We will use $\mathcal{S}$ and $\vec{\mathcal{T}}$ throughout this section. The following is an easy corollary of Lemma 10.2.25

Corollary 10.2.26 *Suppose $\vec{\mathcal{T}}$ is according to $\Sigma_{\mathcal{S}}$, $\pi^{\vec{\mathcal{T}},b}$ exists and for some $\mathcal{Q}_0$, $\vec{\mathcal{T}}_{\geq \mathcal{Q}_0}$ is a normal tree of limit length on $\mathcal{Q}_0$ above $\delta^{\mathcal{Q}_0^b}$. Suppose further that letting $\mathcal{Q} =_{def} \mathcal{M}^+(\vec{\mathcal{T}}_{\geq \mathcal{Q}_0})$, $\mathcal{Q} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". Let $\mathcal{U} \in \mathcal{N}$ be a normal tree on $\mathcal{Q}$ according to $\Sigma_{\mathcal{Q}}$. Then $\vec{\mathcal{T}}^\frown\mathcal{U}$ is $\mathcal{E}$-realizable.*

The next lemma argues that $\mathcal{Q}$-structures appearing in a $\mathcal{E}$-certified iteration are according to Σ.

Lemma 10.2.27 *Suppose $\vec{\mathcal{T}}$ is an $\mathcal{E}$-certified iteration and $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$. Let $\eta > \delta^{\mathcal{Q}^b}$ be such that $\mathcal{M}^+(\mathcal{Q}|\eta) \vDash$ "η is a Woodin cardinal" and let $\mathcal{W} \trianglelefteq \mathcal{Q}$ be an sts mouse over $\mathcal{M}^+(\mathcal{Q}|\eta)$ based on $\mathcal{M}^+(\mathcal{Q}|\eta)$. Suppose $\vec{\mathcal{T}}_{<\mathcal{Q}}$ is according to $\Sigma_{\mathcal{S}}$. Then $\mathcal{W}$ is a $\Sigma^{stc}_{\mathcal{M}^+(\mathcal{Q}|\eta)}$-sts mouse.*

Proof. Towards a contradiction assume that $\mathcal{W}$ is not a $\Sigma^{stc}_{\mathcal{M}^+(\mathcal{Q}|\eta)}$-sts mouse. It follows that $\vec{\mathcal{T}}_{\leq \mathcal{Q}}$ is not according to $\Sigma_{\mathcal{S}}$. If then follows that $\vec{\mathcal{T}}_{<\mathcal{Q}}$ has a last normal component of limit length that is above $\delta^{\mathcal{Q}^b}$. Let then $\mathcal{Q}_0 \in ntn(\vec{\mathcal{T}})$ be such that $\mathcal{Q}_0^b = \mathcal{Q}^b$ and $\mathcal{U} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{Q}_0}$ is a normal tree of limit length that is based on $\mathcal{Q}_0$ and is above $\delta^{\mathcal{Q}_0^b}$. For convenience, we change our notation and set $\mathcal{Q} = \mathcal{M}^+(\mathcal{U})$ and $\vec{\mathcal{T}} = \vec{\mathcal{T}}_{\leq \mathcal{Q}}$. It follows from Definition 10.2.17 that

(1) $\mathcal{W}$ is an initial segment of the fully background construction of $\mathcal{N}$ over $\mathcal{Q}$.

What we need to see is that $\mathcal{W}$ is a $\Sigma^{stc}_{\mathcal{Q}}$-sts mouse over $\mathcal{Q}$. To show this it is enough to show that every stack indexed in $\mathcal{W}$ is according to $\Sigma^{stc}_{\mathcal{Q}}$. To show this, it is enough to show that

(2) if $t = (\mathcal{Q}, \mathcal{U}_0, \mathcal{Q}_1, \vec{\mathcal{U}})$ is a stack of length 2 on $\mathcal{Q}$ appearing in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{Q}$ (done in $\mathcal{N}$) and b is the branch of t indexed in this construction then $t^\frown\{\mathcal{M}^t_b\}$ is according to $\Sigma^{stc}_{\mathcal{Q}}$.

(2) is indeed enough. To see this, notice that if $s = (\mathcal{Q}, \mathcal{U}_0^*, \mathcal{Q}_1^*, \vec{\mathcal{U}}^*)$ is a stack of

length 2 indexed in $\mathcal{W}$ and c is its branch then for some stack $t = (\mathcal{Q}, \mathcal{U}_0, \mathcal{Q}_1, \vec{\mathcal{U}})$ as above if e is the branch of t then $s^\frown\{\mathcal{M}_c^s\}$ is a hull of $t^\frown\{\mathcal{M}_e^t\}$. If t is according to $\Sigma_{\mathcal{Q}}^{stc}$ then it follows from hull condensation of $\Sigma_{\mathcal{Q}}^{stc}$ that s is also according to $\Sigma_{\mathcal{Q}}^{stc}$. We now work towards showing that t is according to $\Sigma_{\mathcal{Q}}^{stc}$.

Suppose first that $\mathcal{U}_0$ is according to $\Sigma_{\mathcal{Q}}^{stc}$. We have that $\vec{\mathcal{U}}$ is a stack based on $\mathcal{Q}_1^b$. Because t is $\mathcal{E}$-certified, we can fix an extender $E \in \mathcal{E}$ such that t is π_E-realizable. We then have $\sigma : \mathcal{Q}_1^b \to \pi_E(\mathcal{R})$ such that $\pi_E \restriction \mathcal{R} = \sigma \circ \pi^{\mathcal{U}_0, b} \circ \pi^{\vec{\mathcal{T}}, b}$. We also have that $\vec{\mathcal{U}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{U}}}\}$ is according to σ-pullback of $\pi_E(\Sigma_{\mathcal{R}})$. Therefore, t is according to $\Sigma_{\mathcal{Q}}^{stc}$.

It remains to show that $\mathcal{U}_0$ is according to $\Sigma_{\mathcal{Q}}^{stc}$. Without loss of generality, we assume that all the initial segments of $\mathcal{U}_0$ are according to $\Sigma_{\mathcal{Q}}^{stc}$. The only way that $\mathcal{U}_0$ could fail to be according to $\Sigma_{\mathcal{Q}}^{stc}$ is if for some $\mathcal{Q}^* \in ntn(\mathcal{U}_0)$ such that $\pi^{\mathcal{U}_0, b}$ exists, $(\mathcal{U}_0)_{\geq \mathcal{Q}^*}$ is above $\delta^{\mathcal{Q}^*}$ and the branch of $(\mathcal{U}_0)_{\geq \mathcal{Q}^*}$ chosen in $\mathcal{U}_0$ is not according to $\Sigma_{\mathcal{Q}}^{stc}$. Let c_0 be this branch. We then have that $\mathcal{Q}(c_0, (\mathcal{U}_0)_{\geq \mathcal{Q}^*})$ exists and appears in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+((\mathcal{U}_0)_{\geq \mathcal{Q}^*})$ (done in $\mathcal{N}$).

Set $\mathcal{Q}_0 =_{def} \mathcal{Q}$, $\mathcal{Q}_2 = \mathcal{M}^+((\mathcal{U}_0)_{\geq \mathcal{Q}^*})$, $\mathcal{W}_0 =_{def} \mathcal{W}$ and $\mathcal{W}_2 = \mathcal{Q}(c_0, (\mathcal{U}_0)_{\geq \mathcal{Q}^*})$. Let $b_0 = \Sigma_{\mathcal{Q}_0}(\mathcal{U}_0)$. Notice that either b_0 has a drop or $\pi_{b_0}^{\mathcal{U}_0}(\delta^{\mathcal{Q}_0}) > \delta^{\mathcal{Q}_2}$. It follows that if we repeat the above argument then we will eventually end up descending indefinitely. □

The following is an easy corollary of Lemma 10.2.21 and Lemma 10.2.27.

Corollary 10.2.28 *Suppose $\vec{\mathcal{T}} \in \mathcal{N}$ is $\mathcal{E}$-certified stack on $\mathcal{S}$. Then if $\mathcal{S}$ is not of lsa type then $\vec{\mathcal{T}}$ is according to $\Sigma_{\mathcal{S}}$ and if $\mathcal{S}$ is of lsa type then $\vec{\mathcal{T}}$ is according to the minimal component of $\Sigma_{\mathcal{S}}^{stc}$ (see Definition 3.9.8).*

We will also need to show that the fully backgrounded $\mathcal{E}$-realizable constructions reach all the necessary $\mathcal{Q}$-structures. This is the goal of the next lemma. We continue with $\mathcal{S}$ and $\vec{\mathcal{T}}$.

Lemma 10.2.29 *Suppose $\mathcal{S}$ is of lsa type, $\vec{\mathcal{T}}$ is $\mathcal{E}$-certified and for some $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$, $\mathcal{T} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{Q}}$ is a normal tree on $\mathcal{Q}$ of limit length above $\delta^{\mathcal{Q}^b}$. Suppose further that $\mathcal{M}^+(\mathcal{M}(\mathcal{T})) \vDash$ "$\delta(\mathcal{T})$ is not a Woodin cardinal" but letting $c = \Sigma_{\mathcal{S}}(\vec{\mathcal{T}})$, $\mathcal{Q}(c, \mathcal{T})$ exists. Then $\mathcal{Q}(c, \mathcal{T})$ is an initial segment of the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{M}(\mathcal{T}))$ (done in $\mathcal{N}$).*

Proof. Suppose the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{M}(\mathcal{T}))$ (done in $\mathcal{N}$) outputs a model of height δ_z. Then because this model is universal and because all stacks indexed in such a model are according to $\Sigma_{\mathcal{M}^+(\mathcal{M}(\mathcal{T}))}$ (see Corollary 10.2.28) we have that $\mathcal{Q}(c, \mathcal{T})$ appears in this construction. Thus, it is enough to show that the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{M}(\mathcal{T}))$ outputs a model of height δ_z.

Suppose this is not the case. We change the notation and let $\mathcal{Q} = \mathcal{M}^+(\mathcal{M}(\mathcal{T}))$. The aforementioned construction can fail to reach a model of height δ_z only because we have encountered an $\mathcal{E}$-certified stack $t = (\mathcal{Q}, \mathcal{U}_0, \mathcal{Q}_1, \vec{\mathcal{U}}_1)$ on $\mathcal{Q}$ but we cannot find a branch c of t such that $t^\frown\{\mathcal{M}^t_c\}$ is $\mathcal{E}$-certified. The rest of the proof is like the proof of Lemma 10.2.27. If $\vec{\mathcal{U}}_1$ exists then we find such a branch following the procedure used in the proof of Lemma 10.2.27. If the troublesome tree is $\mathcal{U}_0$ then this, just like in the proof of Lemma 10.2.27, inevitably leads to an infinite descend. □

10.2.8 $\Lambda_\mathcal{S}$ is total

The goal of this subsection is to show that $\Lambda_\mathcal{S}$, the $\mathcal{E}$-certified strategy of $\mathcal{S}$, is total. Our first lemma shows that $\mathcal{E}$-certified iterations can be continued. Recall that $\mathcal{S} \in \mathcal{N}$ is an initial segment of some $\Sigma_{\mathcal{R}^+}$-iterate of $\mathcal{R}^+$. We start with the following corollary of Lemma 10.2.21 and Lemma 10.2.25.

Corollary 10.2.30 *Suppose $\vec{\mathcal{T}}$ is a stack on $\mathcal{S}$ that is according to both $\Lambda_\mathcal{S}$ and $\Sigma_\mathcal{S}$. Suppose $\pi^{\vec{\mathcal{T}},b}$ exists. Let $\mathcal{M} = \pi^{\vec{\mathcal{T}},b}(\mathcal{R})$. Then $\Sigma_\mathcal{M} \restriction \mathcal{N} \in \mathcal{J}[\mathcal{N}]$ and whenever $\vec{\mathcal{U}}$ is an iteration according to $\Sigma_\mathcal{M}$ then $\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}$ is $\mathcal{E}$-realizable.*

Lemma 10.2.31 *The following holds.*

1. *Suppose $\mathcal{S}$ is not of lsa type or if it is then $\mathcal{J}_1(\mathcal{S}) \vDash$ "$\delta^\mathcal{S}$ is not a Woodin cardinal". Then $\Sigma_\mathcal{S} \restriction \mathcal{N} = \Lambda_\mathcal{S}$.*

2. *Suppose $\mathcal{S}$ is of lsa type and Ψ is the minimal component of $\Sigma^{stc}_\mathcal{S}$ (see Definition 3.9.8). Then $\Psi \restriction \mathcal{N} = \Lambda_\mathcal{S}$.*

Proof. In a sense, the proof has already appeared in previous subsections. Here we only collect the relevant facts.

Because the proof of clause 1 is very similar to the proof of clause 2 and because the proof of clause 2 is more difficult and is really the only new case that is beyond [10], we only give the proof of clause 2. We make an extra benign assumption that $\mathcal{S} = \mathcal{M}^+(\mathcal{S}|\delta^\mathcal{S})$. We need to show that if $\vec{\mathcal{T}}$ is a stack on $\mathcal{S}$ according to $\Lambda_\mathcal{S}$ and $\Sigma^{stc}_\mathcal{S}$ then

1. $\vec{\mathcal{T}} \in b(\Lambda_\mathcal{S}) \leftrightarrow \vec{\mathcal{T}} \in b(\Psi)$,

2. $\Lambda_{\mathcal{S}}$ is total,
3. $\vec{\mathcal{T}} \in m(\Lambda_{\mathcal{S}}) \leftrightarrow \vec{\mathcal{T}} \in m(\Psi)$, and

Verifying these three clauses is enough because it follows from Lemma 10.2.24 that if $\vec{\mathcal{T}}$ is according to $\Lambda_{\mathcal{S}}$ then $\vec{\mathcal{T}} \in dom(\Lambda_{\mathcal{S}})$. Notice that clause 1 and 2 imply clause 3. Fix a stack $\vec{\mathcal{T}}$ on $\mathcal{S}$ according to both $\Lambda_{\mathcal{S}}$ and $\Sigma_{\mathcal{S}}^{stc}$.

We start with clause 1. Suppose $\vec{\mathcal{T}} \in b(\Lambda_{\mathcal{S}})$. Then letting $b = \Lambda_{\mathcal{S}}(\vec{\mathcal{T}})$ we have that $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is $\mathcal{E}$-certified. It follows from Corollary 10.2.28 that $b = \Sigma_{\mathcal{S}}(\vec{\mathcal{T}})$. Suppose now that $b \in b(\Psi)$. We now have two cases.

Suppose first that there is a $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$ such that $\mathcal{T} =_{def} \vec{\mathcal{T}}_{\geq \mathcal{Q}}$ is a normal tree on $\mathcal{Q}$ above $\delta^{\mathcal{Q}^b}$. Because $b \in b(\Psi)$, we must have that $\mathcal{Q}(b, \mathcal{T})$ exists. If $\mathcal{Q}(b,\mathcal{T}) \trianglelefteq \mathcal{M}^+(\mathcal{M}(\mathcal{T}))$ then it follows that $\vec{\mathcal{T}} \in b(\Lambda)$. Assume then that $\mathcal{M}^+(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathcal{Q}(b,\mathcal{T})$. It follows from Lemma 10.2.29 that $\mathcal{Q}(b,\mathcal{T})$ appears in the fully backgrounded $\mathcal{E}$-realizable construction over $\mathcal{M}^+(\mathcal{M}(\mathcal{T}))$ done in $\mathcal{N}$. Therefore, $\vec{\mathcal{T}} \in b(\Lambda_{\mathcal{S}})$.

Suppose then there is $\mathcal{Q} \in B_{\vec{\mathcal{T}}}$ such that the rest of $\vec{\mathcal{T}}_{\geq \mathcal{Q}}$ is an iteration based on $\mathcal{Q}^b$. It follows from Lemma 10.2.25 that $\Lambda_{\mathcal{S}}(\vec{\mathcal{T}})$ is defined. Letting $b = \Lambda_{\mathcal{S}}(\vec{\mathcal{T}})$ we have that $\vec{\mathcal{T}}^\frown\{\mathcal{M}_b^{\vec{\mathcal{T}}}\}$ is $\mathcal{E}$-certified. It follows from Lemma 10.2.28 that $b = \Psi(\vec{\mathcal{T}})$. $\square$

10.2.9 Mixed hod pair constructions

We devote this entire subsection to the definition of construction producing the iterate of $\mathcal{R}^+$. In this construction, we use $\mathcal{E}$-certification method to acquire extenders with critical point $\delta^{\mathcal{R}}$, and we use the total extenders on the sequence of $\mathcal{N}$ to generate extenders with critical point $> \delta^{\mathcal{R}}$. We will use the operators introduced in Section 4.3. Here all operators must be viewed as operators constructed in the model $\mathcal{N}$; however, we will omit $\mathcal{N}$ from our notation. For convenience, we will repeat some of the definitions introduced in Section 4.3. Here we will need two extender operators. First we define $\mathcal{E}$-certified extenders.

Definition 10.2.32 *Suppose $\mathcal{Q} \in \mathcal{N}$ is a hod premouse such that $\Lambda_{\mathcal{Q}}$ (see Definition 10.2.18) is total and $\mathcal{Q}^b = \mathcal{R}$. Suppose F is an extender such that $(\mathcal{Q}, \tilde{F})$ is a reliable lhp where $\tilde{F}$ is the amenable code of F. We say F is $\mathcal{E}$-certified if for some $\mathcal{N}$-strong cardinal λ, for any $E \in \mathcal{E}$ such that $lh(E) > \lambda$, letting $\sigma : \pi_F(\mathcal{R}) \to \pi_E(\mathcal{R})$ be given by*

$$\sigma(x) = \pi_E(f)(\pi^{\Lambda_{\mathcal{Q}}}_{\mathcal{Q},\pi_E(\mathcal{R})}(a))$$

where $f \in \mathcal{R}$ *and* $a \in (\mathcal{Q})^{<\omega}$ *are such that* $x = \pi_F(f)(a)$,

$$(a, A) \in F \leftrightarrow \sigma(a) \in \pi_E(A).$$

We say σ *is the* E*-realizability map.*

Definition 10.2.33 (E^0, E^1, B^0, J^0) *Below we define the four sets* E^0, E^1, B^0 *and* J^0. *First if* $\mathcal{Q} \in \mathsf{E}^0 \cup \mathsf{E}^1 \cup \mathsf{B}^0 \cup \mathsf{J}^0$ *then* $\mathcal{Q}$ *extends* $\mathcal{R}$, $\mathcal{Q}^b = \mathcal{R}$ *and* $\mathcal{N} \vDash$ *"*$\Lambda_{\mathcal{Q}}$ *is a total strategy" (see Definition 10.2.18). In addition we have the following conditions.*

1. $\mathcal{Q} \in dom(\mathsf{E}^0)$ *if* $\mathcal{Q} \in \mathcal{N}$ *is a passive lhp and there is an extender* $F^* \in \mathcal{N}$ *with the property that* $\mathrm{crit}(F^*) > \delta^{\mathcal{R}}$, *an extender* $F \in \mathcal{N}$ *over* $\mathcal{Q}$ *and an ordinal* ν *such that* $\mathcal{N} \vDash$ *"*$\nu(F^*)$ *is an inaccessible cardinal",* $F = F^* \cap [\nu]^{<\omega} \times \mathcal{Q}$, *and* $(\mathcal{Q}, \tilde{F})$ *is a reliable lhp where* $\tilde{F}$ *is the amenable code of* F *and* $\nu^{(\mathcal{Q}, \tilde{F})} = \nu$.

2. $\mathcal{Q} \in dom(\mathsf{E}^1)$ *if* $\mathcal{Q} \in \mathcal{N}$ *is a passive lhp,* $\mathcal{Q}^b = \mathcal{R}$, $\Lambda_{\mathcal{Q}}$ *is total and there is an extender* F *such that* $\mathrm{crit}(F) = \delta^{\mathcal{R}}$, $(\mathcal{Q}, \tilde{F})$ *is a reliable lhp where* $\tilde{F}$ *is the amenable code of* F.

3. $\mathcal{Q} \in dom(\mathsf{B}^0)$ *if* $\mathcal{Q} = \mathcal{J}^{E,f}_{\alpha+\beta} \in \mathcal{N}$ *is a passive lhp such that for some* $\mathcal{R} \in Y^{\mathcal{Q}}$ *such that* $\mathcal{R}$ *is a hod premouse and there is a stack* $\vec{\mathcal{T}} \in \mathcal{Q} - dom(\Sigma^{\mathcal{Q}}_{\mathcal{R}})$ *based on* $\mathcal{R}$ *such that* $\vec{\mathcal{T}}$ *is according to* $\Sigma^{\mathcal{Q}}_{\mathcal{R}}$, $lh(\vec{\mathcal{T}})$ *is not of measurable cofinality in* $\mathcal{Q}$, *and there is some cofinal well-founded branch* $b \in M$ *of* $\vec{\mathcal{T}}$ *such that* $\beta = \sup b$ *and if* $\tilde{b}$ *is such that* $\alpha + \gamma \in \tilde{b}$ *if and only if* $\gamma \in b$ *then* $(\mathcal{Q}, \in, \vec{E}, f^+)$ *is an lhp where* $f^+ = f \cup \{(\mathcal{J}_\omega(\vec{\mathcal{T}}), b)\}$.

4. $\mathcal{Q} \in dom(\mathsf{J}^0)$ *if* $\mathcal{Q}$ *is an lhp and* $\mathcal{Q} \in \mathcal{N} - (dom(\mathsf{E}^0) \cup dom(\mathsf{B}^0))$.

The next definition introduces the *bad lhps.*

(Bad) Suppose $\mathcal{M}$ is an lhp extending $\mathcal{R}$ such that $\mathcal{M}^b = \mathcal{R}$. We say $\mathsf{Bad}(\mathcal{M})$ holds if one of the following conditions hold.

1. $\mathcal{M}$ is unreliable (i.e, for some $k < \omega$, $\mathcal{C}_k(\mathcal{M})$ doesn't exist).
2. $\rho(\mathcal{M}) = \delta^{\mathcal{R}}$.
3. $\mathcal{N} \vDash$ "$\Lambda_{\mathcal{M}}$ is not total".

We will have that $dom(\mathsf{E}_0) \subseteq dom(\mathsf{E}^0)$ and $dom(\mathsf{E}_1) \subseteq dom(\mathsf{E}^1)$ and $dom(\mathsf{B}) \subseteq dom(\mathsf{B}^0)$. All five functions E_0, E_1, B, J and Lim will be defined by induction.

Definition 10.2.34 (Stage 0) *We set.*

1. $\mathsf{J}(0) = \emptyset$.
2. $\mathsf{E}_0(0) = \mathsf{E}_1 = \mathsf{B}(0) = \mathsf{Lim}(0) = \emptyset$.

When defining J, E_0, E_1, B and Lim, we will maintain the following requirements.

Requirements

1. $dom(\mathsf{J})$, $dom(\mathsf{E}_0)$, $dom(\mathsf{E}_1)$, $dom(\mathsf{B})$ and $dom(\mathsf{Lim})$ are subsets of δ.
2. If $\alpha^{\mathcal{N}}_{def} = \sup\{\xi + 1 : \xi \in dom(\mathsf{J}) \cup dom(\mathsf{E}_0) \cup dom(\mathsf{E}_1) \cup dom(\mathsf{B}) \cup dom(\mathsf{Lim}))\}$ then the five sets $dom(\mathsf{J})$, $dom(\mathsf{E}_0)$, $dom(\mathsf{E}_0)$, $dom(\mathsf{B})$ and $dom(\mathsf{Lim})$ form a partition of $\alpha^{\mathcal{N}}$.
3. $\{\beta < \alpha^{\mathcal{N}} : \beta \text{ is a successor ordinal}\} \subseteq dom(\mathsf{J})$.
4. For all $\beta < \alpha^{\mathcal{N}}$, the value of the hpc-operators at β is either undefined or is an lhp $\mathcal{Q}$ such that for every $\mathcal{S} \in Y^{\mathcal{Q}}$, $\mathcal{S}$ is a hod premouse.
5. Given any $\mathcal{Q}$ and $\mathcal{S}$ as in clause 4, we let $\Lambda_{\mathcal{S}}$ be $\mathcal{E}$-certified partial strategy of $\mathcal{S}$ (see Definition 10.2.18). We will have that $\mathcal{N} \vDash$ "$\Lambda_{\mathcal{S}}$ is total".
6. If $\beta \in dom(\mathsf{E}_0) \cup dom(\mathsf{E}_1) \cup dom(\mathsf{B})$ then β is a successor ordinal and $\beta - 1 \in dom(\mathsf{Lim}))$

We start by describing how the operator E_0 works.

Definition 10.2.35 (The first extender operator) *Suppose $\mathsf{J} \restriction \beta$, $\mathsf{E}_0 \restriction \beta$, $\mathsf{E}_1 \restriction \beta$, $\mathsf{B} \restriction \beta$ and $\mathsf{Lim} \restriction \beta$ have been defined, $\beta = \gamma + 1$ and γ is a limit ordinal. Let $\mathcal{Q} = \mathsf{Lim}(\gamma)$.*

1. *Suppose $\mathcal{Q} \notin \mathsf{E}^0$. Then let $\mathsf{E}_0(\beta)$ be undefined.*
2. *Suppose then that $\mathcal{Q} \in \mathsf{E}^0$.*
   (a) *Suppose there is no triple (F^*, F, ν) witnessing that $\mathcal{Q} \in \mathsf{E}^0$ with the additional property that F^* coheres $(\mathsf{J} \restriction \beta, \mathsf{E}_0 \restriction \beta, \mathsf{E}_1 \restriction \beta, \mathsf{B} \restriction \beta, \mathsf{Lim} \restriction \beta)$. Then we let $\mathsf{E}_0(\beta)$ be undefined.*
   (b) *Otherwise let (F^*, F, ν) witness that $\mathcal{Q} \in \mathsf{E}^0$ with the additional property that F^* coheres $(\mathsf{J} \restriction \beta, \mathsf{E}_0 \restriction \beta, \mathsf{E}_1 \restriction \beta, \mathsf{B} \restriction \beta, \mathsf{Lim} \restriction \beta)$. Letting $\tilde{F}$ be the amenable code of F and $\mathcal{M} = (\mathcal{Q}, \tilde{F})$, set*

$$\mathsf{E}_0(\beta) = \begin{cases} \textit{undefined} & : \mathsf{Bad}(\mathcal{M}) \textit{ holds} \\ \mathcal{C}(\mathcal{M}) & : \textit{otherwise.} \end{cases}$$

Definition 10.2.36 (The second extender operator) *Suppose* $\mathsf{J} \restriction \beta$, $\mathsf{E}_0 \restriction \beta$, $\mathsf{E}_1 \restriction \beta$, $\mathsf{B} \restriction \beta$ *and* $\mathsf{Lim} \restriction \beta$ *have been defined,* $\beta = \gamma + 1$ *and* γ *is a limit ordinal. Let* $\mathcal{Q} = \mathsf{Lim}(\gamma)$.

1. *Suppose* $\mathcal{Q} \notin \mathsf{E}^1$. *Then let* $\mathsf{E}_1(\beta)$ *be undefined.*

2. *Suppose then that* $\mathcal{Q} \in \mathsf{E}^1$. *Let* F *witness that* $\mathcal{Q} \in \mathsf{E}^1$ *and set* $\mathcal{M} = (\mathcal{Q}, \tilde{F})$. *Then*

$$\mathsf{E}_1(\beta) = \begin{cases} \textit{undefined} & : \mathsf{Bad}(\mathcal{M}) \textit{ holds} \\ \mathcal{C}(\mathcal{M}) & : \textit{otherwise.} \end{cases}$$

We split the branch operator into three pieces $\mathsf{B}_{\mathsf{nlsa}}$, $\mathsf{B}_{\mathsf{ualsa}}$ and $\mathsf{B}_{\mathsf{alsa}}$. These respectively stand for *non lsa*, *unambiguous lsa* and *ambiguous lsa*. We then let $\mathsf{B} = \mathsf{B}_{\mathsf{nlsa}} \cup \mathsf{B}_{\mathsf{ualsa}} \cup \mathsf{B}_{\mathsf{alsa}}$. Suppose $\mathsf{J} \restriction \beta$, $\mathsf{E}_0 \restriction \beta$, $\mathsf{E}_1 \restriction \beta$, $\mathsf{B} \restriction \beta$ and $\mathsf{Lim} \restriction \beta$ have been defined, $\beta = \gamma + 1$ and γ is a limit ordinal. Let $\mathcal{Q} = \mathsf{Lim}(\gamma)$. The following condition is part of the definition of B.

($\mathsf{B1}$) Suppose $\mathcal{Q} \notin \mathsf{B}^0$. Then let $\mathsf{B}(\beta)$ be undefined.

Suppose then that $\mathcal{Q} = \mathcal{J}^{\vec{E},f}_{\xi+\nu} \in \mathsf{B}^0$ and $\mathcal{S} \in Y^{\mathcal{Q}}$ is least witnessing this. In the next three definitions, we will isolate a stack $\vec{\mathcal{T}}$ based on $\mathcal{S}$ and a branch b of $\vec{\mathcal{T}}$. Then letting $\tilde{b} \subseteq \xi + \nu$ be given by $\xi + \zeta \in \tilde{b} \leftrightarrow \zeta \in b$, set $f^+ = f \cup \{(trc(\vec{\mathcal{T}}), \tilde{b})\}$. If one of the following conditions is satisfied then we will let $\mathsf{B}(\beta)$ be undefined.

($\mathsf{B2}$) $\sup(b) \neq \nu$ or $\mathsf{Bad}(\mathcal{Q}, f^+)$.

Definition 10.2.37 (The non lsa branch operator) *Suppose one of the following holds.*

1. $\mathcal{S} \triangleleft \mathcal{R}$.

2. $\mathcal{S}$ *is not of lsa type.*

3. $\mathcal{S}$ *is of lsa type but* $\mathcal{J}_1(\mathcal{Q}) \vDash$ "$\delta^{\mathcal{S}}$ *is not a Woodin cardinal".*

Let $\vec{\mathcal{T}} \in \mathcal{Q} - dom(\Sigma_{\mathcal{S}}^{\mathcal{Q}})$ be the $\mathcal{Q}$-least stack that is according to $\Sigma_{\mathcal{S}}^{\mathcal{Q}}$, $lh(\vec{\mathcal{T}})$ is not of measurable cofinality in $\mathcal{Q}$,[7] and $\Sigma_{\mathcal{S}}^{\mathcal{Q}}(\vec{\mathcal{T}})$ is not defined. Set $b = \Lambda_{\mathcal{Q}}(\vec{\mathcal{T}})$. If $\mathsf{B2}$ *holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{nlsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{nlsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

The following condition is also part of the definition of B.

($\mathsf{B3}$) Suppose $\mathcal{S}$ is of lsa type and $\mathcal{J}_1(\mathcal{Q}) \vDash$ "$\delta^{\mathcal{S}}$ is a Woodin cardinal". If $\mathcal{Q}$ is not an sts premouse over $\mathcal{S}$ based on $\mathcal{M}^+(\mathcal{S}|\delta^{\mathcal{S}})$ or it is but it is not closed under sharps then let $\mathsf{B}(\beta)$ be undefined.

Suppose then $\mathcal{Q}$ is an sts premouse over $\mathcal{S}$ based on $\mathcal{M}^+(\mathcal{S}|\delta^{\mathcal{S}})$ and $\mathcal{Q}$ is closed under sharps.

Definition 10.2.38 (The ambiguous branch operator) *Suppose $\mathcal{Q}$ is ambiguous and let $t \in \mathcal{Q}$ be the $\mathcal{Q}$-least stack of length 2 witnessing this. Again since $\mathcal{Q} \in \mathsf{B}^0$, we can require $lh(t)$ is not of measurable cofinality in $\mathcal{Q}$. Let $\Lambda_{\mathcal{S}}^{stc}(t) = b$. If* $\mathsf{B2}$ *holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{alsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{alsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

Definition 10.2.39 (The unambiguous branch operator) *Suppose $\mathcal{Q}$ is unambiguous. Suppose there is no $\mathcal{Q}$-terminal $\mathcal{T}$ that has a $\mathcal{Q}$-shortness witness. Then let $\mathsf{B}(\beta)$ be undefined. Suppose then that there is a $\mathcal{Q}$-terminal $\mathcal{T}$ that has a $\mathcal{Q}$-shortness witness and $\mathcal{T}$ is chosen as in the definition of $\mathcal{Q} \in \mathsf{B}^0$. Let $(\mathcal{T}, b) \in \mathcal{Q}$ be the lexicographically $\mathcal{Q}$-least pair such that for some (ξ, ν), $\mathcal{T}$ is $\mathcal{Q}$-terminal and (ξ, ν, b) is a minimal $\mathcal{Q}$-shortness witness. If* $\mathsf{B2}$ *holds of $(b, \mathcal{Q}, f^+)$ then let $\mathsf{B}_{\mathsf{ualsa}}(\beta)$ be undefined. Otherwise set $\mathsf{B}_{\mathsf{ualsa}}(\beta) = \mathcal{C}(\mathcal{Q}, f^+)$.*

Finally set $\mathsf{B}(\beta) = \mathsf{B}_{\mathsf{nlsa}}(\beta) \cup \mathsf{B}_{\mathsf{ualsa}}(\beta) \cup \mathsf{B}_{\mathsf{alsa}}(\beta)$. Next we define the constructibility operator.

Definition 10.2.40 (The constructibility operator) *Suppose $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ and $\mathsf{Lim} \restriction \beta$ have been defined and $\beta = \gamma + 1$. Let*

$$\mathcal{Q} = \begin{cases} \mathsf{E}_0(\gamma) & : \gamma \in dom(\mathsf{E}_0) \\ \mathsf{E}_1(\gamma) & : \gamma \in dom(\mathsf{E}_1) \\ \mathsf{J}(\gamma) & : \gamma \in dom(\mathsf{J}) \\ \mathsf{B}(\gamma) & : \gamma \in dom(\mathsf{B}) \\ \mathsf{Lim}(\gamma) & : \gamma \in dom(\mathsf{Lim}) \end{cases}$$

[7] $\vec{\mathcal{T}}$ exists because $\mathcal{Q} \in \mathsf{B}^0$.

Then

$$\mathsf{J}(\beta) = \begin{cases} \textit{undefined} & : \beta \in \textit{dom}(\mathsf{E}) \cup \textit{dom}(\mathsf{B}) \\ \textit{undefined} & : \beta \notin \textit{dom}(\mathsf{E}) \cup \textit{dom}(\mathsf{B}) \text{ and } \mathsf{Bad}(\mathcal{Q}) \text{ holds} \\ \mathcal{J}_1(\mathcal{Q}) & : \textit{otherwise} \end{cases}$$

Finally we define the limit operator.

Definition 10.2.41 (The limit operator) *Suppose* $\mathsf{J} \restriction \beta$, $\mathsf{E} \restriction \beta$, $\mathsf{B} \restriction \beta$ *and* $\mathsf{Lim} \restriction \beta$ *have been defined and* β *is a limit ordinal. For* $\gamma < \beta$, *let*

$$\mathcal{Q}_\gamma = \begin{cases} \mathsf{E}_0(\gamma) & : \gamma \in \textit{dom}(\mathsf{E}_0) \\ \mathsf{E}_1(\gamma) & : \gamma \in \textit{dom}(\mathsf{E}_1) \\ \mathsf{J}(\gamma) & : \gamma \in \textit{dom}(\mathsf{J}) \\ \mathsf{B}(\gamma) & : \gamma \in \textit{dom}(\mathsf{B}) \\ \mathsf{Lim}(\gamma) & : \gamma \in \textit{dom}(\mathsf{Lim}) \end{cases}$$

Given an ordinal ξ, *we let* $\mathcal{Q}^\xi$ *be the eventual value of* $\mathcal{Q}_\gamma || \xi$ *as* γ *approaches* β *provided this eventual value exists. Then*

$$\mathsf{Lim}(\beta) = \begin{cases} \textit{undefined} & : \text{ for some } \xi,\ \mathcal{Q}^\xi \text{ is undefined} \\ \textit{undefined} & : \mathsf{Bad}(\cup_{\xi \in \mathsf{Ord}} \mathcal{Q}^\xi) \text{ holds} \\ \cup_{\xi \in Ord} \mathcal{Q}^\xi & : \textit{otherwise.} \end{cases}$$

Recall that we set

$$\alpha^{\mathcal{N}}_{def} = \sup\{\xi + 1 : \xi \in \textit{dom}(\mathsf{J}) \cup \textit{dom}(\mathsf{E}_0) \cup \textit{dom}(\mathsf{E}_1) \cup \textit{dom}(\mathsf{B}) \cup \textit{dom}(\mathsf{Lim})\}.$$

We then say $\mathcal{Q}$ appears at stage β if $\mathcal{Q}$ is the value of one of the construction operators at β. We let $\mathcal{Q}_\beta$ be this model and say that $(\mathcal{Q}_\beta, \Lambda_{\mathcal{Q}_\beta} : \beta < \alpha^{\mathcal{N}})$ are the models and strategies of the *mixed hod pair constructions* of $\mathcal{N}$ over $\mathcal{R}$. Here $\Lambda_{\mathcal{Q}_\beta}$ is the $\mathcal{E}$-certified strategy of $\mathcal{Q}_\beta$ (see Definition 10.2.18). The following two condition are our final conditions signaling the halt of the construction.

($\mathsf{Reaching\ LSA}$) If for some limit β, $\mathcal{Q}_\beta$ is of lsa type and $\mathcal{Q}_\beta = Lp^{\Gamma, \Lambda^{stc}_{\mathcal{Q}_\beta}}(\mathcal{M}^+(\mathcal{Q}_\beta | \delta^{\mathcal{Q}_\beta}))$ then stop the construction.
($\mathsf{No\ Strategy}$) If for some $\mathcal{Q}$ appearing in the construction $\Lambda_{\mathcal{Q}}$ is not total or $(\mathcal{Q}, \Lambda_{\mathcal{Q}}) \notin \mathcal{F}$ then stop the construction (see Notation 10.2.12).

Definition 10.2.42 (Mixed hod pair constructions) *The mixed hod pair construction of $\mathcal{N}$ over $\mathcal{R}$ is the sequence $(\mathsf{E}_0^{\mathcal{N}}, \mathsf{E}_1^{\mathcal{N}}, \mathsf{B}^{\mathcal{N}}, \mathsf{J}^{\mathcal{N}}, \mathsf{Lim}^{\mathcal{N}})$. We say that the hod pair construction is successful if $\alpha^{\mathcal{N}} = o(\mathcal{N})$. We say $\mathcal{Q}$ is a model appearing in the hod pair construction of $\mathcal{N}$ if for some $\beta < \alpha^{\mathcal{N}}$,*

$$\mathcal{Q} = \begin{cases} \mathsf{E}_0^{\mathcal{N}}(\beta) & : \beta \in dom(\mathsf{E}_0^{\mathcal{N}}) \\ \mathsf{E}_1^{\mathcal{N}}(\beta) & : \beta \in dom(\mathsf{E}_1^{\mathcal{N}}) \\ \mathsf{B}^{\mathcal{N}}(\beta) & : \beta \in dom(\mathsf{B}^{\mathcal{N}}) \\ \mathsf{J}^{\mathcal{N}}(\beta) & : \beta \in dom(\mathsf{J}^{\mathcal{N}}) \\ \mathsf{Lim}^{\mathcal{N}}(\beta) & : \beta \in dom(\mathsf{Lim}^{\mathcal{N}}) \end{cases}$$

10.2.10 The proof of Theorem 10.2.4

The following is the key step towards the proof of Theorem 10.2.4. We do the proof assuming that $\mathcal{R}^+$ is of lsa type. The remaining cases were either handled in [10] or are very similar and easier. Let Ψ be the minimal component of $\Sigma^{stc}_{\mathcal{R}^+}$ (see Definition 3.9.8).

Lemma 10.2.43 *There is a model $\mathcal{Q}$ appearing in the mixed hod pair construction of $\mathcal{N}$ and a normal tree $\mathcal{T}$ on $\mathcal{R}^+$ such that $(\mathcal{T}, \mathcal{Q}) \in I(\mathcal{R}^+, \Psi)$ and $\Lambda_{\mathcal{Q}} = \Psi_{\mathcal{Q}} \restriction \mathcal{N}$.*

Proof. Assume first that such a tree $\mathcal{T}$ exists. Then it follows from Lemma 10.2.31 that $\Lambda_{\mathcal{Q}} = \Psi_{\mathcal{Q}} \restriction \mathcal{N}$. Thus it is enough to show that such a $\mathcal{T}$ exists.

Suppose then $\mathcal{M}$ is some model appearing in the mixed hod pair construction of $\mathcal{N}$ such that for some normal tree $\mathcal{T}$ on $\mathcal{R}^+$ with last model $\mathcal{Q}$, $\mathcal{M} \lhd \mathcal{Q}$. Suppose further that in the next step of mixed hod pair construction, we either index an extender or a branch. It follows from Lemma 10.2.31 that if we index a branch then this cannot cause a disagreement between $\mathcal{Q}$ and the next model in the construction. It also follows from the stationarity of fully backgrounded constructions (see [10, Lemma 2.11]) that if the next indexed object is an extender with critical point $> \delta^{\mathcal{R}}$ then this too cannot cause a disagreement between $\mathcal{Q}$ and the next model.

Suppose then the next indexed object is an extender with critical point $\delta^{\mathcal{R}}$. We want to show that this also doesn't cause a disagreement between $\mathcal{Q}$ and the next model in the construction. Let F be this extender. It follows that F is $\mathcal{E}$-certified. Let $E \in \mathcal{E}$ be some extender witnessing that $\mathcal{F}$ is $\mathcal{E}$-realizable and let σ be the E-realizability map (see Definition 10.2.32). It follows $\sigma : \pi_F(\mathcal{R}) \to \pi_E(\mathcal{R})$ and $\pi_E \restriction \mathcal{R} = \sigma \circ \pi_F$. We moreover have that

(1) $\sigma \restriction (\pi_F(\mathcal{R}|\delta^{\mathcal{R}}))$ is the iteration embedding according to $\Psi_{\pi_F(\mathcal{R})} \restriction \mathcal{N}$.

(1) follows from the fact that this embedding just depends on $\sigma \restriction \mathcal{M}$ which is the iteration embedding according to $\Lambda_{\mathcal{M}} = \Psi_{\mathcal{M}} \restriction \mathcal{N}$.

We now have two cases. Suppose first that there is no $G \in \vec{E}^{\mathcal{Q}}$ such that $cp(G) = \delta^{\mathcal{R}}$ and $lh(G) \geq o(\mathcal{M})$. Then $\mathcal{Q} = \mathcal{M}^+(\mathcal{M})$ and $\mathcal{Q} \vDash$ "$\delta^{\mathcal{Q}}$ is a Woodin cardinal". This is a contradiction because we index extenders at successor cardinals (implying that $\mathcal{M}$ has a largest cardinal).

Suppose next that there is such an extender G. Let E^* be the background certificate of E and let $\tau : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the canonical factor map. Let $k : \pi_G(\mathcal{R}) \to \pi_{E^*}(\mathcal{R})$ be given by

$$k(x) = \pi_{E^*}(f)(\pi^{\Sigma_{\mathcal{R}}}_{\mathcal{R},\pi_{E^*}(\mathcal{R})}(a))$$

where $f \in \mathcal{R}$ and $a \in (\pi_G(\delta^{\mathcal{R}}))^{<\omega}$ are such that $x = \pi_G(f)(a)$. It follows from Lemma 9.1.9 that

(2) $\tau(\sigma)$-pullback of $\pi_{E^*}(\Sigma_{\mathcal{R}})$ is $\Sigma_{\pi_F(\mathcal{R})}$.

It follows from (1) that

(3) $\tau(\sigma) \restriction \pi_F(\mathcal{R}|\delta^{\mathcal{R}})$ is the iteration embedding according to $\tau(\Psi_{\pi_F(\mathcal{R})})$.

Combining (2) and (3), we get that $\tau(\sigma) = k$. It then follows that $F = G$. This finishes the proof that there is no disagreement between $\mathcal{Q}$ and the next model in the construction, provided we index an object at $\mathcal{M}$.

Next we analyze the situation when the next model in the construction is obtained from $\mathcal{M}$ by not indexing anything. This can cause a disagreement between the next model of the construction and $\mathcal{Q}$ provided there is an object indexed in $\mathcal{Q}$. As before, because $\Lambda_{\mathcal{M}} = \Psi_{\mathcal{M}} \restriction \mathcal{N}$, such a disagreement cannot happen because of strategy disagreements. We claim that such a disagreement cannot happen because of an extender with critical point $\delta^{\mathcal{Q}} = \delta^{\mathcal{R}}$.

Suppose then, toward a contradiction, that we have an extender $F \in \mathcal{E}^{\mathcal{Q}}$ such that $cp(F) = \delta^{\mathcal{R}}$. Suppose further that $\mathcal{M} = \mathcal{Q}|lh(F)$ and either $F \notin \mathcal{N}$ or F is not $\mathcal{E}$-certified (as otherwise we would have to put F on the sequence of our construction). Let λ be a δ_z-strong cardinal of $\mathcal{N}^*_z$ such that λ is also an $\mathcal{N}$-strong cardinal and $F, \mathcal{Q} \in \mathcal{N}^*_z|\lambda$. Let $E \in \mathcal{E}$ be any extender such that $\lambda < lh(E)$. Let E^* be the background certificate of E and let $\tau : Ult(\mathcal{N}, E) \to \pi_{E^*}(\mathcal{N})$ be the canonical factor map.

We define $k : \pi_F(\mathcal{R}) \to \pi_{E^*}(\mathcal{R})$ as above by setting Let $k : \pi_F(\mathcal{R}) \to \pi_{E^*}(\mathcal{R})$ be given by

$$k(x) = \pi_{E^*}(f)(\pi^{\Sigma_{\mathcal{R}}}_{\mathcal{R},\pi_{E^*}(\mathcal{R})}(a))$$

where $f \in \mathcal{R}$ and $a \in (\pi_F(\delta^{\mathcal{R}}))^{<\omega}$ are such that $x = \pi_F(f)(a)$. We have that

(4) $\Psi_{\pi_F(\mathcal{R})}$ is the k-pullback of $\pi_{E^*}(\Sigma_{\mathcal{R}})$ and $k \restriction (\pi_F(\mathcal{R}|\delta^{\mathcal{R}}))$ is the iteration embedding according to $\Psi_{\pi_F(\mathcal{R})}$.

Notice that (5) $\Psi_{\mathcal{M}} \restriction Ult(\mathcal{N}, E) \in Ult(\mathcal{N}, E)$.

(5) follows because λ is a strong cardinal in $Ult(\mathcal{N}, E)$. Because $\mathrm{crit}(\tau) > \lambda$, we have that

(6) $\tau(\Psi_{\mathcal{M}} \restriction Ult(\mathcal{N}, E)) \restriction (\pi_{E^*}(\mathcal{N})|\lambda) = \Psi_{\mathcal{M}} \restriction (\mathcal{N}|\lambda)$.

Again, because λ is a strong cardinal in all relevant models, it follows from (6) that

(7) $\Psi_{\mathcal{M}} \restriction \pi_{E^*}(\mathcal{N}) = \tau(\Psi_{\mathcal{M}} \restriction Ult(\mathcal{N}, E))$.

Because $k \restriction (\pi_F(\mathcal{R}|\delta^{\mathcal{R}}))$ depends only on $k \restriction \mathcal{M}$, it follows from (4) and (7) that

(8) $k \restriction \mathcal{M} \in rng(\tau)$.

It is now routine to check that $m =_{def} \tau^{-1}(k \restriction \mathcal{M}) \in Ult(\mathcal{N}, E)$ defined F as follows:

$$(a, A) \in F \leftrightarrow m(a) \in \pi_E(A).$$

This finishes the proof of Lemma 10.2.43. □

We have now finished proving Theorem 10.2.4.

10.2.11 A proof of Lemma 10.2.5

In this subsection we outline the proof of Lemma 10.2.5. The proof is very similar to the proof of [10, Lemma 6.23]. Suppose that there is no hod pair or an sts hod pair $(\mathcal{P}, \Sigma)$ such that

1. Σ has strong branch condensation and is strongly fullness preserving,

2. $\Gamma(\mathcal{P}, \Sigma) \subseteq \Gamma \subseteq L(\Sigma, \mathbb{R})$

Just like in the proof of [10, Lemma 6.23], it follows from Theorem 10.1.1 that Γ is not a mouse full pointclass (as we are assuming that $L_\alpha(\Gamma, \mathbb{R}) \vDash SMC$). Following the proof of [10, Lemma 6.23], we let A be the set of hod pairs or sts hod pairs $(\mathcal{P}, \Sigma)$ such that $Code(\Sigma) \in \Gamma$ and Σ has strong branch condensation and is strongly fullness preserving. It follows from Claim 1 on page 158 of [10] that $A \neq \emptyset$. It follows from Claim 2 on the same page of [10] that if

$$\Gamma_1 = \cup_{(\mathcal{P},\Sigma)\in A}\Gamma(\mathcal{P}, \Sigma)$$

then

(1) Γ_1 is a mouse full pointclass such that for some limit ordinal α there is a sequence of mouse full pointclasses $(\Gamma_\beta : \beta < \alpha)$ such that for $\beta < \gamma < \alpha$, $\Gamma_\beta \trianglelefteq_{mouse} \Gamma_\gamma$ and $\Gamma_1 = \cup_{\beta<\alpha}\Gamma_\beta$.

It follows from Theorem 10.1.1 that there is a possibly anomalous hod pair $(\mathcal{P}, \Sigma)$ such that either

1. $\mathcal{P}$ is of lsa type and $\Gamma^b(\mathcal{P}, \Sigma) = \Gamma_1$ or

2. $\mathcal{P}$ is not of lsa type and $\Gamma(\mathcal{P}, \Sigma) = \Gamma_1$.

Because $\Gamma \vDash SMC$ and because $\Gamma_1 \triangleleft_{mouse} \wp(\mathbb{R})$, we must have that Σ is strongly fullness preserving (for instance see [10, Lemma 6.21]). Notice that even if clause 1.b of Theorem 10.1.1 applies, we still get a hod pair as apposed to an sts pair. This is because we have good pointclasses beyond Γ.

Notice also that $Code(\Sigma) \notin \Gamma$, as otherwise it follows from Claim 2 on page 158 of [10] that $(\mathcal{P}, \Sigma) \in A$. Thus, it must be the case that $\mathcal{P}$ is an anomalous hod premouse. We now get a contradiction as in page 159 of [10], where it is argued that the computation of $\mathrm{HOD}^{L(\Sigma,\mathbb{R})}$ gives a contradiction.

10.3 A proof of LSA from large cardinals

In this section, we generalize [10, Theorem 6.26].

Theorem 10.3.1 *The theory* $\mathsf{AD}^+ + \mathsf{LSA} + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$ *is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals.*

Proof. Woodin showed that it is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals that there are divergent models of AD^+, i.e., there are sets of reals $A, B \subseteq \mathbb{R}$ such that $L(A,\mathbb{R}) \vDash \mathsf{AD}^+$, $L(B,\mathbb{R}) \vDash \mathsf{AD}^+$, $A \notin L(B,\mathbb{R})$ and $B \notin L(A,\mathbb{R})$. Moreover, his construction shows that we can assume that both $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ satisfy $\mathsf{MC}+\Theta = \theta_0$. Thus, we assume that such a pair of models exists.

Suppose towards a contradiction that there is no inner model satisfying $\mathsf{AD}^+ + \mathsf{LSA} + \mathsf{V} = \mathsf{L}(\wp(\mathbb{R}))$. Let $\Gamma = L(A,\mathbb{R}) \cap L(B,\mathbb{R}) \cap \wp(\mathbb{R})$. It is unpublished theorem of Woodin that $L(\Gamma,\mathbb{R}) \vDash AD_{\mathbb{R}}$. We also have that $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma,\mathbb{R})$. Applying Lemma 10.1.1 in $L(A,\mathbb{R})$ and in $L(B,\mathbb{R})$ we get two hod pairs or sts hod pairs $(\mathcal{P},\Sigma) \in L(A,\mathbb{R})$ and $(\mathcal{Q},\Lambda) \in L(B,\mathbb{R})$ such that both $\mathcal{P}$ and $\mathcal{Q}$ are of limit type and $\Gamma = \Gamma(\mathcal{P},\Sigma) = \Gamma(\mathcal{Q},\Lambda)$.

Let $\mathcal{M}^* = \cup_{(\mathcal{S},\Psi)\in B(\mathcal{P},\Sigma)}\mathcal{M}_\infty(\mathcal{S},\Psi)$ and for $\alpha < \lambda^{\mathcal{M}^*}$ let Ψ_α be the iteration strategy of $\mathcal{M}^*(\alpha)$ obtained from any $(\mathcal{S},\Psi)$ such that $\mathcal{M}^*(\alpha) = \mathcal{M}_\infty(\mathcal{S},\Psi)$. Notice that $\mathcal{M}^*$ and Ψ_α are independent of $(\mathcal{P},\Sigma)$; using $(\mathcal{Q},\Lambda)$ instead of $(\mathcal{P},\Sigma)$ yields the same model $\mathcal{M}^*$ and the same strategy Ψ_α. Let

$$\mathcal{M}_A = (Lp^{\oplus_{\alpha<\lambda^{\mathcal{M}^*}}\Psi_\alpha}(\mathcal{M}^*))^{L(A,\mathbb{R})} \text{ and } \mathcal{M}_B = (Lp^{\oplus_{\alpha<\lambda^{\mathcal{M}^*}}\Psi_\alpha}(\mathcal{M}^*))^{L(B,\mathbb{R})}.$$

We then have that either $\mathcal{M}_A \trianglelefteq \mathcal{M}_B$ or $\mathcal{M}_B \trianglelefteq \mathcal{M}_A$. Without loss of generality we assume that $\mathcal{M}_A \trianglelefteq \mathcal{M}_B$.

Let $\pi : \mathcal{P}^b \to \mathcal{M}$ be the iteration embedding given by Σ. It follows from the proof of Claim 7 appearing in the proof of Theorem 8.2.6 that $\Sigma \in L(\pi[\mathcal{P}],\mathcal{M},\Gamma)$. However, because $\pi[\mathcal{P}]$ is a countable set we have that $\pi[\mathcal{P}] \in L(B,\mathbb{R})$. It follows that $\Sigma \in L(B,\mathbb{R})$. Therefore, $Code(\Sigma) \in \Gamma$ implying that $\Gamma(\mathcal{P},\Sigma) \subset \Gamma$, contradiction! $\square$

Chapter 11

A proof of square in lsa-small hod mice

Definition 11.0.2 *For a cardinal κ and a cardinal $\gamma \leq \kappa$, the principle $\Box_{\kappa,\gamma}$ states that there is a sequence $\langle \vec{C}_\alpha : \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$*

1. *$\vec{C}_\alpha \neq \emptyset$ and for each $C \in \vec{C}_\alpha$, C is a closed unbounded subset of α of order type at most κ,*

2. *$|\vec{C}_\alpha| \leq \gamma$,*

3. *for each $C \in \vec{C}_\alpha$, for each $\beta \in \lim(C_\alpha)$, $C \cap \beta \in \vec{C}_\beta$.*

If $\gamma = 1$, then the principle $\Box_{\kappa,\gamma}$ is simply $\Box_\kappa$.

Pure extender models are models constructed from a canonical sequence of extenders. Jensen (cf. [4]) initiated the program of understanding square principles in pure extender models by proving $L \vDash \forall\kappa\ \Box_\kappa$. Building on works of several people, Schimmerling and Zeman (cf. [17]) give the most optimal characterization of $\Box$ in (short) extender models, namely they prove that in an iterable, short extender model, $\Box_\kappa$ holds if and only if κ is not subcompact. Results on squares in extender models are important in understanding structure theory of such models and have found many applications in set theory. The reader can see, for instance, [16] and [5], for some of the applications of square in extender models in computing lower-bound consistency strength of theories like PFA. Recent advances in the core model induction methods have indicated that to improve the lower-bounds of combinatorial principles like PFA, failure of square at a singular cardinal, the existence of guessing models etc., one way is to prove square holds in the hod mice that are currently being studied and constructed.

All known square proofs in extender models rely heavily on the fine-structure of such models, in particular, they make essential use of condensation properties of these models (cf. [17, Lemma 1.6]). Unfortunately, the full condensation lemma,[17, Lemma 1.6], does not hold in hod mice. However, it is possible to overcome this shortcoming. We present here a proof of $\square_{\kappa,2}$ in an *lsa-small hod mouse* $\mathcal{P}$ for all cardinal κ of $\mathcal{P}$. In this chapter by lsa-small hod mouse, we mean that $\mathcal{P}$ does not contain an active ω Woodin lsa mouse as defined in Definition 8.2.2.

We first set up some terminology. Our hod premice $\mathcal{P}$ are *lsa-small* and hence for no $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha)$ is an lsa hod premouse. Throughout this paper, if $\mathcal{Q}$ is an initial segment of $\mathcal{P}$, we let $\Sigma_{\mathcal{Q}}$ denote the restriction of Σ to $\mathcal{Q}$. If $\mathcal{P}$ is of limit type and has a top window $[\delta_\alpha^{\mathcal{P}}, \delta_{\alpha+1}^{\mathcal{P}})$, then we let $\mathcal{P}^b = \mathcal{P}|({\delta_\alpha^{\mathcal{P}}}^+)^{\mathcal{P}}$. See Section 11.1 for a more detailed discussion of hod mice along with the definitions used in statements of this section. In the definitions below, we adapt the Σ^*-language (see [17]) to hod mice in the obvious way. Let $\rho_{\mathcal{Q}}^n$ be the n^{th}-projectum of $\mathcal{Q}$, and $p_{\mathcal{Q}}^n$ be the n^{th}-standard parameter of $\mathcal{Q}$.[1] Semantically, suppose $\mathcal{Q}$ is an initial segment of $\mathcal{P}$, a relation $A \subset |\mathcal{Q}|$ is $\Sigma_l^{(n)}(\mathcal{Q})$ from p, or $\boldsymbol{\Sigma}_l^{(n)}(\mathcal{Q})$, if it is Σ_l from p (or $\boldsymbol{\Sigma}_l$) over the n^{th}-reduct $\langle H_{\mathcal{Q}}^n, A_{\mathcal{Q}}^n\rangle$ of $\mathcal{Q}$, where $H_{\mathcal{Q}}^n = |\mathcal{Q}|\rho_{\mathcal{Q}}^n|$ and $A_{\mathcal{Q}}^n$ is the n^{th} standard master code (with respect to $p_{\mathcal{Q}}^n$) of $\mathcal{Q}$.

Definition 11.0.3 *Suppose Σ is an iteration strategy for a hod premouse $\mathcal{P}$. Suppose Γ is an inductive-like pointclass. We say that Σ is locally strongly Γ-fullness preserving if Σ is Γ-fullness preserving and if $\mathcal{P}$ is of limit type with a top window and whenever $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{P}, \Sigma)$, and*

$$\pi^{\vec{\mathcal{T}},b} : \mathcal{P}^b \to \mathcal{S}^b \text{ exists,}$$

then letting $\pi = \pi^{\vec{\mathcal{T}},b}$, whenever $\mathcal{S}^b \lhd \mathcal{W} \unlhd \mathcal{S}$ is such that for some n and some cardinal κ of $\mathcal{W}$,

$$o(\mathcal{S}^b) \leq \omega\rho_{\mathcal{W}}^{n+1} \leq \kappa < \omega\rho_{\mathcal{W}}^n,$$

and $\tau : \mathcal{R} \to \mathcal{W}$ is cardinal preserving and $\Sigma_0^{(n)}$ and $\omega\rho_{\mathcal{R}}^n > cr(\tau) \geq \omega\rho_{\mathcal{R}}^{n+1} = \omega\rho_{\mathcal{W}}^{n+1}$, then the τ-pullback of the strategy $\Sigma_{\mathcal{W},\vec{\mathcal{T}}}$ is Γ-fullness preserving.

[1] Other notations for the n^{th}-projectum and n^{th}-standard parameter of $\mathcal{Q}$ used elsewhere in this book are $\rho_n(\mathcal{Q})$ and $p_n(\mathcal{Q})$ respectively. For this chapter, we stick to the more compact notations $\rho_{\mathcal{Q}}^n$ and $p_{\mathcal{Q}}^n$.

Definition 11.0.4 *Suppose Σ is an iteration strategy for a hod premouse $\mathcal{P}$. We say that Σ has locally strong branch condensation if Σ has branch condensation and if $\mathcal{P}$ is of limit type with a top window and κ is a cardinal of $\mathcal{P}$ such that*

$$o(\mathcal{P}^b) \leq \kappa,$$

and $\mathcal{Q}$ is such that $\mathcal{P}^b \lhd \mathcal{Q} \unlhd \mathcal{P}$, and n is such that $\omega\rho_{\mathcal{Q}}^{n+1} \leq \kappa < \omega\rho_{\mathcal{Q}}^{n}$, and $\mathcal{S}$ is a $\Sigma_{\mathcal{Q}}$-iterate along a stack $\vec{\mathcal{T}}$ such that $\pi^{\vec{\mathcal{T}},b}$ exists, and $\tau : \mathcal{Q} \to \mathcal{R}$ is a cardinal preserving, $\Sigma_0^{(n)}$-embedding such that $\mathcal{R} \unlhd \mathcal{S}$ and $(\mathcal{Q}^*)^b = \mathcal{R}^b$ for some non-dropping $\Sigma_{\mathcal{Q}}$-iterate $\mathcal{Q}^*$ of $\mathcal{Q}$. Suppose also that letting $j : \mathcal{Q} \to \mathcal{Q}^*$ be the iteration map, then $j \restriction \mathcal{Q}^b = \tau \restriction \mathcal{Q}^b$. Then $\Sigma^{\tau}_{\mathcal{R},\vec{\mathcal{T}}} = \Sigma_{\mathcal{Q}}$.

We seem to need to strengthen the usual notions of fullness preservation and branch condensation (as in Definitions 11.0.3 and 11.0.4) to ensure that various phalanx comparison arguments go through in the proof of Theorem 11.0.5. In most (but not all) applications, the map π in Definitions 11.0.3 and 11.0.4, is the identity and τ is the uncollapse map associated to a sufficiently elementary hull. The main theorem is the following.

Theorem 11.0.5 *Suppose $(\mathcal{P}, \Sigma)$ is an lsa-small hod pair such that Σ has locally strong branch condensation and is locally strongly Γ-fullness preserving for some inductive-like pointclass Γ that satisfies "$\mathsf{AD}^+ + \mathsf{SMC}$". Then $\mathcal{P} \vDash \forall\kappa\ \square_{\kappa,2}$.*[2]

Many techniques in the proof of 11.0.5 come from the Schimmerling-Zeman's proof in [17]. In Section 11.1, we import some results from the theory of hod mice we need. In Section 11.2, we will import some terminology, results from [17] that we need here. We also explain in this section why a straightforward adaptation of [17] fails in the context of hod mice. In Section 11.3, we give the actual proof of Theorem 11.0.5.

Finally, we remark that hod pairs constructed in practice (those constructed in sufficiently strong AD^+ models or in the core model induction settings) do have the properties in the hypothesis of Theorem 11.0.5. The main application of Theorem 11.0.5 in this book is to improve the lower-bound consistency strength of various theories such as PFA to that of LSA (see Chapter 12).

[2] The assumption that $\mathcal{P}$ is lsa-small implies that there are no subcompact cardinals in $\mathcal{P}$ and all extenders on the $\mathcal{P}$-sequence are short.

11.1 Ingredients from hod mice theory

We summarize some definitions and results of the hod mice theory developed above that we need to prove Theorem 11.0.5. Suppose $(\mathcal{P}, \Sigma)$ is an lsa-small hod pair. $\mathcal{P}$ is constructible from a sequence of extenders and a sequence of strategies of its own initial segments. There are two ways in which an initial segment $\mathcal{Q}$ of $\mathcal{P}$ can be active: B-active and E-active. $\mathcal{Q}$ is *B-active* the top predicate for $\mathcal{Q}$ (amenably) codes a branch for some tree on an initial segment of $\mathcal{Q}$. $\mathcal{Q}$ is *E-active* if the top predicate of $\mathcal{Q}$ codes an extender. Otherwise, we say that $\mathcal{Q}$ is *passive*. B-active levels and passive levels are more or less treated the same way in the proof of Theorem 11.0.5.

A few words about how the B-predicate codes up branches for an iteration tree $\mathcal{T}$ in $\mathcal{P}$ is in order. Suppose $\lambda = \mathrm{lh}(\mathcal{T})$ is limit and $\mathcal{P}|\gamma$ is B-active such that $B^{\mathcal{P}|\gamma}$ codes a cofinal branch b of $\mathcal{T}$. The traditional way that B codes b is that letting $\gamma^* + \lambda = \gamma$, $B^{\mathcal{P}|\gamma} = \{\gamma^* + \alpha \mid \alpha \in b\}$. While this approach is sufficient for developing the basic theory of strategic premice and certainly is sufficient for the theory of hod mice we have developed so far, it seems to create significant obstructions in the proof $\square$ in this chapter. So instead, we use the coding method developed in [20]. Using [20, Definition 2.26], we let $\mathcal{P}|\gamma = \mathfrak{B}(\mathcal{P}|\gamma^*, \mathcal{T}, b)$. The reader is advised to consult [20] for the precise definition of $\mathfrak{B}(\mathcal{P}|\gamma^*, \mathcal{T}, b)$. Roughly, for every $0 < \alpha < \lambda$, $\mathcal{P}|(\gamma^* + \omega\alpha)$ is B-active and $B^{\mathcal{P}|(\gamma^*+\omega\alpha)}$ codes the branch $[0, \alpha)_{\mathcal{T}}$ and $B^{\mathcal{P}|\gamma}$ codes b in the manner described above. The use of the $\mathfrak{B}$-operator in coding branches of iteration trees will be explained in Section 11.3.

We briefly discuss indexing schemes for extenders on the $\mathcal{P}$-sequence. Suppose κ is a cardinal limit of cutpoints of $\mathcal{P}$, and if E is an extender on the $\mathcal{P}$'s sequence such that $\mathrm{cr}(E) = \kappa$, then the index of E is γ where γ is the successor cardinal of the least cutpoint above κ in $\mathrm{Ult}(\mathcal{P}, E)$ (we call this *cutpoint indexing scheme*). It turns out that such extenders are all total over $\mathcal{P}$. Suppose E is an extender with critical point ξ and E is indexed according to the cutpoint indexing scheme. Then according to [22], for all $\gamma < \mathrm{lh}(E)$, $E \restriction \gamma$ is not on the $\mathcal{P}$-sequence, though $E \restriction \gamma \in \mathcal{P}$ (for γ below the sup of the generators of E) and the trivial completion of $E \restriction \gamma$ is on the $\mathcal{P}$-sequence for various γ (this is similar to the initial segment condition for Jensen indexing). Also, the set of indices of extenders with a fixed critical point ξ indexed according to the cutpoint indexing scheme is nowhere continuous. For other extenders on $\mathcal{P}$'s sequence, we use *the Jensen indexing scheme* (this is for convenience of importing terminology and results from [17]; the result we prove here for hod mice with the Jensen indexing scheme will also hold for hod mice with the Mitchell-Steel indexing scheme by results of [3]). Suppose E is an extender with critical point ξ on the sequence of $\mathcal{P}$ and E is indexed by the Jensen indexing scheme, that is the index of E in $\mathcal{P}$ is the successor cardinal of $i_E(\xi)$ in $\mathrm{Ult}(\mathcal{P}, E)$. For a summary of the fine structure, see [17, Section 1]. A couple of remarks regarding the adaptation of [17, Section 1] into our situation are in order. First, we still demand extenders indexed according to the Jensen indexing to satisfy the *initial segment condition* (ISC) in the sense of [17, Section 1.4]; that is for all $\gamma < \mathrm{lh}(E)$, if γ is a cutpoint of E, then $E \restriction \gamma \in |\mathcal{P}|\mathrm{lh}(E)|$. Secondly, under this initial segment condition, using the assumption that our hod premice are lsa-small, it's easy to see that these extenders E are all of type A, that is the set of cutpoints is empty; this is because there are no superstrong cardinals in lsa-small hod mice. The initial segment condition (for both indexing schemes) is needed to prove comparisons terminate.

If $\mathcal{P}$ is a hod premouse, we let $\lambda^{\mathcal{P}}$ denote the order type of the Woodin cardinals of $\mathcal{P}$ and $(\delta^{\mathcal{P}}_\alpha : \alpha < \lambda^{\mathcal{P}})$ enumerate the closure of the set of Woodin cardinals in $\mathcal{P}$. If $\mathcal{P}$ has a largest Woodin cardinal, we denote that $\delta^{\mathcal{P}}$. Recall we use $\mathcal{P}^b$ to denote the "bottom part" of $\mathcal{P}$ in the case that $\mathcal{P}$ has a top window $[\kappa = \delta^{\mathcal{P}}_\alpha, \delta^{\mathcal{P}})$, where κ is either a Woodin or limit of Woodins in $\mathcal{P}$. In this case, $\mathcal{P}^b = \mathrm{Lp}^{\Sigma^{\mathcal{P}}_\kappa, \mathcal{P}}(\mathcal{P}|\kappa)$, where $\Sigma^{\mathcal{P}}_\kappa = \oplus_{\beta<\alpha}\Sigma^{\mathcal{P}}_{\mathcal{P}(\beta)}$. In the case α is a limit ordinal, $\mathcal{P}^b = \mathcal{P}|((\kappa)^+)^{\mathcal{P}}$. In this case, if κ happens to be measurable in $\mathcal{P}$, then all extenders E on the $\mathcal{P}$ sequence with critical point κ are indexed according to the cutpoint indexing scheme. Notice that since $\mathcal{P}$ is lsa-small, κ is a cutpoint (but not a strong cutpoint) in $\mathcal{P}$, though κ is a strong cutpoint in $\mathcal{P}^b = \mathcal{P}|(\kappa^+)^{\mathcal{P}}$. Let $o(\kappa)$ be the supremum of the indices of extenders on the $\mathcal{P}$ sequence with critical point κ. If $\mathcal{P}$ is of *limit type* ($\kappa < o(\kappa)$) or of *lsa* type ($\kappa < o(\kappa) = \delta^{\mathcal{P}}$), then there may be local large cardinals in the interval $(\kappa, o(\kappa))$, e.g. there may be a $\gamma \in (\kappa, o(\kappa))$ which is Woodin in some initial segment $\mathcal{Q}$ of $\mathcal{P}$; such large cardinals are witnessed by the extender sequence and *the short tree strategy* of initial segments of $\mathcal{Q}$, but not the full strategy. This point is crucial in many arguments given below (see Lemma 11.1.1).

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is Γ-fullness preserving for some inductive-like pointclass and has branch condensation. Suppose $\mathcal{R} \lhd \mathcal{P}$ is an initial segment of $\mathcal{P}$, then we let $\Sigma_{\mathcal{R}}$ denote the restriction of Σ to trees based on $\mathcal{R}$. Let $I(\mathcal{P}, \Sigma)$ denote the set of $(\vec{\mathcal{T}}, \mathcal{R})$ where $\vec{\mathcal{T}}$ is a stack according to Σ with last model $\mathcal{R}$. In this case, the "$\vec{\mathcal{T}}$-tail" of Σ, denoted $\Sigma_{\vec{\mathcal{T}},\mathcal{R}}$, is a strategy for $\mathcal{R}$. We let $B(\mathcal{P}, \Sigma)$ denote the set of $(\vec{\mathcal{T}}, \mathcal{R})$ where $\vec{\mathcal{T}}$ is according to Σ and $\mathcal{R}$ is a strict hod-initial segment of $\mathcal{N}^{\vec{\mathcal{T}}}$, the last model of $\vec{\mathcal{T}}$. We let $\Gamma(\mathcal{P}, \Sigma)$ be the set of $A \subseteq \mathbb{R}$ such that $A <_w \Sigma_{\vec{\mathcal{T}},\mathcal{R}}$ for some $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{P}, \Sigma)$. Note that $\Gamma(\mathcal{P}, \Sigma)$ is a Wadge initial segment of Γ. We say that $\mathcal{P}$ *generates* Ω if $\Gamma(\mathcal{P}, \Sigma) = \Omega$.

The following fact will be used in many places throughout this chapter, and whose proof is essentially that of 4.9.2.

Lemma 11.1.1 (No strategy disagreement) *Suppose $(\mathcal{P}, \Sigma)$ is an lsa-small hod pair such that $\mathcal{P}$ has a top window $[\delta_\alpha^{\mathcal{P}}, \delta^{\mathcal{P}})$ and $\delta_\alpha^{\mathcal{P}}$ is not a strong cutpoint of $\mathcal{P}$, Σ has locally strong branch condensation and is locally strongly Γ-fullness preserving for some constructibly closed pointclass $\Gamma \vDash$ "$\mathsf{AD}^+ + \mathsf{SMC}$". Suppose $\pi : \mathcal{P}' \to \mathcal{P}^*$ for some cardinal preserving, $\Sigma_0^{(n)}$ map π such that $\mathcal{P}^b \lhd \mathcal{P}^* \unlhd \mathcal{P}$, and $\omega\rho^n_{\mathcal{P}^*} > cr(\pi) =_{\text{def}} \gamma > \omega\rho^{n+1}_{\mathcal{P}'} = \omega\rho^{n+1}_{\mathcal{P}^*} \geq o(\mathcal{P}^b)$ and $\rho^{n+1}_{\mathcal{P}^*}$ is a cardinal of $\mathcal{P}$. Then letting $\Lambda = \Sigma^\pi_{\mathcal{P}^*}$, the comparison of the phalanx $(\mathcal{P}^*, \mathcal{P}', \gamma)$ (using Λ) versus $\mathcal{P}^*$ (using $\Sigma_{\mathcal{P}^*}$) does not involve disagreements of strategies.*

Lemma 11.1.1 is useful since it reduces such comparisons to ordinary extender comparisons. Such phalanx comparisons will appear in many places in the proof of Theorem 11.0.5. A corollary of this is the following version of the Condensation Lemma for hod mice (cf. [35, Lemma 9.3.2]). For notations used in the statement of the lemma, see [17, Section 1.3].

Theorem 11.1.2 *Suppose $(\mathcal{P}, \Sigma)$ is an lsa-small hod pair such that $\mathcal{P}$ has a top window $[\delta_\alpha^{\mathcal{P}}, \delta^{\mathcal{P}})$ and $\delta_\alpha^{\mathcal{P}}$ is not a strong cutpoint of $\mathcal{P}$, Σ has locally strong branch condensation and is locally strongly Γ-fullness preserving for some constructibly closed pointclass $\Gamma \vDash$ "$\mathsf{AD}^+ + \mathsf{SMC}$". Suppose $\mathcal{P}^b \lhd \mathcal{M} \unlhd \mathcal{P}$, $\tilde{\mathcal{M}}$ is a hod premouse, and $\sigma : \tilde{\mathcal{M}} \to \mathcal{M}$ is a cardinal preserving and $\Sigma_0^{(n)}$ embedding such that $\sigma \restriction \omega\rho^{n+1}_{\tilde{\mathcal{M}}} = \mathrm{id}$, where $\omega\rho^{n+1}_{\tilde{\mathcal{M}}} = \omega\rho^{n+1}_{\mathcal{M}} \geq o(\mathcal{P}^b)$ is a cardinal of $\mathcal{P}$.[3][4] Then $\tilde{\mathcal{M}}$ is solid and $p_{\mathcal{M}}$ is k-universal for all $k \in \omega$. Furthermore, if $\tilde{\mathcal{M}}$ is sound above $\nu = \mathrm{cr}(\sigma)$ the one of the following holds:*

(a) $\tilde{\mathcal{M}} = \mathcal{M}$ and $\sigma = \mathrm{id}$.

(b) $\tilde{\mathcal{M}}$ is a proper initial segment of $\mathcal{M}$.

(c) $\tilde{\mathcal{M}} = \mathrm{Ult}^*(\mathcal{M}||\eta, \mathrm{E}^{\mathcal{M}}_\alpha)$ where $\nu \leq \eta < o(\mathcal{M})$, $\alpha \leq \omega\eta$ and $\nu = (\kappa^+)^{\mathcal{M}||\eta}$ where $\kappa = \mathrm{cr}(E^{\mathcal{M}}_\alpha)$; moreover, $E^{\mathcal{M}}_\alpha$ has a single generator κ.

(d) $\tilde{\mathcal{M}}$ is a proper initial segment of $\mathrm{Ult}(\mathcal{M}, E^{\mathcal{M}}_{\mathrm{cr}(\sigma)})$.

Remark 11.1.3 *If δ_α is a strong cutpoint of $\mathcal{P}$, then it follows simply from the definition of hod premice that for all $\kappa \in [\delta_\alpha, \delta)$, $\square_\kappa$ holds in $\mathcal{P}$; this is because $\mathcal{P}$ is a $\Sigma^{\mathcal{P}}_\alpha$-premouse ($\Sigma_\alpha$ is the strategy for $\mathcal{P}(\alpha)$) and the $\square$ proof of [17] adapts*

[3] In the Mitchell-Steel language, one requires σ to be a weak n-embedding such that $\sigma'' T^{\bar{\mathcal{M}}}_n \subseteq T^{\mathcal{M}}_n$.

[4] If $\omega\rho^{n+1}_{\tilde{\mathcal{M}}} = \omega\rho^{n+1}_{\mathcal{M}} = o(\mathcal{P}^b)$, then since $o(\mathcal{P}^b)$ is a cardinal of $\mathcal{P}$, $\mathrm{cr}(\sigma) > o(\mathcal{P}^b)$. Equality can happen in other cases.

straightforwardly. On the other hand, if δ_α is not a strong cutpoint of $\mathcal{P}$, then Theorem 11.1.2 is false if one required that the embedding σ have critical point δ_α. This is because of the fact that no partial extenders of critical point δ_α are indexed on the interval $[\delta_\alpha, \delta_\alpha^+]$ in $\mathcal{P}$.

The proof of the theorem is essentially that of [35, Theorem 9.3.2]. The idea is one compares the phalanx $(\mathcal{M}, \tilde{\mathcal{M}}, \mathrm{cr}(\sigma))$ against $\mathcal{M}$. Depending on how the comparison terminates, one gets one of the four possibilities in the statement of the theorem. Using locally strong Γ-fullness preservation and the fact that $\mathrm{cr}(\sigma) > \omega\rho_{\mathcal{M}}^{n+1}$, Lemma 11.1.1 shows that the comparison is an extender comparison (no strategy disagreements are encountered). This puts us the in the situation to apply the proof of [35, Theorem 9.3.2] (the Dodd-Jensen-like property we assume as part of locally strong branch condensation is enough to carry out the proof of [35, Theorem 9.3.2]). To illustrate the main ideas, we present a proof of a special case, which often shows up in the $\Box$-constructions.

Proof.[Proof of a special case] We assume $\tilde{\mathcal{M}}$ is sound. Let $\tilde{\tau} = \mathrm{cr}(\sigma)$ and let $\tau = \sigma(\tilde{\tau})$. We further assume that: letting $\kappa = \omega\rho_{\mathcal{M}}^{n+1}$, $\tau = (\kappa^+)^{\mathcal{M}}$, and hence $\tilde{\tau} = (\kappa^+)^{\tilde{\mathcal{M}}}$. In this case, we prove that $\tilde{\mathcal{M}} \lhd \mathcal{M}$. The reader can see [35, Theorem 9.3.2] for the full argument.

Claim 11.1.4 *Let $\Lambda = \Sigma_{\mathcal{M}}^{\sigma}$. Then the comparison of the phalanx $(\mathcal{M}, \tilde{\mathcal{M}}, \tilde{\tau})$ and $\mathcal{M}$ using Λ and $\Sigma_{\mathcal{M}}$ respectively is successful. Furthermore, the main branch on the phalanx side doesn't drop (in model or degree) and is above $\tilde{\mathcal{M}}$, and the $\mathcal{M}$ side doesn't move.*

Proof. Using strong fullness preservation of Σ, Λ is fullness preserving; so the comparison can be carried out. By Lemma 11.1.1, the comparison is an extender comparison (no strategy disagreements show up in the comparison). Now we use strong branch condensation to prove the claim. The proof is a fairly standard argument. Let $\mathcal{T}$ and $\mathcal{U}$ be the trees on $(\mathcal{M}, \tilde{\mathcal{M}}, \tilde{\tau})$ and $\mathcal{M}$ respectively that are generated by the comparison (via Λ and $\Sigma_{\mathcal{M}}$ respectively). The comparison terminates successfully with $\mathcal{Q}$ being the last model of $\mathcal{T}$ and $\mathcal{S}$ being the last model of $\mathcal{U}$.

Let $\sigma\mathcal{T}$ be the copy tree and $\sigma^* : \mathcal{Q} \to \mathcal{Q}^*$ be the copy map, where $\mathcal{Q}^*$ is the last model of $\sigma\mathcal{T}$. Note then that $\sigma\mathcal{T}$ is via $\Sigma_{\mathcal{M}}$.

Suppose $\mathcal{Q}$ is above $\mathcal{M}$. We prove this case is impossible. Suppose $\mathcal{Q} \lhd \mathcal{S}$ and hence the branch embedding $\pi^{\mathcal{T}}$ exists. Note that $(\mathcal{Q}^*)^b \lhd \mathcal{Q}$ and $\mathcal{Q}^*$ is a non-dropping $\Sigma_{\mathcal{M}}$ iterate. Hence by strong branch condensation,

$$\Sigma_{\mathcal{Q},\mathcal{T}}^{\pi^{\mathcal{T}}} = \Sigma_{\mathcal{M}}.$$

The usual Dodd-Jensen argument yields a contradiction. The main point is that the tree $\pi^{\mathcal{T}}\mathcal{U}$ is via $\Sigma_{\mathcal{Q},\mathcal{U}}$.

Suppose now $\mathcal{S} \lhd \mathcal{Q}$ and hence the branch embedding $\pi^{\mathcal{U}}$ exists. Note then that $\sigma^*(\mathcal{S}) \lhd \mathcal{Q}^*$. Again, by strong branch condensation,

$$\Sigma^{\sigma^* \restriction \mathcal{S} \circ \pi^{\mathcal{U}}}_{\sigma^*(\mathcal{S}),\sigma\mathcal{T}} = \Sigma_{\mathcal{M}}.$$ [5]

The usual Dodd-Jensen argument then yields a contradiction. The main point is that $(\sigma^* \restriction \mathcal{S} \circ \pi^{\mathcal{U}})\sigma\mathcal{T}$ is according to $\Sigma_{\mathcal{Q}^*,\sigma\mathcal{T}}$.

The above arguments easily give us that: $\mathcal{Q} = \mathcal{S}$ and $\pi^{\mathcal{T}}$, $\pi^{\mathcal{U}}$ both exist and they are equal. We can then find a pair of extenders (E, F) used on $\mathcal{T}$ and $\mathcal{U}$ respectively such that E and F are compatible. By a standard argument, this is not possible.

Hence $\mathcal{Q}$ is on the main branch above $\tilde{\mathcal{M}}$. Note then that if $\pi^{\mathcal{T}}$ exists, then $\mathrm{cr}(\pi^{\mathcal{T}}) > \tilde{\tau}$. Say b is the main branch of $\mathcal{T}$. Then b cannot drop (in model or degree) as otherwise, we have $\mathcal{S} \lhd \mathcal{Q}$ and $\pi^{\mathcal{U}}$ exists. As before, $\sigma^*(\mathcal{S}) \lhd \mathcal{Q}^*$ and $\Sigma_{\mathcal{M}} = \Sigma^{\pi^{\mathcal{U}} \circ \sigma^* \restriction \mathcal{S}}_{\mathcal{Q}^*,\sigma\mathcal{T}}$. We get a contradiction as before.

So b doesn't drop. Since $\tilde{\mathcal{M}}$ is κ-sound, $\rho^{\omega}_{\tilde{\mathcal{M}}} = \kappa < \tilde{\tau}$ and the branch b is above $\tilde{\tau}$ and does not drop in model or degree, we get that $b = \emptyset$. And hence $\mathcal{Q} = \tilde{\mathcal{M}}$. Now it's not the case that $\mathcal{S}$ is a strict segment of $\mathcal{Q} = \tilde{\mathcal{M}}$; otherwise, $\pi^{\mathcal{U}}$ exists and

$$\sigma^* \circ \pi^{\mathcal{U}} : \mathcal{M} \to \sigma^*(\mathcal{S}) \lhd \mathcal{Q}^*.$$

We get a contradiction as before.

If $\mathcal{S} = \mathcal{Q} = \tilde{\mathcal{M}}$, then $\mathcal{U}$'s main branch doesn't drop. This is because $\tilde{\mathcal{M}}$ is sound. Note also that $\mathcal{U} \neq \emptyset$ since otherwise, $\mathcal{M} = \tilde{\mathcal{M}}$ which is impossible (after all, $\tau = (\kappa^+)^{\mathcal{M}} > \tilde{\tau} = (\kappa^+)^{\tilde{\mathcal{M}}}$). Now $\rho^{\omega}_{\mathcal{M}} = \rho^{\omega}_{\tilde{\mathcal{M}}} = \kappa$ and if there is an extender E used along the main branch of $\mathcal{U}$ such that $\nu(E) > \kappa$ [6] then $\mathcal{S}$ is not κ-sound. Contradiction.

So for all E used along the main branch of $\mathcal{U}$, $\nu(E) \leq \kappa$. If for all such E, $\nu(E) < \kappa$, then since $\mathcal{M}|\kappa = \tilde{\mathcal{M}}|\kappa = \mathcal{Q}|\kappa$, $\mathcal{S}|\kappa \neq \mathcal{Q}|\kappa$. Contradiction. If there is some such E such that $\nu(E) = \kappa$, then $\mathcal{U}$ must drop since otherwise, $\rho^{\omega}_{\mathcal{S}} > \kappa$. Contradiction.

So $\mathcal{Q} \lhd \mathcal{S}$. We claim that $\mathcal{Q} = \tilde{\mathcal{M}} \lhd \mathcal{M}$. It suffices to show $\mathcal{U} = \emptyset$. Otherwise, let $E = E^{\mathcal{U}}_0$. Then

$$\mathrm{lh}(E) \geq \tilde{\tau} \text{ and } \mathrm{lh}(E) < o(\mathcal{Q}) = o(\tilde{\mathcal{M}}). \tag{11.1}$$

[5] Note that in this case, $\mathcal{S}^b = \sigma^*(\mathcal{S})^b$. The last equality follows from the fact that σ^* has critical point $> o(\mathcal{Q}^b) \geq o(\mathcal{S}^b)$.

[6] $\nu(E)$ is the sup of the generators of E.

Note that $\mathrm{lh}(E)$ is a cardinal of $\mathcal{S}$ strictly larger than κ and $|o(\mathcal{Q})|^{\mathcal{S}} = \rho^{\omega}_{\tilde{\mathcal{M}}} = \kappa$. This contradicts 12.2. This completes the proof of Claim 11.1.4. $\square$

Using the claim, it is easy to see that $\tilde{\mathcal{M}} \lhd \mathcal{M}$ (that is, case (b) holds). This is because the branch embedding on the phalanx side must have critical point $> \kappa$ and $\tilde{\mathcal{M}}$ is κ-sound, so the branch is trivial with end model $\tilde{\mathcal{M}}$. $\square$

11.2 Ingredients from the Schimmerling-Zeman construction

In this section, we briefly remind the reader of the $\square$-construction in [17]. First, the reader should recall from [17] the notions of a *protomouse* and a *pluripotent level* of $L[E]$ (we give definitions of these notions in the context of hod premice in Section 11.3.1). See the beginning of [17, Section 2] for a fairly detailed discussion on how protomice appear in interpolation arguments. Basically, protomice arise in interpolation arguments where the target structure is a pluripotent level. The reader should see the definition of *divisor*, [17, Section 2.1], and *strong divisor*, [17, Section 2.4] (these notions are also defined in Section 11.3.1 for hod premice). Divisors identify protomice in interpolation arguments and (canonical) strong divisors in some sense are those (amongst many possible divisors of a given collapsing structure) that one uses in the course of the construction.

We proceed to briefly outline the proof of $\square_\kappa$ in $L[E]$ as done in [17]. To get the main ideas across in a reasonable amount of space, we will be imprecise at various places. The reader can see [17, Section 3] for a precise construction of the $\square_\kappa$-sequence $(C_\tau : \tau < \kappa^+)$. The proof starts by choosing the collapsing structure $\mathcal{N}_\tau$ for $\kappa < \tau < (\kappa^+)^{L[E]}$: $\mathcal{N}_\tau$ is the first level of $L[E]$ that satisfies "$\tau = \kappa^+$" and $\rho^{\omega}_{\mathcal{N}_\tau} = \kappa$. There is a club $\mathcal{S} \subset \kappa^+$ of such τ in $L[E]$. We further require that for each $\tau \in \mathcal{S}$, $\mathcal{J}^E_\tau \prec \mathcal{J}^E_{\kappa^+}$. For each $\tau \in \mathcal{S}$, let $\mathcal{S}_1 \subseteq \mathcal{S}$ be the set of τ for which the strong divisors of $\mathcal{N}_\tau$ exists (and let $(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau))$ be the canonical strong divisor and $\mathcal{N}_\tau(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau))$ be the unique associated protomouse as defined at the end of [17, Section 2]). Let $\mathcal{S}_0 = \mathcal{S} - \mathcal{S}_1$.

For $\tau \in \mathcal{S}_0$, the associated club $C_\tau \subset \tau$ can be constructed by Jensen's method of constructing $\square$-sequences in L. In this case, C_τ is a tail-end of the set B_τ of all $\bar{\tau} \in \mathcal{S}_0 \cap \tau$ such that:

- $\mathcal{N}_{\bar{\tau}}$ is a premouse of the same type as $\mathcal{N}_\tau$ and $n_{\bar{\tau}} = n_\tau$, where for a $\sigma \in \mathcal{S}$, n_σ is the least n such that $\omega\rho^{n+1}_{\mathcal{N}_\sigma} \leq \kappa < \omega\rho^{n}_{\mathcal{N}_\sigma}$.

- There is a map $\sigma_{\bar{\tau},\tau} : \mathcal{N}_{\bar{\tau}} \to \mathcal{N}_\tau$ that is $\Sigma_0^{(n_\tau)}$-preserving with respect to the language of premice and such that: $\bar{\tau} = \mathrm{cr}(\sigma_{\bar{\tau},\tau})$, $\sigma_{\bar{\tau},\tau}(\bar{\tau}) = \tau$, $\sigma_{\bar{\tau},\tau}(p(\mathcal{N}_{\bar{\tau}})) = p(\mathcal{N}_\tau)$, and each $\alpha \in p(\mathcal{N}_\tau)$ has a generalized witness with respect to $(\mathcal{N}_\tau, p(\mathcal{N}_\tau))$ in the range of $\sigma_{\bar{\tau},\tau}$. Here, and later, $p(\mathcal{N}_\tau)$ is the n_τ^{th}-standard parameter of $\mathcal{N}_\tau$.

For $\tau \in \mathcal{S}_1$, the set C_τ will be a tail-end of the set B_τ of $\bar{\tau} \in \mathcal{S}_1 \cap \tau$ that satisfies:

- $(\mu(\mathcal{N}_{\bar{\tau}}), |q(\mathcal{N}_{\bar{\tau}})|) = (\mu(\mathcal{N}_\tau), |q(\mathcal{N}_\tau)|)$; here by definition of divisors, $q(\mathcal{N}_\tau)$ is a bottom initial segment of $d(\mathcal{N}_\tau)$, the Dodd-parameter of $\mathcal{N}_\tau$.

- There is a map $\sigma_{\bar{\tau},\tau} : \mathcal{N}_{\bar{\tau}}(\mu(\mathcal{N}_{\bar{\tau}}), q(\mathcal{N}_{\bar{\tau}})) \to \mathcal{N}_\tau(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau))$ that is Σ_0-preserving with respect to the language for coherent structures such that: $\bar{\tau} = \mathrm{cr}(\sigma_{\bar{\tau},\tau})$, $\sigma_{\bar{\tau},\tau}(\bar{\tau}) = \tau$, $\sigma_{\bar{\tau},\tau}(q(\mathcal{N}_{\bar{\tau}})) = q(\mathcal{N}_\tau)$, and each $\alpha \in q(\mathcal{N}_\tau)$ has a generalized witness (with respect to $(\mathcal{N}_\tau(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau)), q(\mathcal{N}_\tau))$ in the range of $\sigma_{\bar{\tau},\tau}$.

Now we focus on the key point: the proof that B_τ is unbounded in τ if $\tau \in \mathcal{S}^1$ and $\mathrm{cof}(\tau) > \omega$ in $L[E]$. Fix such a τ and let $\kappa < \gamma < \tau$ be arbitrary. We want to find a $\gamma < \bar{\tau} < \tau$ in B_τ. Working in $L[E]$, fix some $\theta >> \kappa$ and let $X \prec H_\theta$ be countable such that all relevant objects are in X, in particular $\{\kappa, \tau, \gamma\} \in X$. Let $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ be the uncollapse map of $X \cap \mathcal{M}$, where $\mathcal{M} = \mathcal{N}(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau))$. We write $\sigma^{-1}(x) = \bar{x}$ for each x in the range of σ. Let $\tilde{\tau} = \sup(\sigma''\bar{\tau})$. Let $\tilde{\sigma} : \bar{\mathcal{M}} \to \tilde{\mathcal{M}}$ come from the $(\mathrm{cr}(\sigma), \tilde{\tau})$-extender derived from σ. Also, let $\sigma' : \tilde{\mathcal{M}} \to \mathcal{M}$ be given by the interpolation lemma [17, Lemma 1.2]. In this case, $\tilde{\mathcal{M}} = (\mathcal{N}, \tilde{F})$ is a protomouse (even if $\mathcal{N}(\mu(\mathcal{N}_\tau), q(\mathcal{N}_\tau)) = \mathcal{N}_\tau$ since in this case, $\mathcal{N}_\tau$ is a pluripotent level of $L[E]$ and the map σ' is not cofinal). The way one shows $\tilde{\tau} \in B_\tau$ is as follows. Let $\mathcal{M}^*$ be the largest segment of $\mathcal{N}$ such that $\tilde{F}$ measures all sets in $\mathcal{M}^*$. One then shows that $\mathrm{Ult}(\mathcal{M}^*, \tilde{F})$ is $\mathcal{N}_{\tilde{\tau}}$. Say $\mathcal{M} = (\mathcal{M}^-, F)$. This is accomplished by applying the condensation lemma [17, Lemma 1.6] to $\phi : \mathrm{Ult}(\mathcal{M}^*, \tilde{F}) \to \pi_F(\mathcal{M}^*)$ such that

$$\phi(\pi_{\tilde{F}}(f)(a)) = \sigma'(f)(\pi_F(a))$$

where π_F is the F-ultrapower embedding applied to the largest initial segment of $\mathcal{M}^-$ that makes sense.

The key for the proof above is that we can always compare two iterable pure extender models; in this case, we compare the phalanx $(\pi_F(\mathcal{M}^*), \mathrm{Ult}(\mathcal{M}^*, \tilde{F}), \tilde{\tau})$ against $\pi_F(\mathcal{M}^*)$. If one adapted this argument to hod mice, it fails because the hod mice $\pi_F(\mathcal{M}^*)$ and $\mathrm{Ult}(\mathcal{M}^*, \tilde{F})$ generally belong to two different pointclasses, and hence cannot be directly compared (though if the map σ' has critical point $\geq o(\mathcal{P}^b)$ for some hod premouse $\mathcal{P}$ with a top window, then these objects can be compared).

The fix for this, as done in the next section, is to sometimes allow for the collapsing structure of τ, $\mathcal{N}_\tau$, to *not be an initial segment of* the hod mouse and incorporate this kind of collapsing structures into the construction. It is this aspect that forces the construction to yield a weaker result, i.e. $\square_{\kappa,2}$, rather than $\square_\kappa$.

One other new situation in the hod mouse case that does not come up in the $L[E]$ case is the following. Suppose in the above, $\mathcal{M} = \mathcal{N}_\tau$ is B-active. Then the way branches are coded into the model (using the $\mathfrak{B}$-operator as discussed in the previous section) allows us to show that $\tilde{M}$ is a B-active hod premouse. If one used the traditional coding of branches, then $\tilde{M}$ may fail to be a hod premouse; this is the reason we switch to the coding of branches via the $\mathfrak{B}$-operator. We will discuss this in more details in the next section.

11.3 The proof

We give a proof of Theorem 11.0.5, making use of the notions, notations, and proofs in [17] whenever applicable. We only focus on the details that are new in our situation and direct the reader to constructions in [17] that are obviously generalizable to our situation.

11.3.1 Some set-up

We will use the fine-structure terminology and notations from [17, Section 1], generalized to our context in an obvious way. For example, notions in [17] that are defined using the language of premice are defined here using the language of hod premice; when we talk about a *coherent structure* in this paper, we mean a structure M of the form $(\mathcal{Q}, F)$ where $\mathcal{Q}$ is an amenable structure in the language of hod premice and F is a whole extender at (κ, λ) (in the language of [17, Section 1]) with $\mathrm{dom}(F) = \wp(\kappa) \cap \mathcal{Q}|\bar{\alpha}$ for some $\bar{\alpha} \leq (\kappa^+)^M$ and $\mathcal{Q} \trianglelefteq \mathrm{Ult}_n(\mathcal{Q}|\bar{\alpha}, F) = N$, where n is the least such that $\rho^{n+1}_{\mathcal{Q}|\bar{\alpha}} = \kappa$. We say N is *the hod premouse associated with* M. The notion of *a generalized witness* for some ordinal α with respect to a pair (M, s) where M is a coherent structure, s is a finite set of ordinals (or a generalized witness for an ordinal α with respect to a hod premouse N associated with M and some finite set of ordinals $r \cup s$) in [17] is generalized in an obvious way to our context.[7] A *protomouse* $\mathcal{P} = (\mathcal{Q}, F)$ is a coherent structure where F is an extender with critical

[7] Let M, N, κ, λ be as above and $s \subset \lambda$ is finite. The *standard witness* $W^{\alpha,s}_M$ for α with respect to M and s to be the transitive collapse of $h_M(\alpha \cup \{s\})$, where h_M is the canonical Σ_1-Skolem function of the coherent structure M. Similarly, $W^{\alpha,r\cup s}_N$ denotes the standard witness for α with respect to N and $r \cup s$ and is the transitive collapse of $\tilde{H}^{n+1}_N(\alpha \cup \{r \cup s\})$, where $\tilde{h}^{n+1}_N$ is the canonical

κ such that F does not measure $\wp(\kappa)^{\mathcal{Q}}$. A *pluripotent level* of a hod premouse $\mathcal{P}$ is an E-active initial segment $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathrm{cr}(E^{\mathcal{Q}}_{\mathrm{top}}) < \kappa$ and $\omega\rho^1_{\mathcal{Q}} = \kappa$, where κ is a cardinal of $\mathcal{P}$.

Fix $(\mathcal{P}, \Sigma)$ as in the hypothesis of Theorem 11.0.5. Fix $\kappa \geq \delta^{\mathcal{P}^b}$, a cardinal of $\mathcal{P}$. Working in $\mathcal{P}$, let $\mu = \delta^{\mathcal{P}^b}$ and $\mathcal{S} \subset \kappa^+$ be the club of $\kappa < \tau < \kappa^+$ such that $\mathcal{P}|\tau \prec \mathcal{P}|\kappa^+$. Let $\mathcal{N}^*_\tau \lhd \mathcal{P}$ be the collapsing level for τ, that is $\mathcal{N}^*_\tau$ the least initial segment $\mathcal{N}$ of $\mathcal{P}$ such that $\mathcal{N} \vDash \tau = \kappa^+$ and $\rho^\omega_{\mathcal{N}} = \kappa$. Let γ_τ be the sup of indexes of extenders E on the sequence of $\mathcal{N}^*_\tau$ such that $\mathrm{cr}(E) = \mu$. Without loss of generality, we may assume throughout this paper that

$$\mu \text{ is measurable in } \mathcal{P};\ \kappa \geq o(\mathcal{P}^b);\text{ and } \sup_{\tau\in\mathcal{S}}(\gamma_\tau) \geq \kappa^+.$$ [8]

The following follow easily from the definitions and our assumption.

Proposition 11.3.1 *1. $o(\mathcal{N}^*_\tau) > \tau$.*

2. $\gamma_\tau \geq \tau$.

Extenders E with $\mathrm{cr}(E) = \mu$ play a special role in this construction. Recall these extenders are indexed according to the cutpoint indexing scheme. Note that μ is a strong cutpoint of $\mathcal{P}^b$, that is, there are no partial extenders with critical point μ on the sequence of $\mathcal{P}$. This is the main difference between our situation and the $L[E]$-situation.

Some discussions regarding protomice and divisors are in order. Following [17], for a hod premouse $\mathcal{N}$ such that $\omega\rho^{n+1}_{\mathcal{N}} \leq \kappa < \omega\rho^n_{\mathcal{N}}$, we say that (ν, q) is a *divisor* of $\mathcal{N}$ if and only if there is an ordinal $\lambda = \lambda_{\mathcal{N}}(\nu, q)$ such that letting $p_{\mathcal{N}}$ be the $(n+1)$ standard parameter of $\mathcal{N}$, setting $r = p_{\mathcal{N}} - q$, the following hold:

(a) $\nu \leq \kappa < \lambda < \omega\rho^n_{\mathcal{N}}$;

(b) $q = p_{\mathcal{N}} \cap \lambda$;

$\Sigma_1^{(n)}$-Skolem function of the hod premouse N. A *generalized witness* for α with respect to M and s is a pair (Q, t), where $t \subset Q$ is a finite set of ordinals and such that for any $\xi_1, \ldots, \xi_l < \alpha$, if $M \vDash \Phi(i, \xi_1, \ldots, \xi_l, s)$ then $Q \vDash \Phi(i, \xi_1, \ldots, \xi_l, s)$, where Φ is the universal Σ_1-formula. A generalized witness for α with respect to N and $r \cup s$ is a pair (Q, t), where $t \subset Q$ is a finite set of ordinals such that given any $\xi_1, \ldots, \xi_l < \alpha$, if $N \vDash \Phi(i, \xi_1, \ldots, \xi_l, r \cup s)$ then $Q \vDash \Phi(i, \xi_1, \ldots, \xi_l, r \cup s)$, where Φ is the universal $\Sigma_1^{(n)}$-formula.

[8] If $\kappa = \delta^{\mathcal{P}}_\alpha$, where $\delta^{\mathcal{P}} = \delta^{\mathcal{P}}_{\alpha+1}$, then since $\mathcal{P}^b = \mathcal{P}|(\kappa^+)^{\mathcal{P}} = \mathrm{Lp}^{\oplus_{\beta<\alpha}\Sigma^{\mathcal{P}}_{\mathcal{P}(\beta)}}(\mathcal{P}|\delta^{\mathcal{P}}_\alpha)$, then $\mathcal{P} \vDash \Box_\kappa$ since κ is a strong cutpoint cardinal of $\mathcal{P}^b$. If $\kappa > \delta^{\mathcal{P}}_\alpha$ and $\sup_{\tau\in\mathcal{S}}(\gamma_\tau) < \kappa^+$, then the proof is significantly easier. One constructs the $\Box_\kappa$-sequence using points $\tau \in \mathcal{S}$ above $\sup_{\tau\in\mathcal{S}}(\gamma_\tau)$ mimicking essentially the Schimmerling-Zeman construction and use Theorem 11.1.2.

(c) $\tilde{h}^{n+1}_{\mathcal{N}}(\nu \cup \{r\}) \cap \omega\rho^n_{\mathcal{N}}$ is cofinal in $\omega\rho^n_{\mathcal{N}}$;

(d) $\lambda = \min(\mathrm{OR} \cap \tilde{h}^{n+1}_{\mathcal{N}}(\nu \cup \{r\}) - \nu)$.

As in [17], both ν and λ are (inaccessible) cardinals in $\mathcal{N}$. Let $\mathcal{N}^*(\nu, q)$ be the transitive collapse of $\tilde{h}^{n+1}_{\mathcal{N}}(\nu \cup \{r\})$.

The notion of strong divisors in [17] generalize in an obvious way to our context. We recall it now. A divisor (μ, q) of $\mathcal{N}$ is *strong* if and only if for every $\xi < \mu$ and every x of the form $\tilde{h}^{n+1}_{\mathcal{N}}(\xi, p_{\mathcal{N}})$ we have $x \cap \mu \in \mathcal{N}^*(\mu, q)$. If $\mathcal{N}$ is pluripotent, we define the notion of strong divisor in the same way, but with $h^*_{\mathcal{N}}$ (the Σ_1-Skolem function of $\mathcal{N}$ computed in the language of coherent structures) and $d_{\mathcal{N}}$ (the Dodd-parameter of $\mathcal{N}$) in place of $\tilde{h}^{n+1}_{\mathcal{N}}$ and $p_{\mathcal{N}}$, respectively. As in [17], if $\mathcal{N}$ has strong divisors, *the canonical strong divisor* $(\mu_{\mathcal{N}}, q_{\mathcal{N}})$ of $\mathcal{N}$ is chosen as follows: $q_{\mathcal{N}}$ is the shortest initial segment of $p_{\mathcal{N}}$ such that for some ν, $(\nu, q_{\mathcal{N}})$ is a strong divisor of $\mathcal{N}$ and $\mu_{\mathcal{N}}$ is the largest ν such that $(\nu, q_{\mathcal{N}})$ is a strong divisor of $\mathcal{N}$. Now we define our collapsing structure $\mathcal{N}_\tau$ for $\tau \in \mathcal{S}$.

Definition 11.3.2 *Suppose $\mathcal{P}$ is a hod premouse and δ is a Woodin cardinal in some $\mathcal{Q} \trianglelefteq \mathcal{P}$. We say that δ is a layer Woodin in $\mathcal{Q}$ if there is some $\mathcal{R} \in Y^{\mathcal{Q}}$ such that $\delta = \delta^{\mathcal{R}}$.*

Definition 11.3.3 *Fix $\tau \in \mathcal{S}$. Suppose there is a pointclass $\Omega \subsetneq \Gamma$ such that there is a hod pair $(\mathcal{R}, \Sigma_{\mathcal{R}})$ such that*

- $\mathcal{N}^*_\tau | \gamma_\tau \triangleleft \mathcal{R}$,
- $\rho^\omega_{\mathcal{R}} = \kappa$,
- *$\mathcal{R}$ is sound,*
- *γ_τ is a cutpoint of $\mathcal{R}$ and $\Sigma_{\mathcal{R}|\gamma_\tau} = \Sigma_{\mathcal{P}|\gamma_\tau}$,*
- *the order type of $\mathcal{R}$'s layer Woodin cardinals above γ_τ is a limit ordinal,*
- *$\mathcal{R}$ has a strong divisor of the form (μ, q) where $p_{\mathcal{R}} = q \cup r$ for r above the supremum λ of the layer Woodin cardinals of $\mathcal{R}$ and max(q) is below $(\gamma^+_\tau)^{\mathcal{R}}$,*
- *$\Sigma_{\mathcal{R}}$ has branch condensation, is Ω-fullness preserving, and $(\mathcal{R}, \Sigma_{\mathcal{R}})$ generates Ω; that is $\Gamma(\mathcal{R}, \Sigma_{\mathcal{R}}) = \Omega$.*

We call $(\mathcal{R}, \Sigma_{\mathcal{R}})$ with the above properties the pointclass generator of Ω. Let Γ_τ be the Wadge-minimal such pointclass and $\mathcal{N}_\tau$ be the pointclass generator of Γ_τ, $(\mu_\tau, q_\tau, \lambda_\tau)$ be the (μ, q, λ) associated with $\mathcal{N}_\tau$ as above (note that $\mathcal{N}_\tau$ must be distinct from $\mathcal{N}^*_\tau$ in this case). If $(\Gamma, \mathcal{R}, \mu, q, \lambda)$ doesn't exist, we let $\mathcal{N}_\tau = \mathcal{N}^*_\tau$.

The properties of pointclass generators seem technical; these properties are abstracted from various situations in interpolation arguments. It seems hard to do with much less. The following proposition justifies the uniqueness of pointclass generators.

Proposition 11.3.4 *Let $\mathcal{P}, \tau, \Omega$ be as in Definition 11.3.3. Let $(\mathcal{R}_0, \Sigma_0)$ and $(\mathcal{R}_1, \Sigma_1)$ be pointclass generators of Ω. Then $(\mathcal{R}_0, \Sigma_0) = (\mathcal{R}_1, \Sigma_1)$.*

Proof. We compare the pair $(\mathcal{R}_0, \Sigma_0)$ against $(\mathcal{R}_1, \Sigma_1)$, lining up the models and the strategies (as done in Section 4.6). The comparison is possible by the assumption and is above γ_τ. The end model is, say, $\mathcal{S}$ and the tail strategies of Σ_0 and Σ_1 on $\mathcal{S}$ are the same. The usual proof using the fact that $\mathcal{R}_0$ and $\mathcal{R}_1$ are γ_τ-sound and the comparison is above γ_τ shows that $\mathcal{S} = \mathcal{R}_0 = \mathcal{R}_1$ (the comparison is trivial) and $\Sigma_0 = \Sigma_1$. $\square$

We simply use the notations from [17, page 49] in the definition of our square sequence below. For instance, (μ_τ, q_τ) denotes the canonical strong divisor of $\mathcal{N}_\tau$ (if exists) in the case $\mathcal{N}_\tau = \mathcal{N}_\tau^*$ and denotes the (μ_τ, q_τ) in Definition 11.3.3 in the case $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$ (note that (μ_τ, q_τ) is the unique strong divisor of $\mathcal{N}_\tau$ with the properties as in Definition 11.3.3). If $\mathcal{N}_\tau = \mathcal{N}_\tau^*$ is a pluripotent level that has no strong divisors, then (μ_τ, q_τ) denotes $(\mathrm{cr}(E^{\mathrm{top}}_{\mathcal{N}_\tau}), p(\mathcal{N}_\tau))$.

Suppose (ν, q) is a divisor of $\mathcal{N}_\tau$; let r, λ, n be as in the definition of divisor. Let $\pi : \mathcal{N}_\tau^*(\nu, q) \to \tilde{h}^{n+1}_{\mathcal{N}}(\nu \cup \{r\})$ be the uncollapse map. We let the associated protomouse $\mathcal{N}_\tau(\nu, q)$ be the coherent structure $(\mathcal{N}_\tau|\xi, F)$ where $\xi = \pi((\nu^+)^{M^*})$ and $F = E_\pi \restriction (\wp(\nu) \cap \mathcal{N}_\tau^*(\nu, q))$, if $\nu > \mu$. If $\nu = \mu$, in which case $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$, then we let $\mathcal{N}_\tau(\nu, q)$ be the coherent structure $(\mathcal{N}_\tau|\xi, F)$ where $\xi = (\gamma_\tau^+)^{\mathcal{N}_\tau}$ and $F = E_\pi \restriction (\wp(\nu) \cap M^*)$.

The following proposition is easy to see and justifies that the structure $\mathcal{N}_\tau(\nu, q)$ are protomice (and not hod premice). See [17, Section 2.1] for a detailed discussion and proof.

Proposition 11.3.5 *Suppose (ν, q) be a divisor of $\mathcal{N}_\tau$ and $\pi : \mathcal{N}_\tau^*(\nu, q) \to \tilde{h}^{n+1}_{\mathcal{N}_\tau}(\nu \cup \{r\})$ be the uncollapse map (and in the case $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$, assume $\nu = \mu$). Then $\wp(\nu) \cap \mathcal{N}_\tau^*(\nu, q) \subsetneq \wp(\nu) \cap \mathcal{N}_\tau$. Furthermore, ν is an (inaccessible) cardinal of $\mathcal{N}_\tau^*(\nu, q)$ and a limit cardinal of $\mathcal{N}_\tau$, and $\lambda_{\mathcal{N}_\tau}(\nu, q)$ is an (inaccessible) cardinal of $\mathcal{N}_\tau$.*

We let $\mathcal{M}_\tau = \mathcal{N}_\tau(\mu_\tau, q_\tau)$ be the protomouse associated with (μ_τ, q_τ).

Definition 11.3.6 *Let $\mathcal{S}^1 \subset \mathcal{S}$ be the set of τ such that (μ_τ, q_τ) is defined and $\mathcal{S}^0 = \mathcal{S} - \mathcal{S}^1$.*

Suppose $\mathcal{N}_\tau = \mathcal{N}_\tau^*$, then no divisors of $\mathcal{N}_\tau$ are of the form (μ, q). This is because otherwise, $\lambda = \lambda_{\mathcal{N}_\tau}(\mu, q)$ is a limit of Woodin cardinals. Let $\gamma_0 < \gamma_1$ be consecutive Woodin cardinals in the interval (μ, λ); then by definition of $\mathcal{P}$, $\mathcal{P}|\gamma_1$ is a Λ^{sts}-mouse where Λ is the strategy of $M^+(\mathcal{P}|\gamma_0)$. On the other hand, by elementarity, $\mathcal{P}|\gamma_1$ is a Λ-mouse. Contradiction. [9]

A similar argument applies to show that no divisors for $\mathcal{N}_\tau$ are of the form (ξ, q) for $\xi < \mu$; though we don't need this fact in our construction as no divisors (ν, q) in this paper will have the property that $\nu < \mu$. So if (ν, q) is a divisor of $\mathcal{N}_\tau$, then $\nu > \mu$. This allows us to simply quote results of [17, Section 2] in this case (in light of Theorem 11.1.2). In the case that $\mu_\tau = \mu$ (so $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$), more care needs to be taken since it's not obvious that all results in [17, Section 2.4] can be generalized to this case.

Using the remarks above, it is easy to see that if $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$, then $\tau \in \mathcal{S}^1$ and in fact $\mathcal{N}_\tau$ is not an initial segment of $\mathcal{P}$ (though $\mathcal{N}_\tau \in \mathcal{P}$ by Proposition 11.3.7); also, if $\mathcal{N}_\tau = \mathcal{N}_\tau^*$ is pluripotent, then $\tau \in \mathcal{S}^1$. For $\tau \in \mathcal{S}^0$, $\mathcal{N}_\tau = \mathcal{N}_\tau^*$ is not pluripotent and does not admit a strong divisor.

The following lemma allows us to define our $\square_\kappa$-sequence in a uniform manner.

Proposition 11.3.7 *Suppose $\mathcal{N}_\tau \neq \mathcal{N}_\tau^*$. Then $\mathcal{N}_\tau$ is definable over $\mathcal{P}$ (in fact, over any $\mathcal{N}_\xi^*$ or $\mathcal{N}_\xi$ for $\xi > \tau$) unformly from $\{\tau, \gamma_\tau\}$.*

Proof. Fix $\xi > \tau$. We first claim that $\gamma_\xi > \gamma_\tau$. To see this, note that $\tau \leq \gamma_\tau \leq o(\mathcal{N}_\tau^*) < \xi$. This is because ξ is a cardinal (successor of κ) in $\mathcal{N}_\xi$ while there is a surjection from κ onto γ_τ in $\mathcal{N}_\xi$. Since $\xi \leq \gamma_\xi$, the claim follows.

Now let E be the extender on the $\mathcal{N}_\xi$-sequence such that $\mathrm{cr}(E) = \mu$, $\mathrm{lh}(E) > \gamma_\tau$, and is the least such.[10] Let $\mathcal{S} = \mathrm{Ult}(\mathcal{N}_\xi, E)$ (this is a Σ_0-ultrapower). Let $i : \mathcal{S} \to \mathcal{S}_\infty$ be an $\mathbb{R}$-genericity iteration (above γ_τ). Now it is easy to see that in the derived model of $\mathcal{S}_\infty$ (at the sup of its Woodin cardinals), the pointclass Ω in the definition of $\mathcal{N}_\tau$ is a strict Wadge initial segment of $\wp(\mathbb{R})$ and is definable there from $\{\tau, \gamma_\tau\}$. Then $\mathcal{N}_\tau \in \mathcal{S}_\infty$ and in fact is definable there from parameters $\{\tau, \gamma_\tau\}$. The same holds in $\mathcal{S}$ by elementarity and the fact that $\mathrm{cr}(i) > \gamma_\tau$. Finally, $\mathcal{N}_\tau \in \mathcal{N}_\xi$ and is definable there from parameters $\{\tau, \gamma_\tau, E\}$. But E is definable in $\mathcal{N}_\xi$ from $\{\tau, \gamma_\tau\}$. So $\mathcal{N}_\tau$ is definable in $\mathcal{N}_\xi$ from $\{\tau, \gamma_\tau\}$. $\square$

[9] Another argument is as follows. Note that each Woodin cardinal in the interval (μ, λ) is $> (\mu^+)^{\mathcal{P}}$, and hence μ is strong to λ (in $\mathcal{P}$) by the initial segment condition. This contradicts the definition and smallness assumption on $\mathcal{P}$ since one can easily find an active ω Woodin lsa mouse in $\mathcal{P}$ (as defined in Definition 8.2.2).

[10] We note that the set of indices for extenders with critical point μ is nowhere continuous.

Remark 11.3.8 *By our smallness assumption on $\mathcal{P}$, the set $\mathfrak{A} = \{\xi \mid \kappa < \xi < \kappa^+ \wedge \mathcal{P}|\xi$ is E-active$\}$ is non-stationary in $\mathcal{P}$. The reason is $\mathfrak{A} = \mathfrak{A}_0 \cup \mathfrak{A}_1$. Here $\mathfrak{A}_0$ consists of ξ's such that the top extender of $\mathcal{P}|\xi$ has critical point μ and $\mathfrak{A}_1 = \mathfrak{A} - \mathfrak{A}_0$. $\mathfrak{A}_0$ in nonstationary by Footnote 10. $\mathfrak{A}_1$ is nonstationary because otherwise, κ is subcompact by [17]. As in [17], the fact that $\mathfrak{A}$ is nonstationary is crucial in our construction. We use this fact in various arguments to follow.*

11.3.2 Approximation of a $\square_{\kappa,2}$ sequence

We use the notation established in the previous section. Below, as in [17], n_τ is the unique n such that $\rho^{n+1}_{\mathcal{N}_\tau} = \kappa < \rho^n_{\mathcal{N}_\tau}$ and p_τ is the standard parameter of $\mathcal{N}_\tau$. Let also p^*_τ be the standard parameter of $\mathcal{N}^*_\tau$.

Definition 11.3.9 *Suppose $\tau \in \mathcal{S}^0$, let $\vec{B}_\tau = \{B^0_\tau\}$ be the set of $\bar{\tau} \in \mathcal{S} \cap \tau$ satisfying:*

- *$\mathcal{N}_{\bar{\tau}}$ is a hod premouse of the same type as $\mathcal{N}_\tau$.* [11]
- *$n_\tau = n_{\bar{\tau}}$.*
- *There is a map $\sigma^0_{\bar{\tau}\tau} : \mathcal{N}^*_{\bar{\tau}} \to \mathcal{N}_\tau$ that is $\Sigma_0^{(n_\tau)}$-preserving with respect to the language of hod premice such that*
  - *(a) $\bar{\tau} = \mathrm{cr}(\sigma^0_{\bar{\tau}\tau})$ and $\sigma^0_{\bar{\tau}\tau}(\bar{\tau}) = \tau$.*
  - *(b) $\sigma^0_{\bar{\tau}\tau}(p^*_{\bar{\tau}}) = p_\tau$.*
  - *(c) for each $\alpha \in p_\tau$, there is a generalized witness for α with respect to $\mathcal{N}_\tau$ and p_τ in the range of $\sigma_{\bar{\tau}\tau}$.*

Note that if $\tau \in \mathcal{S}^0$, then $\mathcal{N}^*_\tau = \mathcal{N}_\tau$ and either $\mathrm{crt}(E^{top}_{\mathcal{N}_\tau}) \geq \kappa$ or $\rho_1^{\mathcal{N}_\tau} > \kappa$. Recall the definition of (μ_τ, q_τ), p_τ, d_τ for $\tau \in \mathcal{S}^1$ in Section 11.3.1. Below, m_τ is $|q_\tau|$. We also let $r_\tau = d_\tau - q_\tau$ be the top part of d_τ.

Definition 11.3.10 *Suppose $\tau \in \mathcal{S}^1$. Let B^1_τ be the set of $\bar{\tau} \in \mathcal{S}^1 \cap \tau$ satisfying:*

- *$(\mu_{\bar{\tau}}, m_{\bar{\tau}}) = (\mu_\tau, m_\tau)$.*

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[11] In this case, it simply means: $\mathcal{N}_\tau$ is E (B)-active if and only if $\mathcal{N}_{\bar{\tau}}$ is E (B)-active. If $\mathcal{N}_\tau$ is E-active (equivalently, $\mathcal{N}_{\bar{\tau}}$ is E-active), then $E^{\text{top}}_{\mathcal{N}_\tau}$ is indexed according to the cutpoint (Jensen) indexing scheme if and only if $E^{\text{top}}_{\mathcal{N}_{\bar{\tau}}}$ is indexed according to the cutpoint (Jensen, respectively) indexing scheme. Recall that all E-active hod mice, where E is indexed according to the Jensen indexing scheme, in our paper will be of type A.

- *There is a map* $\sigma^1_{\bar\tau\tau} : \mathcal{M}_{\bar\tau} \to \mathcal{M}_\tau$ *that is* Σ_0*-preserving with respect to the language of coherent structures such that*
  - *(a)* $\bar\tau = \mathrm{cr}(\sigma^1_{\bar\tau\tau})$ *and* $\sigma^1_{\bar\tau\tau}(\bar\tau) = \tau$.
  - *(b)* $\sigma^1_{\bar\tau\tau}(q_{\bar\tau}) = q_\tau$.
  - *(c) for each* $\alpha \in q_\tau$*, there is a generalized witness for* α *with respect to* $\mathcal{N}_\tau$ *and* q_τ *in the range of* $\sigma^1_{\bar\tau\tau}$.

Suppose in addition that either crt$(E^{top}_{\mathcal{N}^*_\tau}) \geq \kappa$ *or* $\rho_1^{\mathcal{N}^*_\tau} > \kappa$*, let* B^0_τ *be the set of* $\bar\tau \in \mathcal{S} \cap \tau$ *satisfying:*

- $\mathcal{N}^*_{\bar\tau}$ *is a hod premouse of the same type as* $\mathcal{N}^*_\tau$.
- $n_\tau = n_{\bar\tau}$.
- *There is a map* $\sigma^0_{\bar\tau\tau} : \mathcal{N}^*_{\bar\tau} \to \mathcal{N}^*_\tau$ *that is* $\Sigma_0^{(n_\tau)}$*-preserving with respect to the language of hod premice such that*
  - *(a)* $\bar\tau = \mathrm{cr}(\sigma^0_{\bar\tau\tau})$ *and* $\sigma^0_{\bar\tau\tau}(\bar\tau) = \tau$.
  - *(b)* $\sigma^0_{\bar\tau\tau}(p^*_{\bar\tau}) = p_\tau$.
  - *(c) for each* $\alpha \in p_\tau$*, there is a generalized witness for* α *with respect to* $\mathcal{N}^*_\tau$ *and* p_τ *in the range of* $\sigma^0_{\bar\tau\tau}$.

Finally, if B^0_τ *exists, let* $\vec{B}_\tau = \{B^0_\tau, B^1_\tau\}$*. Otherwise, let* $\vec{B}_\tau = \{B^1_\tau\}$.

As in [17], it is easy to see that in both cases $\sigma_{\bar\tau\tau}, \sigma^0_{\bar\tau\tau}, \sigma^1_{\bar\tau\tau}$ (if exist) are uniquely determined, Σ_0 (and not Σ_1), and non-cofinal. By [17, Lemma 3.3], for each $\tau \in \mathcal{S}$ such that B^0_τ is defined, and $\bar\tau \in B^0_\tau$,

$$B^0_\tau \cap \bar\tau = B^0_{\bar\tau} - \mathrm{min} B^0_\tau. \tag{11.2}$$

And similarly, if B^1_τ is defined, then for all $\bar\tau \in B^1_\tau$,

$$B^1_\tau \cap \bar\tau = B^1_{\bar\tau} - \mathrm{min} B^0_\tau. \tag{11.3}$$

The following is the key lemma (cf. [17, Lemma 3.5]).

Lemma 11.3.11 *For each* $\tau \in \mathcal{S}$ *of uncountable cofinality, for* $i \in \{0, 1\}$*, if* B^i_τ *is defined, then* B^i_τ *is a club subset of* τ *on a tail end. That is, there is a* $\bar\tau < \tau$ *such that* $B^i_\tau - \bar\tau$ *is closed and unbounded in* τ*. If* $i = 0$*, we can take* $\bar\tau = 0$.

Using the lemma and 11.2, 11.3, by the argument on [17, pg 52-55] , we can construct a $\square'_{\kappa,2}$-sequence on $\mathcal{S}$. We summarize the construction next. First for $\tau \in \mathcal{S}$, for i such that B^i_τ is defined, let

- $\tau^i(0) = \tau$;
- $\tau^i(j+1) = \min(B^i_{\tau(j+1)})$;
- $l^i_\tau =$ the least j such that $B^i_{\tau(j)} = \emptyset$.

Now let

- $B^{i,*} = B^i_{\tau^i(0)} \cup \cdots \cup B^i_{\tau^i(l^i_\tau - 1)}$;
- $\sigma^{i,*}_{\bar{\tau}\tau} = \sigma^i_{\tau^i(1)\tau^i(0)} \circ \cdots \circ \sigma^i_{\tau^i(j)\tau^i(j-1)} \circ \sigma^i_{\bar{\tau}\tau^i(j)}$ whenever $\bar{\tau} \in B^{i,*}_\tau$ and j is such that $\bar{\tau} \in B^i_{\tau(j)}$.

By the exact same proof as in [17, Lemma 3.4], we get the coherency of the $B^{i,*}_\tau$ sets.

Lemma 11.3.12 *For $\tau \in \mathcal{S}$, for i such that B^i_τ is defined, suppose $\bar{\tau} \in B^{i,*}_\tau$. Then $B^i_{\bar{\tau}}$ is defined and $B^{i,*}_{\bar{\tau}} = B^{i,*}_\tau \cap \bar{\tau}$.*

For each $\tau \in \mathcal{S}$, for i such that B^i_τ is defined, let β^i_τ be the least β in $B^{i,*}_\tau \cup \{\tau\}$ such that $B^{i,*}_\tau - \beta$ is closed in τ. Using Lemmata 11.3.11 and 11.3.12, we easily get that letting

$$C^{i,*}_\tau = B^{i,*}_\tau - \beta^i_\tau, \tag{11.4}$$

then for $\bar{\tau} \in \beta^{i,*}_\tau$, $\bar{\tau} \geq \beta_\tau$,

$$\beta^i_\tau = \beta^i_{\bar{\tau}} \text{ and } C^{i,*}_\tau \cap \bar{\tau} = C^{i,*}_{\bar{\tau}}. \tag{11.5}$$

Now note that if $C^{0,*}_\tau$ is defined, then o.t.$(C^{0,*}_\tau)$ may not be $\leq \kappa$, while if $C^{1,*}_\tau$ is defined then o.t.$(C^{0,*}_\tau) \leq \kappa$. As in [17, pg 54-55], we can shrink $C^{0,*}_\tau$ to a set $C^{0,\prime}_\tau \subseteq C^{0,*}_\tau$ such that

- o.t.$(C^{0,\prime}_\tau) \leq \kappa$;
- $C^{0,\prime}_\tau$ is a closed subset of $\mathcal{S} \cap \tau$ and if $\mathrm{cof}(\alpha) > \omega$, then $C^{0,\prime}_\tau$ is also unbounded in τ;
- $C^{0,\prime}_\tau \cap \bar{\tau} = C^{0,\prime}_{\bar{\tau}}$.

So letting $\vec{C}'_\tau = \{C^{i,\prime}_\tau \mid i \in \{0,1\} \wedge C^{i,\prime}_\tau \text{ is defined}\}$, we get that the sequence $\langle \vec{C}'_\tau \mid \tau < \kappa^+ \rangle$ is a $\square'_{\kappa,2}$-sequence on $\mathcal{S}$. Since $\mathcal{S}$ is club subset of κ^+, by a standard combinatorial argument (cf. [2]), the $\square'_{\kappa,2}$-sequence on $\mathcal{S}$ can be turned into a $\square_{\kappa,2}$-sequence. Our main task is to prove Lemma 11.3.11. This will take up the rest of the section.

Remark 11.3.13 *It's clear from [17, pg 54-55], Definitions 11.3.9 and 11.3.10 and Proposition 11.3.7 that the square sequence $\square_{\kappa,2}$ is definable from κ in $\mathcal{P}$ and the definition is uniform in κ.*

11.3.3 When $\tau \in \mathcal{S}^0$

Fix $\tau \in \mathcal{S}^0$. Assume τ is a limit point of $\mathcal{S}$ uncountable cofinality. Recall B^0_τ is defined to be the set of $\bar{\tau} \in \mathcal{S}$ such that

- $n_\tau = n_{\bar{\tau}}$.
- $\mathcal{N}^*_{\bar{\tau}}$ is a hod premouse of the same type as $\mathcal{N}_\tau$.
- There is an embedding $\sigma^0_{\bar{\tau}\tau} : \mathcal{N}^*_{\bar{\tau}} \to \mathcal{N}_\tau$ such that $\sigma^0_{\bar{\tau}\tau}$ is $\Sigma_0^{(n_\tau)}$-preserving (in the language of hod premice) and
  - (a) $\bar{\tau} = \mathrm{cr}(\sigma^0_{\bar{\tau}\tau})$ and $\sigma^0_{\bar{\tau}\tau}(\bar{\tau}) = \tau$.
  - (b) $\sigma^0_{\bar{\tau}\tau}(p^*_{\bar{\tau}}) = p_\tau$, where recall $p^*_{\bar{\tau}}$ is the standard parameter of $\mathcal{N}^*_{\bar{\tau}}$.
  - (c) for each $\alpha \in p_\tau$, there is a generalized witness for α with respect to $\mathcal{N}_\tau$ and p_τ in the range of $\sigma_{\bar{\tau}\tau}$.

To simplify the notation, let D denote B^0_τ and $\sigma_{\bar{\tau},\tau}$ denote $\sigma^0_{\bar{\tau},\tau}$.

Lemma 11.3.14 *D is unbounded in τ.*

Proof. Given $\tau' < \tau$, we find $\tilde{\tau} \geq \tau'$ in D. In $\mathcal{P}$, form an elementary hull of $\{\mathcal{N}_\tau, \tau', \mathcal{S}\}$ in $H_{(\kappa^{++})}$ (in which everything relevant is present). Let H be the transitive collapse of the hull and $\sigma_0 : H \to H_{\kappa^{++}}$ be the uncollapse map. Set:

- $\bar{x} = \sigma_0^{-1}(x)$ for any x in range of σ_0,
- $\sigma = \sigma_0 \restriction \bar{\mathcal{N}}_\tau : \bar{\mathcal{N}}_\tau \to \mathcal{N}_\tau$,
- $\tilde{\tau} = \sup(\sigma''\bar{\tau})$.

Note that since $\tau \in \mathcal{S}^0$, $\mathcal{N}_\tau = \mathcal{N}_\tau^*$ and either $\mathrm{cr}(E^{\mathrm{top}}_{\mathcal{N}_\tau}) \geq \kappa$ or $\omega\rho^1_{\mathcal{N}_\tau} > \kappa$. Set $n = n_\tau$. Using the interpolation lemma (Lemma [17, Lemma 1.2]), we can find a map $\tilde{\sigma} : \bar{\mathcal{N}}_\tau \to \tilde{\mathcal{N}}$ which is $\Sigma_0^{(n)}$-preserving and cofinal (the map $\tilde{\sigma}$ is the ultrapower map via the $(\mathrm{cr}(\sigma), \tilde{\tau})$-extender derived from σ). Note that $\tilde{\tau} = (\kappa^+)^{\tilde{\mathcal{N}}}$. Also, by the interpolation lemma, there is a map $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ satisfying $\sigma' \restriction \tilde{\tau} = \mathrm{id}$, $\sigma'(\tilde{\tau}) = \tau$, and $\sigma' \circ \tilde{\sigma} = \sigma$.

We have that

- $\tilde{\mathcal{N}}$ is a hod premouse of the same type as $\mathcal{N}_\tau$.
- $\tilde{\mathcal{N}}$ is sound.
- $\omega\rho^\omega_{\tilde{\mathcal{N}}} = \omega\rho^{n+1}_{\tilde{\mathcal{N}}} \leq \kappa$.

The above follow from the proof of [17, Lemma 3.7] for the most part, except for the first item in the case when $\mathcal{N}_\tau$ is B-active. In this case, the first item follows from [20, Lemma 2.36] and hull condensation of Σ.[12]

It remains to see that $\tilde{\mathcal{N}}$ is indeed $\mathcal{N}^*_{\tilde{\tau}}$. We apply Theorem 11.1.2. (a) cannot hold since $\tilde{\tau} = \mathrm{cr}(\sigma') = (\kappa^+)^{\tilde{\mathcal{N}}} < \tau = (\kappa^+)^{\mathcal{N}_\tau}$. (c) cannot hold because $\tilde{\mathcal{N}}$ is sound. (d) cannot hold since $\tilde{\tau}$ is a cardinal in $\mathrm{Ult}(\mathcal{N}_\tau, E^{\mathcal{N}_\tau}_{\tilde{\tau}})$ while $\tilde{\mathcal{N}}$ definably collapses $\tilde{\tau}$. So (b) holds. This easily implies $\tilde{\mathcal{N}} = \mathcal{N}^*_{\tilde{\tau}}$. □

Lemma 11.3.15 *D is a closed subset of τ.*

Proof. Let $\tilde{\tau}$ be a limit point of D. We show that $\tilde{\tau} \in D$. Form the direct limit $\langle \tilde{\mathcal{N}}, \sigma_{\bar{\tau}\tilde{\tau}} \mid \bar{\tau} \in D \cap \tilde{\tau} \rangle$ of the system $\langle \mathcal{N}^*_{\bar{\tau}}, \sigma_{\tau^*\bar{\tau}} \mid \tau^* \leq \bar{\tau} \wedge \tau^*, \bar{\tau} \in D \cap \tilde{\tau} \rangle$. The direct limit is well-founded and there is a Σ_0 embedding $\sigma : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ (defined by $\sigma(\sigma_{\bar{\tau}\tilde{\tau}}(x)) = \sigma_{\bar{\tau},\tau}(x)$). It is easy to check that:

(a) $\sigma \circ \sigma_{\bar{\tau}\tilde{\tau}} = \sigma_{\bar{\tau}\tau}$.

(b) $\tilde{\tau} = \sigma_{\bar{\tau}\tilde{\tau}}(\bar{\tau})$, $\sigma_{\tilde{\tau}\tau}(\tilde{\tau}) = \tau$, and $\tilde{\tau} = \mathrm{cr}(\sigma)$.

(c) σ is $\Sigma_0^{(n)}$ preserving where $n = n_\tau$ (with respect to the language of coherent structures).

[12] Indexing branches using the $\mathfrak{B}$-operator allows the proof of [20, Lemma 2.36] to go through in this situation. The traditional approach to indexing branches does not seem to imply that $\tilde{N}$ is a hod premouse.

We need to see that $\tilde{\mathcal{N}} = \mathcal{N}^*_{\tilde{\tau}}$. First, we show that $\tilde{\mathcal{N}}$ is a hod premouse of the same type as $\mathcal{N}_\tau$. Note that Π_2-properties which hold on a tail end are upward preserved under direct limit maps (cf. [17, pg 8-9]). Furthermore, $\mathcal{N}^*_{\bar{\tau}}$ is of the same type as $\mathcal{N}_\tau$ for each $\bar{\tau} \in D \cap \tau$. So $\tilde{\mathcal{N}}$ is of the same type as $\mathcal{N}_\tau$ (as either a passive hod premouse, or a B-active hod premouse, or an E-active hod premouse with $\mathrm{cr}(E^{\mathrm{top}}_{\tilde{\mathcal{N}}}) > \mu$, in which case $\tilde{\mathcal{N}}$ is of type A, or else an E-active hod premouse with $\mathrm{cr}(E^{\mathrm{top}}_{\tilde{\mathcal{N}}}) = \mu$, in which case $\omega\rho^1_{\tilde{\mathcal{N}}} > \kappa$; these statements can be expressed in a Π_2-fashion).

Recall that for $\bar{\tau} \in D$, we use $\tilde{h}_{\bar{\tau}}$ to denote $\tilde{h}^{(n_{\bar{\tau}}+1)}_{\mathcal{N}^*_{\bar{\tau}}}$, the $\Sigma_1^{(n_{\bar{\tau}})}$-Skolem function of $\mathcal{N}^*_{\bar{\tau}}$. Here note that $n_{\bar{\tau}} = n_\tau = n$. Let $\tilde{p} = \sigma_{\bar{\tau}\tilde{\tau}}(p^*_{\bar{\tau}})$ for $\bar{\tau} \in D \cap \tilde{\tau}$. Given any $x \in \tilde{\mathcal{N}}$, there is $\bar{\tau} \in D \cap \tilde{\tau}$ and $\bar{x} \in \mathcal{N}^*_{\bar{\tau}}$ such that $x = \sigma_{\bar{\tau}\tau}(\bar{x})$. There is $\xi < \kappa$ such that $\bar{x} = \tilde{h}_{\bar{\tau}}(\xi, p_{\bar{\tau}})$. This $\Sigma_1^{(n)}$-statement is preserved by $\sigma_{\bar{\tau}\tilde{\tau}}$, so $x = \tilde{h}^{n+1}_{\tilde{\mathcal{N}}}(\xi, \tilde{p})$. So $\tilde{\mathcal{N}} = \tilde{h}^{n+1}_{\tilde{\mathcal{N}}}(\kappa \cup \{\tilde{p}\})$.

This gives $\omega\rho^{n+1}_{\tilde{\mathcal{N}}} = \omega\rho^\omega_{\tilde{\mathcal{N}}} \leq \kappa$. But κ is a cardinal in $\mathcal{P}$, so we indeed have equality. For each $\alpha \in p_\tau$, there is a generalized witness for α with respect to $(\mathcal{N}_\tau, p_\tau)$ in range of σ. This is because $\mathrm{rng}(\sigma)$ contains $\mathrm{rng}(\sigma_{\bar{\tau},\tau})$ for any $\bar{\tau} \in D \cap \tilde{\tau}$ and $\mathrm{rng}(\sigma_{\bar{\tau},\tau})$ contains such a witness. This takes care of (c) in the definition of D. This easily implies that $\tilde{\mathcal{N}}$ is sound and $\tilde{p}$ is the standard paramter of $\tilde{\mathcal{N}}$. We can now apply Theorem 11.1.2 as in the proof of Lemma 11.3.14 to conclude that $\tilde{\mathcal{N}} = \mathcal{N}^*_{\tilde{\tau}}$. $\square$

Lemmata 11.3.14, 11.3.15 together complete the proof of Lemma 11.3.11 in the case $\tau \in \mathcal{S}^0$.

11.3.4 When $\tau \in \mathcal{S}^1$

Fix $\tau \in \mathcal{S}^1$ a limit point of $\mathcal{S}$ of uncountable cofinality. If B^0_τ is defined, then as in the previous section, using the fact that $\mathrm{crt}(E^{top}_{\mathcal{N}^*_\tau}) \geq \kappa$ or $\rho_1^{\mathcal{N}^*_\tau} > \kappa$, we can show that B^0_τ is closed and unbounded in τ. So let us now focus on the case B^1_τ is defined. Define $D \subset \tau$ to be the set of $\bar{\tau} \in \mathcal{S}$ such that

- $(\mu_\tau, q^*_{\bar{\tau}})$ is a strong divisor of $\mathcal{N}_{\bar{\tau}}$ where $q^*_{\bar{\tau}}$ is the bottom segment of $p_{\bar{\tau}}$ of length m_τ (recall m_τ is the length of q_τ).
- Letting $\mathcal{M}^*_{\bar{\tau}}$ be the protomouse of $\mathcal{N}_{\bar{\tau}}$ associated with $(\mu_\tau, q^*_{\bar{\tau}})$, there is a map $\sigma_{\bar{\tau}\tau} : \mathcal{M}^*_{\bar{\tau}} \to \mathcal{M}_\tau$ that is Σ_0-preserving (with respect to the language of coherent structures) such that

  (a) $\bar{\tau} = \mathrm{cr}(\sigma_{\bar{\tau}\tau})$ and $\sigma_{\bar{\tau}\tau}(\bar{\tau}) = \tau$.

  (b) $\sigma_{\bar{\tau}\tau}(q^*_{\bar{\tau}}) = q_\tau$.

(c) for each $\alpha \in q_\tau$, there is a generalized witness for α with respect to $\mathcal{N}_\tau$ and q_τ in the range of $\sigma_{\bar{\tau}\tau}$ (in the language of coherent structures).

We will show that there is some $\bar{\tau} < \tau$ such that $B_\tau - \bar{\tau} = D - \bar{\tau}$. Part of this is to show that for all sufficiently large $\bar{\tau} \in D$, $(\mu_\tau, q^*_{\bar{\tau}}) = (\mu_{\bar{\tau}}, q_{\bar{\tau}})$.

Lemma 11.3.16 *D is unbounded in τ.*

Proof. Let $\tau' < \tau$. As before, we find $\tilde{\tau} \in D$ above τ'. We note that $\mathcal{M}_\tau$ may be $\mathcal{N}_\tau$; this happens when $\mathcal{N}_\tau = \mathcal{N}^*_\tau$ is pluripotent. Since protomice are present, we carry out the argument in the language of coherent structures.

We let σ_0, H be defined as in Lemma 11.3.14. Again, we denote $\bar{x}$ for $\sigma_0^{-1}(x)$. We let $\sigma : \bar{\mathcal{M}}_\tau \to \mathcal{M}_\tau$ and $\tilde{\tau} = \sup \sigma''\bar{\tau}$. As before, $\tau' \leq \tilde{\tau} < \tau$. Let $\tilde{\sigma} : \bar{\mathcal{M}}_\tau \to \tilde{\mathcal{M}}$ be the $(\mathrm{cr}(\sigma), \tilde{\tau})$-ultrapower map derived from σ and $\sigma' : \tilde{\mathcal{M}} \to \mathcal{M}_\tau$ be the factor map. As in [17, Lemma 3.10], we have:

- $\tilde{\sigma}(\bar{\kappa}, \bar{\tau}) = (\kappa, \tilde{\tau})$.
- $\mathrm{cr}(\sigma') = \tilde{\tau}$ and $\sigma(\tilde{\tau}) = \tau$.
- $h_{\tilde{\mathcal{M}}}(\kappa \cup \{\tilde{q}\}) = \tilde{\mathcal{M}}$ where $\tilde{q} = \tilde{\sigma}(\bar{q_\tau})$; in other words, $\tilde{\mathcal{M}}$ is Σ_1-generated by $\kappa \cup \{\tilde{q}\}$.
- $\omega\rho^\omega_{\tilde{\mathcal{M}}} = \omega\rho^1_{\tilde{\mathcal{M}}} = \kappa$ and $\tilde{q} \in R_{\tilde{\mathcal{M}}}$, the set of very good parameters for $\tilde{\mathcal{M}}$.
- The range of $\tilde{\sigma}$ contains a generalized solidity witness for α with respect to $(\mathcal{M}_\tau, q_\tau)$ for each $\alpha \in q_\tau$.
- $\tilde{q} = p_{\tilde{\mathcal{M}}}$ and $\tilde{\mathcal{M}}$ is solid and sound.

Note that as in Lemma 11.3.14, $\tilde{\sigma}$ is Σ_0 (but not Σ_1) and is not cofinal. This implies that $\tilde{\mathcal{M}}$ is a protomouse, even if $\mathcal{M}_\tau$ is a hod premouse (in which case, $\mathcal{M}_\tau = \mathcal{N}_\tau$ is pluripotent).

We show $\tilde{\mathcal{M}} = \mathcal{N}_{\tilde{\tau}}(\mu_\tau, \tilde{q})$. Let $\mathcal{R}_0, \mathcal{R}_1$ be the hod premice associated with $\mathcal{M}_\tau, \tilde{\mathcal{M}}$, respectively. We have that $\mathcal{R}_0 = \mathrm{Ult}_n(\mathcal{N}^*_0, F)$, where F is the top extender (fragment) of $\mathcal{M}_\tau$ and $\mathcal{N}^*_0$ is largest (strict) segment of $\mathcal{M}_\tau$ such that $\omega\rho^{n+1}_{\mathcal{N}^*_0} \leq \mathrm{cr}(F)$ and F measures all sets in $\mathcal{N}^*_0$; in the other case, $\mathcal{R}_1 = \mathrm{Ult}_k(\mathcal{N}^*_1, \tilde{F})$, where $\tilde{F}$ is the top extender (fragment) of $\tilde{\mathcal{M}}$ and $\mathcal{N}^*_1$ is the largest (strict) segment of $\tilde{\mathcal{M}}$ (equivalently, of $\mathcal{M}_\tau$) such that $\omega\rho^{k+1}_{\mathcal{N}^*_1} \leq \mathrm{cr}(\tilde{F})$ and $\tilde{F}$ measures all sets in $\mathcal{N}^*_1$. Let $\pi_i : \mathcal{N}^*_i \to \mathcal{R}_i$ be the ultrapower maps and $\pi_2 : \mathcal{R}_1 \to \pi_0(\mathcal{N}^*_1)$ be the factor map

$$\pi_2(\pi_1(f)(a)) = \pi_0(f)(\tilde{\sigma}(a)).$$

Note that $\pi_2 \restriction \tilde{\tau} = \tilde{\sigma} \restriction \tilde{\tau} = \mathrm{id}$.

Note that $p_{\mathcal{R}_1} = \pi_1(p_{\mathcal{N}_1^*}) \cup p_{\tilde{\mathcal{M}}}$ (cf. [17, Lemma 2.16, 2.19]). In the case $\mathcal{N}_\tau^* \neq \mathcal{N}_\tau$, and hence $\mu_\tau = \mu$, $\pi_1(p_{\mathcal{N}_1^*})$ is the part of $p_{\mathcal{R}_1}$ above $\pi_1(\mu)$, the supremum of $\mathcal{R}_1$'s layer Woodin cardinals, and $p_{\tilde{\mathcal{M}}}$ is the part below $\pi_1(\mu)$.

The argument in [17, Lemma 3.10] then shows that $(\mu_\tau, \tilde{q})$ is a strong divisor of $\mathcal{R}_1$.[13] To show $\tilde{\mathcal{M}} = \mathcal{N}_{\tilde{\tau}}(\mu_\tau, \tilde{q})$, we show $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}}$. This then will show $\tilde{\tau} \in D$ as desired. There are two cases to consider.

Case 1. $\mathcal{N}_\tau = \mathcal{N}_\tau^*$.

If $\mathrm{cr}(F) = \mathrm{cr}(\tilde{F}) > \mu$, then it is easy to see that $\mathcal{P}^b \lhd \mathcal{R}_0, \mathcal{R}_1$. Note that in this case, $\mathcal{R}_0 = \mathcal{N}_\tau = \mathcal{N}_\tau^*$. So we can apply Theorem 11.1.2 as in the proof of Lemma 11.3.14 and conclude that $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}} = \mathcal{N}_{\tilde{\tau}}^*$. Now suppose $\mathrm{cr}(F) = \mu$ (so $\mu_\tau = \mu$). Recall from the discussion above that we know $(\mu_\tau, \tilde{q})$ is a strong divisor of $\mathcal{R}_1$ and $\tilde{q}$ is the bottom part of the standard parameter of $\mathcal{R}_1$ below $\pi_1(\mathrm{cr}(\tilde{F}))$. We show that $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}} \neq \mathcal{N}_{\tilde{\tau}}^*$ by the following claims (note that we already know that $(\mu, \tilde{q})$ is a strong divisor of $\mathcal{R}_1$). We also will get then that $(\mu_\tau, \tilde{q}) = (\mu_{\tilde{\tau}}, q_{\tilde{\tau}})$ in this case.

Let γ_τ be defined as in Definition 11.3.3 for $\mathcal{N}_\tau$; let $\gamma_{\tilde{\tau}}, \tilde{\gamma}$ be defined similarly for $\mathcal{N}_{\tilde{\tau}}^*, \mathcal{R}_1$, respectively. Let Λ be $\mathcal{R}_0$'s iteration strategy.

Claim 11.3.17 $\tilde{\gamma} = \gamma_{\tilde{\tau}}$.

Proof. Suppose not. Assume $\tilde{\gamma} < \gamma_{\tilde{\tau}}$ (the other case is similar). Let E be least on the extender sequence of $\mathcal{N}_{\tilde{\tau}}$ such that

- $\mathrm{cr}(E) = \mu$,
- $\mathrm{lh}(E) \geq \tilde{\gamma}$.

Let $\mathcal{S} = \mathrm{Ult}(\mathcal{R}_0, E)$. Note that $\tilde{\gamma}$ is a cutpoint of $\mathcal{S}$ and $i_E(\mu)$ is a limit of Γ-full Woodin cardinals above $\tilde{\gamma}$. By SMC in Γ, we can conclude that $\mathcal{R}' \in \mathcal{S}$, where $\mathcal{R}'$ is a sound hod premouse extending $\mathcal{R}_1|\tilde{\gamma}$, having $\tilde{\tau} = \kappa^+$, $\tilde{\gamma}$ as a cutpoint, and projects to κ. [14]

[13] The proof of this fact does not depend on whether $\mu_\tau > \mu$.

[14] By genericity iterations, without loss of generality, we may assume that a real witnessing the Wadge reduction of Λ^{π_2} to Λ is generic over $\mathcal{S}$. In $\mathcal{S}$'s derived model at $i_E(\mu)$, we can find $\mathcal{R}_1$. This means, in the derived model of $\mathcal{S}$, there is some hod mouse $\mathcal{R}$ extending $\mathcal{R}_1|\tilde{\gamma}$, having $\tilde{\tau} = \kappa^+$, $\tilde{\gamma}$ as a cutpoint, and projects to κ; furthermore, we can demand that $(\mu_\tau, \tilde{q})$ is a strong divisor of $\mathcal{R}$ and $\tilde{q}$ is the bottom part of the standard parameter of $\mathcal{R}$ below the supremum of $\mathcal{R}$'s layer Woodin cardinals. Let Ω be the Wadge-minimal pointclass that has a pointclass generator with these properties. Note that this determines the unique pointclass generator $\mathcal{S}_\Omega$ for Ω. This implies that $\mathcal{S}_\Omega \in \mathcal{S}$.

Fix $\mathcal{R}' \in \mathcal{S}$ as above. $\mathcal{R}'$ defines a surjection f from κ onto $\tilde{\tau}$. Since $\mathcal{R}' \in \mathcal{S}$, $f \in \mathcal{S}$. This contradicts the fact that $\mathcal{S} \vDash \tilde{\tau} = \kappa^+$. □

Claim 11.3.18 *There is a pointclass Ω with pointclass generator a sound hod mouse that projects to κ, extends $\mathcal{P}|\tilde{\gamma}$, having $\tilde{\tau} = \kappa^+$, $\tilde{\gamma}$ as a cutpoint, and the set of layer Woodin cardinals above $\tilde{\gamma}$ has limit order type. $\mathcal{R}_1$ is the generator for the Wadge minimal such pointclass.*

Proof. Clearly, such Ω exists since the pointclass generated by $\mathcal{R}_1$ is such. Let Ω_0 be the pointclass $\mathcal{R}_1$ generates and Ω_1 be a pointclass satisfying the hypothesis of the claim. Let $\mathcal{N}$ generate Ω_1 with the properties in the statement of the claim. Note that at this point, we know $\mathcal{R}_1$ and $\mathcal{N}$ are: sound, projects to κ, extends $\mathcal{P}|\gamma_{\tilde{\tau}}$, satisfies $\kappa^+ = \tilde{\tau}$, and have $\gamma_{\tilde{\tau}}$ as cutpoint.

We claim that $\Omega_0 = \Omega_1$. Suppose for contradiction that $\Omega_0 \subsetneq \Omega_1$ (the other case is similar). Then, using $\mathbb{R}$-genericity iteration and elementarity, in the derived model of $\mathcal{N}$ (at the supremum of its Woodin cardinals) there is a pointclass with a generator $\mathcal{S}$ that is sound, projects to κ, extends $\mathcal{P}|\gamma_{\tilde{\tau}}$, satisfies $\kappa^+ = \tilde{\tau}$, and have $\gamma_{\tilde{\tau}}$ as cutpoint. Some such $\mathcal{S}$ is in $\mathcal{N}$ by a similar argument as in Footnote 14. This implies as in Claim 11.3.17 that $\tilde{\tau}$ is not a cardinal in $\mathcal{N}$. Contradiction.

Now we can compare $\mathcal{R}_1$ against $\mathcal{N}$. The comparison is an extender comparison, is successful, and is above $\gamma_{\tilde{\tau}}$. Since both models are κ-sound, projects to κ, and $\kappa < \gamma_{\tilde{\tau}}$. We conclude that $\mathcal{N} = \mathcal{R}_1$. □

Using the claims and the fact that $(\mu_\tau, \tilde{q})$ is a strong divisor of $\mathcal{R}_1$ (note that $\max(\tilde{q}) < (\gamma^+_{\tilde{\tau}})^{\mathcal{R}_1}$ and $\tilde{q}$ is the bottom part below $\pi_1(\mathrm{cr}(\tilde{F}))$ of the standard parameter of $\mathcal{R}_1$) we easily verify that $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}}$ and hence $\tilde{\mathcal{M}} = \mathcal{N}_{\tilde{\tau}}(\mu_\tau, \tilde{q})$. Hence $\tilde{\tau} \in D$ as desired.

Case 2. $\mathcal{N}_\tau \neq \mathcal{N}^*_\tau$.

In this case, $\mu_\tau = \mu$. As above, $\mathcal{R}_0 = \mathcal{N}_\tau$ and (μ_τ, q_τ) is a strong divisor of $\mathcal{N}_\tau$. We aim to show that $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}}$. As above, $(\mu, \tilde{q})$ is a strong divisor of $\mathcal{R}_1$ by the proof of [17, Lemma 3.10]; also $\max(\tilde{q}) < (\tilde{\gamma}^+)^{\mathcal{R}_1} = (\gamma^+_{\tilde{\tau}})^{\mathcal{R}_1}$ and $\tilde{q}$ is the bottom part below $\mathcal{R}_1$'s limit of layer Woodin cardinals $\pi_1(\mathrm{cr}(\tilde{F}))$ of the standard parameter of $\mathcal{R}_1$. This easily implies, using Claim 11.3.18, that $\mathcal{N}_{\tilde{\tau}} \neq \mathcal{N}^*_{\tilde{\tau}}$, $\mathcal{R}_1 = \mathcal{N}_{\tilde{\tau}}$, $\mu = \mu_{\tilde{\tau}}$, $\tilde{q} = q_{\tilde{\tau}}$ and hence $\tilde{\mathcal{M}} = \mathcal{N}_{\tilde{\tau}}(\mu_{\tilde{\tau}}, q_{\tilde{\tau}})$. So $\tilde{\tau} \in D$ as desired. □

Lemma 11.3.19 *D is a closed subset of τ.*

Proof. Let $\tilde{\tau}$ be a limit point of D. We show that $\tilde{\tau} \in D$. As in Lemma 11.3.15, form the direct limit $\langle \tilde{\mathcal{M}}, \sigma^1_{\bar{\tau}\tilde{\tau}} \mid \bar{\tau} \in D \cap \tilde{\tau} \rangle$ of the system $\langle \mathcal{M}^*_{\bar{\tau}}, \sigma^1_{\tau^*\bar{\tau}} \mid \tau^* \leq \bar{\tau} \wedge \tau^*, \bar{\tau} \in D \cap \bar{\tau} \rangle$. The direct limit is well-founded (so we identify $\tilde{\mathcal{M}}$ with its transitive collapse) and there is a Σ_0 embedding $\sigma : \tilde{\mathcal{M}} \to \mathcal{M}_\tau$ (defined by $\sigma(\sigma^1_{\bar{\tau}\tilde{\tau}}(x)) = \sigma^1_{\bar{\tau},\tau}(x)$). It is easy to check that (cf. [17, Lemma 3.11]):

(a) $\tilde{\mathcal{M}}$ is a coherent structure.

(b) $\sigma \circ \sigma^1_{\bar{\tau}\tilde{\tau}} = \sigma^1_{\bar{\tau}\tau}$.

(c) $\tilde{\tau} = \sigma^1_{\bar{\tau}\tilde{\tau}}(\bar{\tau})$, $\sigma^1_{\tilde{\tau}\tau}(\tilde{\tau}) = \tau$, and $\tilde{\tau} = \mathrm{cr}(\sigma)$.

(d) $h_{\tilde{\mathcal{M}}}(\kappa \cup \{\tilde{q}\}) = \tilde{\mathcal{M}}$ where $\tilde{q} = \sigma^1_{\bar{\tau}\tilde{\tau}}(q^*_{\bar{\tau}})$, so $\omega\rho^\omega_{\tilde{\mathcal{M}}} = \omega\rho^1_{\tilde{\mathcal{M}}} = \kappa$ and $\tilde{q} \in R_{\tilde{\mathcal{M}}}$.

(e) For every $\alpha \in q_\tau$, there is a generalized witness for α with respect to $(\mathcal{M}_\tau, q_\tau)$ in the range of σ. Hence $\tilde{q} = p_{\tilde{\mathcal{M}}} = \sigma^{-1}(q_\tau)$ and $\tilde{\mathcal{M}}$ is sound and solid.

The first four clauses are clear. The last follows from the fact that the direct limit $\tilde{\mathcal{M}}$ satisfies Π_2-statements which hold on a tail-end of $D \cap \tilde{\tau}$.

Note that $\tilde{\mathcal{M}}$ is always a protomouse (this is because σ is not cofinal). If $\mu_\tau > \mu$ (or equivalently $\mathcal{N}_\tau = \mathcal{N}^*_\tau$), we can appeal to the proof of [17, Lemma 3.11] to get that $\tilde{\mathcal{M}} = \mathcal{N}_{\tilde{\tau}}(\mu_\tau, \tilde{q})$ and $(\mu_\tau, \tilde{q})$ is a strong divisor of $\tilde{\mathcal{M}}$. Otherwise, the same conclusion follows from the proof of Claim 11.3.18.

The previous paragraph gives $\tilde{\tau} \in D$ as desired. $\square$

Lemma 11.3.20 *There is a $\bar{\tau} < \tau$ such that for all $\tau' \in D - \bar{\tau}$, $(\mu_\tau, q^*_{\tau'}) = (\mu_{\tau'}, q_{\tau'})$. Consequently, $B^1_\tau - \bar{\tau} = D - \bar{\tau}$.*

Proof. We need to prove that there is $\bar{\tau} < \tau$ such that for every $\tau' \in D - \bar{\tau}$, $(\mu_\tau, q^*_{\bar{\tau}}) = (\mu_{\tau'}, q_{\tau'})$. Assume for contradiction that there is a sequence $\langle \tau_i \mid i < \delta \rangle$ that is increasing, cofinal in τ such that $(\mu_{\tau_i}, q_{\tau_i}) \neq (\mu_\tau, q^*_{\tau_i})$. We may assume without loss of generality that the sequence $\langle \mu_{\tau_i} \mid i < \delta \rangle$ is monotonic and all q_{τ_i}'s have the same length, say m.

If $\mu_\tau = \mu$, then we claim that for each $i < \delta$, $(\mu_{\tau_i}, q_{\tau_i}) = (\mu_\tau, q^*_{\tau_i})$. This follows from the proof of Lemma 11.3.16, where we prove that in this case, $\mathcal{N}_\tau \neq \mathcal{N}^*_\tau$ and so for each $i < \delta$, $\mathcal{N}_{\tau_i} \neq \mathcal{N}^*_{\tau_i}$ and $\mu_{\tau_i} = \mu = \mu_\tau$ and $q_{\tau_i} = q^*_{\tau_i}$. This contradicts the assumption that $(\mu_{\tau_i}, q_{\tau_i}) \neq (\mu_\tau, q^*_{\tau_i})$. So we must have that $\mu_\tau > \mu$, so $\mathcal{N}_\tau = \mathcal{N}^*_\tau$. This implies that for each $i < \delta$, $\mathcal{N}_{\tau_i} = \mathcal{N}^*_{\tau_i}$ (again, by remarks in Section 11.3.1 and the argument in Lemma 11.3.16). So it must be the case then that $\mu_{\tau_i} > \mu$

(recall that $\mathcal{N}^*_{\tau_i}$ cannot have divisors of the form (μ, q) for some q) and so $(\mu_{\tau_i}, q_{\tau_i})$, by definition, is the canonical strong divisor of $\mathcal{N}_{\tau_i}$.

By the definition of $(\mu_{\tau_i}, q_{\tau_i})$, each q_{τ_i} is a bottom part of $q^*_{\tau_i}$, say $q^*_{\tau_i} = q_{\tau_i} \cup s_{\tau_i}$ (s_{τ_i} may be empty). Recall we have shown $\mu_\tau, \mu_{\tau_i} > \mu$ (so we can freely quote results of [17, Section 2.4 and Lemma 3.12] in the arguments that follow). Now we observe that $\mu_{\tau_i} > \mu_\tau$ for all $i < \delta$. This is because the argument in [17, Lemma 3.12] shows: if $q_{\tau_i} = q^*_{\tau_i}$, then μ_{τ_i} must be $> \mu_\tau$ by maximality of μ_{τ_i} for $\mathcal{N}_{\tau_i}$ and the assumption that $(\mu_\tau, q^*_{\tau_i}) \neq (\mu_{\tau_i}, q_{\tau_i})$; otherwise, q_{τ_i} is a strict bottom segment of $q^*_{\tau_i}$, and hence[17, Lemma 2.26] shows that no $\nu \leq \mu_\tau$ is such that (ν, q_{τ_i}) is a strong divisor of $\mathcal{N}_{\tau_i}$.

Set for some (equivalently for all sufficiently large) $i < \delta$, $q = \sigma_{\tau_i \tau}(q_{\tau_i})$, $s = \sigma_{\tau_i \tau}(s_{\tau_i})$, $r = r_\tau$, $\nu = \sup_{i<\delta} \mu_{\tau_i}$. Now (ν, q) is a divisor of $\mathcal{N}_\tau$ (see [17, Lemma 3.12]). Since $\nu > \mu_\tau > \mu$, (ν, q) cannot be a strong divisor of $\mathcal{N}_\tau$. Then a calculation as in [17, Lemma 3.12] shows that for some $i < \delta$, $(\mu_{\tau_i}, q_{\tau_i})$ is not a strong divisor of $\mathcal{N}_{\tau_i}$. Contradiction. $\square$

Lemmata 11.3.16, 11.3.19, 11.3.20 together complete the proof of Lemma 11.3.11 in the case $\tau \in \mathcal{S}^1$.

Chapter 12

LSA from PFA

For a cardinal κ, let $\wp_0(\kappa) = \kappa$; $\wp_{n+1}(\kappa) = 2^{\wp_n(\kappa)}$ for all $n < \omega$. We prove the following theorem.

Definition 12.0.21 *A sequence $\langle \vec{C}_\alpha \mid \alpha \in \lambda \rangle$ is a $\square(\kappa, \lambda)$ sequence if it satisfies the following properties.*

(i) $0 < |\vec{C}_\alpha| < \kappa$ *for all limit* $\alpha \in \lambda$.

(ii) $C \subseteq \alpha$ *is club in* α *for all limit* $\alpha \in \lambda$ *and* $C \in \vec{C}_\alpha$.

(iii) $C \cap \beta \in \vec{C}_\beta$ *for all limit* $\alpha \in \lambda$, $C \in \vec{C}_\alpha$ *and* $\beta \in \mathrm{Lim}(C)$.

(iv) There is no club $D \subseteq \lambda$ *such that* $D \cap \alpha \in \vec{C}_\alpha$ *for all* $\alpha \in \mathrm{Lim}(D)$.

We say that $\square(\kappa, \lambda)$ *holds if a* $\square(\kappa, \lambda)$*-sequence exists.*

Clearly, $\square_{\lambda, <\kappa}$ implies $\square(\kappa, \lambda^+)$ and if $\kappa \leq \kappa'$, then $\square(\kappa, \lambda)$ implies $\square(\kappa', \lambda)$. $\square(2, \lambda)$ is $\square(\lambda)$.

Theorem 12.0.22 *Suppose* κ *is a cardinal such that* $\kappa^\omega = \kappa$. *Suppose for all* $\alpha \in [(\wp_0(\kappa))^+, (\wp_3(\kappa))^+]$, $\neg\square(3, \alpha)$. *Then there is a model* M *containing* $\mathrm{OR} \cup \mathbb{R}$ *such that* $M \vDash \mathsf{LSA}$.

As in [31], we immediately have the following corollary.

Corollary 12.0.23 *Assume one of the following theories.*

1. PFA.

2. *There is a strongly compact cardinal.*

3. *There is a cardinal* $\lambda \geq (\wp_3(\aleph_2))^+$ *such that the set* $\{X \prec H_{\lambda^+} \mid X^\omega \subset X \wedge X$ *is* ω_2*-guessing* $\wedge\ |X| = \aleph_2\}$ *is stationary.*

Then there is a model M *containing* $\mathrm{OR} \cup \mathbb{R}$ *such that* $M \vDash \mathsf{LSA}$*.*

Proof. For (1) and (3), let $\kappa = \aleph_2$. It is well-known that both (1) and (3) imply the hypothesis of Theorem 12.0.22 (cf. [32] and [33] for (3)). For (2), let κ be a strong limit cardinal of uncountable cofinality above a strongly compact cardinal. The hypothesis for Theorem 12.0.22 holds at κ by the construction in [21]. □

Theorem 12.0.22 obtains models of LSA from a combinatorial principle that does not involve large cardinal properties. Therefore, in contrast to the previous chapter where one shows there are LSA models inside the derived model at some limit of Woodin cardinals, here we use the core model induction method to construct some model of determinacy (which plays the role of the derived model in the previous section) that satisfies LSA. The proof of Theorem 12.0.22 is built on that of [31], which in turns is inspired by [25] and [14].

The rest of the chapter is dedicated to proving Theorem 12.0.22. We assume the hypothesis of Theorem 12.0.22 along with the following simplifying assumption on cardinal arithmetic:

$$\forall \alpha \in [\kappa, \kappa^{++}]\ 2^\alpha = \alpha^+. \tag{12.1}$$

Note that assumption 12.1 implies that

$$\forall \alpha \in [\kappa^+, \kappa^{+++}],\ \alpha^\omega = \alpha.$$

This is because $\kappa^\omega = \kappa$. We will use this fact many times later on. Later, we show how to get rid of assumption 12.1. Our smallness assumption throughout this section is:

(†) : in $V[G]$, there is no model M containing $\mathbb{R} \cup \mathrm{OR}$ such that $M \vDash$ "$\mathsf{ZF} + \mathsf{AD}^+ + \Theta = \theta_{\alpha+2} + \theta_{\alpha+1}$ is the largest Suslin cardinal below $\theta_{\alpha+1}$ ".

Before plunging in the the details, we give a very rough outline of the proof of Theorem 12.0.22. Fix κ as in the hypothesis of Theorem 12.0.22. We operate under assumptions (†) and 12.1. Let $\mathbb{P} = Col(\omega, \kappa)$. In $V^{\mathbb{P}}$, let Ω be the "maximal pointclass of determinacy" (as defined in [31]). Let $\mathcal{P}^-$ be the direct limit of hod pairs $(\mathcal{M}, \Sigma)$ such that $\Sigma \restriction \mathrm{HC} \in \Omega$ and Σ is Ω-fullness preserving and has branch condensation. Let $\mathcal{P}$ be the appropriate "Lp"-closure of $\mathcal{P}^-$ (defined in Section 12.1). So $\mathcal{P}^-$ is an initial segment of $\mathcal{P}$. [31] shows that $\mathcal{P} \vDash o(\mathcal{P}^-)$ is a regular limit of Woodin cardinals. In $V^{\mathbb{P}}$, we carry out a hybrid K^c-construction over $\mathcal{P}$ (to be explained in Section 12.2). Either the construction stops prematurely (before stage κ^{+++} for various reasons to be specified in Section 12.2), in which case we show that a model of LSA has been reached; otherwise, we reach a model $\mathcal{P}^+$ (extending $\mathcal{P}$) of height κ^{+++}. Then we consider the stack $\mathcal{S}$ of (appropriately defined hod) mice over $\mathcal{P}^+$. Using the proof of [5, Theorem 3.4], we show that $\text{cof}(o(\mathcal{S})) \geq \kappa^{+++}$. Using the fact that $\mathcal{S} \in V$, we show that $\text{cof}(o(\mathcal{S})) < \kappa^{+++}$. Contradiction.

12.1 Some core model induction backgrounds

We continue to assume (†) and 12.1 in this section. We recall some notions and results from [31]. In $V[G]$, where $\mathbb{P} = Col(\omega, \kappa)$ and $G \subseteq \mathbb{P}$ is V-generic, let

$$\Omega = \bigcup\{\wp(\mathbb{R}) \cap M \mid \mathbb{R} \cup \text{OR} \subset M \wedge M \vDash \mathsf{AD}^+\}.$$

[31] shows that, under (†),[1] the Solovay sequence $\langle \theta^\Omega_\alpha \mid \alpha \leq \gamma\rangle$ of Ω is of limit length. Furthermore, if $A \in \Omega$, then there is a hod pair (or sts hod pair) $(\mathcal{P}, \Sigma) \in \Omega$ such that $A \in \Gamma^b(\mathcal{P}, \Sigma)$.

Let $\mathcal{P}^-$ be the direct limit of all hod pairs $(\mathcal{M}, \Sigma)$ such that $\mathcal{M}$ is countable in $V^{\mathbb{P}}$ and Σ is an $(\omega_1, \omega_1 + 1)$-strategy of $\mathcal{M}$ that is Ω-fullness preserving, positional, commuting, has branch condensation, and $\Sigma \restriction \text{HC} \in \Omega$. We will say that a pair $(\mathcal{P}, \Sigma)$ with these properties is *nice* and let $\mathcal{F}$ be the direct limit system of all nice hod pairs. [31] shows that if $(\mathcal{M}, \Sigma \restriction V) \in V$, then Σ can be uniquely extended to a $(\kappa^{+4}, \kappa^{+4})$-strategy Σ^+ (and hence $\Sigma^+ \restriction V \in V$). Say $\mathcal{M}$ iterates (via Σ^+) to $\mathcal{P}^-(\alpha)$ for some $\alpha < \gamma = \lambda^{\mathcal{P}^-}$, we let Σ_α be the Σ^+-tail of Σ^+. Σ_α only depends on α and does not depend on any particular choice of $(\mathcal{M}, \Sigma^+)$ as long as Σ^+ is nice. Let

$$\Sigma = \oplus_{\alpha < \lambda^{\mathcal{P}}} \Sigma_\alpha,$$

and

$$\mathcal{P} = \text{Lp}^\Sigma(\mathcal{P}^-).$$

That is $\mathcal{P}$ is the collection of $\mathcal{P}^- \lhd \mathcal{M}$ such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) \leq o(\mathcal{P}^-)$, then $\mathcal{M}$ is a Σ-premouse over $\mathcal{P}^-$ and for every countable, transitive $\mathcal{M}^*$ embeddable into $\mathcal{M}$ via π, $\mathcal{M}^*$ is $(\omega_1 + 1)$-iterable as an (anomalous) hod mouse with strategy Λ such that $\Lambda \restriction \text{HC} \in \Omega$.

[1] [31] uses a stronger assumption, namely no models of "$\mathsf{AD}_{\mathbb{R}}$+Θ is regular" exist. But the same proof works using (†); the main point is that the HOD analysis now can be carried out up to models of LSA.

Lemma 12.1.1 *Let λ be the ordinal height of Ω, so $\lambda = o(\mathcal{P}^-) = \delta^{\mathcal{P}}$.*

1. *No levels of $\mathcal{P}$ projects across λ. Hence $\rho_\omega(\mathcal{P}) = o(\mathcal{P})$ and $\mathcal{P} \vDash \mathsf{ZFC}^-$.*

2. *$\mathcal{P} \vDash \delta^{\mathcal{P}}$ is a regular limit of Woodin cardinals*

3. *$\lambda \leq \kappa^{++}$.*

4. *In V, $o(\mathcal{P}) < \lambda^+$ and cof$(o(\mathcal{P})) \leq \kappa$.*

Proof. (1) and (2) follow from [31, Lemma 3.78]. (3) follows from $2^\kappa = \kappa^+$ and the fact that $\omega_1 = (\kappa^+)^V$ in $V[G]$.

For (4), first note that $\mathcal{P} \in V$. Let $\vec{C}$ be the $\square_\lambda$-sequence built in $\mathcal{P}$, where $\lambda = o(\mathcal{P}^-)$ is the ordinal height of Ω as defined above. $\vec{C}$ is not threadable (by the maximality of $\mathcal{P}$). So if $o(\mathcal{P}) = \lambda^+$ or $\mathrm{cof}(o(\mathcal{P})) \geq \kappa^+$, then using our hypothesis $\forall \alpha \in [\kappa^+, \kappa^{+4}], \neg\square(\alpha)$, we can find a thread for $\vec{C}$ by standard arguments. Contradiction. $\square$

Remark 12.1.2 *As in Chapter 9, we let $\phi(U, V)$ be the formula that expresses the fact that U is a mousefull pointclass with all the properties that Ω has and V is a hod pair $(\mathcal{Q}, \Lambda)$ such that $Code(\Lambda) \in U$ and Λ is U-fullness preserving and has branch condensation. Then the $\mathcal{F}$ above is $\mathcal{F}_{\phi,\Omega}$ etc. From this point on, we will often suppress ϕ, Ω from our notations; this should not be confusing since all the notations that come into the following constructions are relative to (ϕ, Ω).*

In $V[G]$, as done in the previous section, for each $X \subseteq \wp_{\omega_1}(\mathcal{P})$, we let $\mathcal{Q}_X$ be the transitive collapse of X, $\delta_X = \delta^{\mathcal{Q}_X}$, and $\pi_X : \mathcal{Q}_X \to \mathcal{P}$ be the uncollapse map. Let Σ_X be the π_X-pullback strategy for $\mathcal{Q}_X$.[2] For $X \subseteq Y \in \wp_{\omega_1}(\mathcal{P})$, let $\pi_{X,Y} = \pi_Y^{-1} \circ \pi_X$ and $\sigma_{X,Y} : \mathcal{Q}_Y \to \mathcal{P}$ be given by

$$\sigma_{X,Y}(q) = \pi_X(f)(\pi^{\Sigma_Y}_{\mathcal{Q}_Y,\infty}(a))$$

where $a \in (\mathcal{Q}_Y|\delta^{\mathcal{Q}_Y})^{<\omega}$ and $q = \pi_{X,Y}(f)(a)$.

Let $\mathfrak{S} \in V$ be the set of $X \prec H_{\kappa^{+++}}$ such that $X^\omega \subseteq X$, $\kappa + 1 \subset X$, $|X| = \kappa$, $X \cap \mathcal{P}$ is cofinal in $o(\mathcal{P})$. Note that $\mathfrak{S}$ is stationary.

Recall the notions of $((\phi, \Omega))$-condensing sets and honest extensions discussed in Chapter 9. The following facts follow easily from [31] and Chapter 9.

[2]Typically, $X = X^* \cap \mathcal{P}$ for some countable $X^* \prec H_{\kappa^{+++}}$. And Σ_X is the π_{X^*}-realization map, where π_{X^*} is the uncollapse map of X^*.

Lemma 12.1.3 *(i) Lower part covering holds for (ϕ,Ω). (ϕ,Ω) is maximal, homogeneous, and captured by a stationary set $\mathfrak{S}_{\phi,\Omega}\subseteq\mathfrak{S}$.*

(ii) $\forall^* X'\in\mathfrak{S}_{\phi,\Omega}$, $X=X'\cap\mathcal{P}$ is a condensing set.

(iii) Suppose Y is an honest extension of a condensing set X and there are maps $i:\mathcal{Q}_Y\to\mathcal{R}$ and $\sigma:\mathcal{R}\to\mathcal{P}$ such that $\sigma\circ i=\pi_Y$ and every $x\in\mathcal{R}$ has the form $i(f)(a)$ for $f\in\mathcal{Q}_Y$ and $a\in[\delta^{\mathcal{R}}]^{<\omega}$. Then letting Λ be the τ-pullback strategy of $\mathcal{R}$, and $\tau(i(f)(a))=\pi_Y(f)(\pi^{\Lambda}_{\mathcal{R}|\delta^{\mathcal{R}},\infty}(a))$, then τ is well-defined, (sufficiently) elementary, and $\tau\restriction\mathcal{R}|\delta^{\mathcal{R}}=\pi^{\Lambda}_{\mathcal{R}|\delta^{\mathcal{R}},\infty}\restriction\mathcal{R}|\delta^{\mathcal{R}}$.

(iv) Suppose X is condensing and Y,Z are honest extensions of X such that $\mathcal{Q}_Y=\mathcal{Q}_Z$, then $\Sigma_Y=\Sigma_Z$.

Remark 12.1.4 *Let X be as in (i) of the lemma. Then it is easy to se that any $Y=Y^*\cap\mathcal{P}$ where $Y^*\prec H_{\kappa^{+++}}$ is such that Y^* is countable (in $V[G]$) is an honest extension of X.*

12.2 Hybrid K^c-constructions and stacking mice

In this section, we proceed to describe the hybrid K^c-construction over $\mathcal{P}$. We use the notations and definitions from the previous section. We fix a condensing set $X\in V$ (X exists by the previous section); and we assume that $X=X'\cap\mathcal{P}$ where $X'\prec H_{\kappa^{+++}}$ is of size κ in V. We build in $V[G]$ a sequence $(\mathcal{N}^*_\xi,\mathcal{N}_\xi:\xi\leq\Upsilon)$ of levels of our K^c-construction such that $\mathcal{N}_0=\mathcal{N}^*_0=\mathcal{P}$, $\mathcal{N}_\xi=\mathcal{C}_\omega(\mathcal{N}^*_\xi)$ for all $\xi\leq\Upsilon$ and $\Upsilon\leq\kappa^{+++}$. Though it is clear from the construction that $\mathcal{N}_\xi,\mathcal{N}^*_\xi\in V$ for all ξ.

Before defining the sequence, we discuss the kind of background extenders being used in this construction. Suppose $\mathcal{N}_\xi$ has been constructed and is in V, is passive, $\wp(\delta^{\mathcal{P}})^{\mathcal{N}_\xi}=\wp(\delta^{\mathcal{P}})^{\mathcal{P}}$, and suppose F is a $(\mathrm{cr}(F),o(\mathcal{N}_\xi))$-extender that coheres the sequence of $\mathcal{N}_\xi$. Suppose $Y\prec H_{\kappa^{+++}}$ (or $Y\prec H_\lambda$ for $\lambda\geq\kappa^{+++}$) is in V and is countable (in $V[G]$) and Y contains all relevant objects. Let π_Y be the corresponding uncollapse map. We say Y is *good* if $Y^\omega\subseteq Y$ in V, $\kappa\cup X\subset Y$ and Y is an honest extension of X.[3] Let $(\mathcal{P}^Y,\mathcal{N}^Y_\xi)=\pi_Y^{-1}(\mathcal{P},\mathcal{N}_\xi)$. Suppose $\mathcal{N}^Y_\xi$ has a unique π_Y-realization strategy Σ^Y_ξ such that $\Sigma^Y_\xi\restriction\mathrm{HC}\in\Omega$ (these properties will be maintained during the course of our construction). We say that F is *correctly backgrounded* if one of the following holds:

[3] Technically, we should write $Y\cap\mathcal{P}$ is an honest extension of X, but we will be sloppy here and from now on.

- if $\mathrm{cr}(F) = \delta^{\mathcal{P}}$ and the least cutpoint above $\delta^{\mathcal{P}}$ is the largest cardinal in $\mathcal{N}_\xi$, then $(a, A) \in F$ if and only if for all good Y such that $(a, A) \in Y$, $\pi^{\Sigma^Y_\xi}_{\mathcal{N}_\xi,\infty}(a) \in A$. In this case, we say that F *is* π_Y*-certified over* $(\mathcal{N}^Y_\xi, \Sigma^Y_\xi)$.

- if $\mathrm{cr}(F) > \delta^{\mathcal{P}}$, then say, $\lambda = F(\mathrm{cr}(F))$, F is *certified by a collapse* in the sense of [5], that is, there is $Z \prec H^V_{\kappa^{+++}}$ (in V) such that $|Z|$ is κ^*, where $\kappa^* = |o(\mathcal{P})|$, $o(\mathcal{P}) + 1 \subset Z$, $Z^\kappa \subseteq Z$,[4] and letting $\pi : M_Z \to Z$ be the uncollapse, we have: $\mathcal{N}_\xi|\mathrm{cr}(F) \in M_Z$, $\mathrm{cr}(\pi_Z) = \mathrm{cr}(F)$, and

$$F \text{ is the trivial completion of } (\pi \restriction \wp(\mathrm{cr}(\pi_Z)) \cap \mathcal{N}_\xi) \restriction \lambda.$$

We continue with the notations of the previous paragraph. Let γ_ξ be the supremum of indices of extenders on the $\mathcal{N}_\xi$-sequence with critical point $\delta^{\mathcal{P}}$. Suppose $\gamma_\xi < o(\mathcal{N}_\xi)$ and let $\gamma_\xi \leq \lambda_\xi \leq o(\mathcal{N}_\xi)$[5] be such that $\rho_\omega(\mathcal{N}_\xi) \geq \lambda_\xi$ and there is a stack $\vec{\mathcal{T}} \in \mathcal{N}_\xi$ based on $\mathcal{N}_\xi|\lambda_\xi$ according to the internal strategy $\Sigma^{\mathcal{N}_\xi}_{\lambda_\xi}$ such that $\Sigma^{\mathcal{N}_\xi}_{\lambda_\xi}(\vec{\mathcal{T}})$ is undefined. Suppose also $\vec{\mathcal{T}}$ is such that the theory developed above (Chapter 3) dictates that a cofinal branch b for $\vec{\mathcal{T}}$ needs to be added to $\mathcal{N}_\xi$ and $\mathcal{N}_\xi$ is so that $(\mathcal{N}_\xi, B_b)$ is amenable.[6] We call such a tuple $(\mathcal{N}_\xi, \lambda_\xi, \vec{\mathcal{T}})$ *appropriate.*

We now discuss how a branch b is chosen to extend $\mathcal{N}_\xi$ for an appropriate tuple $(\mathcal{N}_\xi, \lambda_\xi, \vec{\mathcal{T}})$. Suppose $Y \prec H_{\kappa^{+++}}$ is good and contains all relevant objects. Let $(\mathcal{N}^Y_\xi, \lambda^Y_\xi, \gamma^Y_\xi, \vec{\mathcal{T}}^Y) = \pi_Y^{-1}(\mathcal{N}_\xi, \lambda_\xi, \gamma_\xi, \vec{\mathcal{T}})$. Then the B-sequence of $\mathcal{N}^Y_\xi$ [7] above λ^Y_ξ is according to:

(a) either a short tree strategy of $\mathcal{N}^Y_\xi|\lambda^Y_\xi$, which we denote $\Sigma^{sts}_{Y,\xi}$ if γ^Y_ξ is definably Woodin over $\mathcal{N}^Y_\xi|\lambda^Y_\xi$;

(b) or the strategy $\Sigma_{Y,\xi}$ of $\mathcal{N}^Y_\xi|\lambda^Y_\xi$, where $\Sigma_{Y,\xi}$ is the canonical Q-structure guided strategy of $\mathcal{N}^Y_\xi|\lambda^Y_\xi$ if $o(\mathcal{N}^Y_\xi) < o(\mathcal{M}_2^{\Sigma_{Y,\xi},\sharp})$;

(c) or else the canonical $\Sigma_{Y,\xi}$-strategy $\Lambda_{Y,\xi}$ of $\mathcal{N}^Y_\xi|\lambda^Y_\xi = \mathcal{M}_2^{\Sigma_{Y,\xi},\sharp}(\mathcal{N}^Y_\xi|\epsilon)$, where $\Sigma_{Y,\xi}$ is the canonical Q-structure guided, π_Y-realization strategy of $\mathcal{N}^Y_\xi|\epsilon$.

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[4] Note that $|o(\mathcal{P})| = |\Omega| < \kappa^{+++}$. Futhermore, $Z[G]^\omega \subseteq Z[G]$ in $V[G]$.

[5] In fact, the theory developed in Section 2, 3 dictates that $\lambda_\xi > \gamma_\xi$. This is because we don't activate new strategies above γ_ξ until we reach the first E-active $\mathcal{N}_\xi|\lambda$ above γ_ξ. At that point, we activate the short-tree strategy of $\mathcal{N}_\xi|\lambda$.

[6] B_b is a code for b as done in [20, Section 2] and outlined in Chapter 11. We only note that this amenable coding ensures condensation under very weak hull embeddings, cf. [20, Lemma 3.10] and this fact is in turns used to show that $\Box_{\kappa,2}$ holds in hod mice.

[7] Recall, this is the sequence of branch predicates that codes up some internal strategy of $\mathcal{N}^Y_\xi$.

We let $\Psi_{Y,\xi}$ denote $\Sigma^{sts}_{Y,\xi}$ in case (a), $\Sigma_{Y,\xi}$ in case (b), and $\Lambda_{Y,\xi}$ in case (c).

Remark 12.2.1 *We will discuss the construction of $\Psi_{Y,\xi}$ in the next section. At this point, we assume it exists and just want to extend the internal strategy of $\mathcal{N}_\xi$ one more step.*

Let $b^Y = \Psi_{Y,\xi}(\vec{\mathcal{T}}^Y)$ and c^Y be the downward closure of $\pi_Y[b^Y] \subset \mathcal{T}$. We remind the reader that in case (a), the stack $\vec{\mathcal{T}}^Y$ has the form $(\mathcal{R}_0, \mathcal{T}, \mathcal{R}_1, \vec{\mathcal{U}})$ and is an $\mathcal{N}^Y_\xi$-authenticated stack (of length 2), where $\mathcal{R}_0 = \mathcal{N}^Y_\xi|\lambda^Y_\xi$ and $\vec{\mathcal{U}}$ is a stack on $\mathcal{M}_2^{\Sigma_{\mathcal{R}_1(\alpha)},\sharp}$ for some $\alpha < \delta^{\mathcal{R}_1} - 1$. The branch b^Y in this case is given according to the canonical strategy of $\mathcal{M}_2^{\Sigma_{\mathcal{R}_1(\alpha)},\sharp}$. We note that at this point of the construction, the π_Y-realization strategy for $\mathcal{R}_1(\alpha)$ and that of $\mathcal{M}_2^{\Sigma_{\mathcal{R}_1(\alpha)},\sharp}$ have been constructed. Similarly, in cases (b) and (c) we have constructed $\Psi_{Y,\xi}$ and hence can define b^Y.

Remark 12.2.2 *The reason we have case (c) as well as feeding in stacks on $\mathcal{M}_2^{\Sigma_{\mathcal{R}_1(\alpha)},\sharp}$ in case (a) is because we want our hod mice to be g-organized in the sense of [20]. g-organization ensures that S-constructions go through as discussed in Chapter 6.*

In the following, we write $\forall^* Y$ to mean "for some club $\mathfrak{C}$, $Y \in \mathfrak{C} \cap \mathfrak{S}_{\phi,\Omega}$". Now, [31, Lemma 3.62] shows that for any $\nu < lh(\mathcal{T})$,

$$\text{either } \forall^* Y\ \nu \in c^Y \text{ or } \forall^* Y\ \nu \notin c^Y.$$

We then define b as follows: for all $\nu < lh(\mathcal{T})$,

$$\nu \in b \text{ if and only if } \forall^* Y\ \nu \in c^Y.$$

We say that b is *suitable* for $(\mathcal{N}_\xi, \lambda_\xi, \vec{\mathcal{T}})$. Letting $\mathcal{R} = (\mathcal{N}_\xi, B_b)$, we say b is according to $\Sigma^{\mathcal{R}}_{\lambda_\xi}$, which is a "one step extension" of $\Sigma^{\mathcal{N}_\xi}_{\lambda_\xi}$.

The procedure above allows us to define the object $\mathrm{Lp}^{\Sigma_{\mathcal{N}_\xi}}(\mathcal{N}_\xi)$ in the case $\gamma_\xi < \lambda_\xi = o(\mathcal{N}_\xi)$ as follows.

Definition 12.2.3 *Suppose $\gamma_\xi < \lambda_\xi = o(\mathcal{N}_\xi)$. We let $Lp^{\Sigma_{\mathcal{N}_\xi}}(\mathcal{N}_\xi)$ be the union of $\mathcal{N}_\xi \lhd \mathcal{M}$ such that $\rho_\omega(\mathcal{M}) \leq o(\mathcal{N}_\xi)$ and for a club of good Y that contains all relevant objects, $\pi_Y^{-1}(\mathcal{M}) \lhd \mathrm{Lp}^{\Psi_{Y,\xi},\Omega}(\mathcal{N}^{Y}_\xi)$.*

Now suppose $\mathcal{N}_\xi$ is such that either $\gamma_\xi = o(\mathcal{N}_\xi)$ or $\gamma_\xi < \lambda_\xi = o(\mathcal{N}_\xi)$. If $\gamma_\xi = o(\mathcal{N}_\xi)$, we let $\mathcal{N}^+_\xi = \mathcal{J}_\gamma[\mathcal{N}_\xi]$ for γ being least such that $\mathcal{J}_\gamma[\mathcal{N}_\xi]$ is not sound or else $\mathcal{N}^+_\xi = (\mathcal{N}_\xi)^\sharp$, the least E-active mouse extending $\mathcal{N}_\xi$ (so $\mathcal{N}^+_\xi$ is the least mouse of the form

$(\mathcal{J}_\alpha[\mathcal{N}_\xi], E)$ for some E). Otherwise, let $\mathcal{N}_\xi^+$ be the largest sound (not just $\mathcal{N}_\xi$-sound) $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma_{\mathcal{N}_\xi}}(\mathcal{N}_\xi)$ such that either $\mathcal{M}$ defines a failure of Woodinness of γ_ξ or else $\rho_\omega(\mathcal{M}) < \rho_\omega(\mathcal{N}_\xi)$ if such an $\mathcal{M}$ exists, or else, we let $\mathcal{N}_\xi^+$ be the largest sound level of $\mathrm{Lp}^{\Sigma_{\mathcal{N}_\xi}}(\mathcal{N}_\xi)$ ($\mathcal{N}_\xi^+$ may be $\mathrm{Lp}^{\Sigma_{\mathcal{N}_\xi}}(\mathcal{N}_\xi)$).

Definition 12.2.4 *The models $\mathcal{N}_\xi, \mathcal{N}_\xi^*$ are defined as follows: for all $\xi \leq \Upsilon$,*

(a) if ξ is limit, let $\mathcal{N}_\xi$ be $\lim_{\xi^* \to \xi} \mathcal{N}_{\xi^*}$;

(b) if $\xi = \xi^* + 1$, there are a couple of cases:

(i) if $\mathcal{N}_{\xi^*}$ is passive and there is a correctly backgrounded extender F that coheres the $\mathcal{N}_{\xi^*}$-sequence, then let $\mathcal{N}_\xi^* = (\mathcal{N}_{\xi^*}, F)$.[8]

(ii) if $\mathcal{N}_{\xi^*}$ is passive and case (i) does not hold, then $\mathcal{N}_\xi = \mathcal{N}_{\xi^*}^+$.

The $\mathcal{N}_\xi^*, \mathcal{N}_\xi$ constructed above are hod (sts)-premice in the sense of the previous chapters.

Let $Y \prec \mathcal{N}_\xi^*$ be an honest extension of X. Let $\mathcal{N}_\xi^Y$ be the transitive collapse of Y and π_Y be the uncollapse map. We say that Λ is the *π_Y-realization strategy of* $\mathcal{N}_\xi^Y$ if whenever $i = \pi^{\vec{\mathcal{T}}} : \mathcal{N}_\xi^Y \to \mathcal{Q}$ is the iteration map along stack $\vec{\mathcal{T}}$ according to Λ, then the map $k : \mathcal{Q} \to \mathcal{N}_\xi^*$ defined as: for $f \in \mathcal{N}_\xi^Y$, $a \in (\delta^{\mathcal{Q}})^{<\omega}$,

$$k(i(f)(a)) = \pi_Y(f)(\pi_{\mathcal{Q},\infty}^{\Lambda_{\vec{\mathcal{T}},\mathcal{Q}}}(a))$$

is well-defined, elementary[9], $k \circ i = \pi_Y$, and $k \restriction \delta^{\mathcal{Q}} = \pi_{\mathcal{Q},\infty}^{\Lambda_{\vec{\mathcal{T}},\mathcal{Q}}} \restriction \delta^{\mathcal{Q}}$.

We maintain as part of the construction the following:

1. For every Y countable substructure of $\mathcal{N}_\xi^*$ (in $V[G]$) that contains all relevant objects (e.g. X) and Y is an honest extension of X, let $\pi_Y, \mathcal{N}_\xi^Y$ be defined as above, then if $\mathcal{N}_\xi^Y$ is lsa-small and is not of lsa type, then there is a unique π_Y-realization strategy Σ_ξ^Y for $\mathcal{N}_\xi^Y$. Additionally, $\Sigma_\xi^Y \restriction \mathrm{HC} \in \Omega$ and is locally Ω-fullness preserving,[10] has local strong branch condensation (in the sense of Chapter 9). We will define Σ_ξ^Y in the next section.

[8]If there are F_0 such that $\mathrm{cr}(F_0) = \delta^{\mathcal{P}}$ and F_1 such that $\mathrm{cr}(F_1) > \delta^{\mathcal{P}}$ such that $(\mathcal{N}_{\xi^*}, F_0)$ and $(\mathcal{N}_{\xi^*}, F_1)$ are both hod premice, then we give priority to F_0. The uniqueness of extenders F with $\mathrm{crt}(F) = \delta^{\mathcal{P}}$ is clear from the definition of F. The uniqueness of extenders F with $\mathrm{crt}(F) > \delta^{\mathcal{P}}$ is proved by the usual bicephalus argument.

[9]By this, we mean if $\vec{\mathcal{T}}$ is a k-maximal stack then k is a weak k-embedding in the sense of [8].

[10]Ω-fullness preserving means whenever $i : \mathcal{N}_\xi^Y \to \mathcal{Q}$ is an iteration map according to Σ_ξ^Y, then $\mathcal{Q}^b$ is Ω-full.

2. $\rho_\omega(\mathcal{N}_\xi^*) \geq o(\mathcal{P})$ for all $\xi \leq \Upsilon$. In other words, $o(\mathcal{P})$ is $(\delta^{\mathcal{P}})^+$ in $\mathcal{N}_\xi^*$ and in $\mathcal{N}_\xi$ for all $\xi \leq \Upsilon$.

3. $\mathcal{N}_\xi^*$ is solid and universal for all $\xi \leq \Upsilon$. So $\mathcal{N}_\xi$ is sound for all such ξ.

Definition 12.2.5 (Relevant extender) *Suppose F is on the $\mathcal{N}_\xi$-extender sequence for some $\xi \leq \Upsilon$. We say that F is* ***relevant*** *if the resurrection of F (in the sense of [8]) is correctly backgrounded.*

Granting that (1)-(3) are maintained for each ξ. We say the construction *stops prematurely* when Υ is the least such that $\mathcal{N}_\Upsilon$ satisfies the following:

(i) There is an increasing sequence $(\delta_n : n < \omega)$ of Woodin cardinals above $\delta^{\mathcal{P}}$ such that $\delta^{\mathcal{P}}$ is the least $< \delta_0$-strong and $(\delta_n : n \geq 1)$ are the Woodin cardinals above δ_0.

(ii) There are no (relevant) extenders E on the $\mathcal{N}_\Upsilon$-sequence such that there is some n such that $\mathrm{cr}(E) \leq \delta_i < \mathrm{lh(E)}$.

(iii) I am a sts hod premouse over $\mathcal{M}^+(\mathcal{N}_\Upsilon|\delta_0) =_{\mathrm{def}} (\mathcal{N}_\Upsilon|\delta_0)^\sharp$.

(iv) I am E-active with top extender F such that $\mathrm{cr}(F) > \delta_n$ for all $n < \omega$.

From this, we then show that there must be a model of LSA.

Remark 12.2.6 *In (1)-(3) above, suppose now $\mathcal{N}_\xi^Y$ is such that $\mathcal{N}_\xi^Y \vDash$ "the largest cardinal δ is Woodin and $\delta^{\mathcal{P}}$ is strong to δ", and $\Psi_{Y,\gamma}$ is the short tree strategy of $(\mathcal{N}_\xi^Y|\delta)^\sharp \lhd \mathcal{N}_\xi^Y$. That is, $\mathcal{N}_\xi^Y$ is of lsa type. Then it is not clear that there is a π_Y-realization strategy for $\mathcal{N}_\xi^Y$. However, if the construction above $\mathcal{N}_\xi^Y$ does not define a failure of Woodinness of $\gamma_\xi = \delta$ (the construction will be the construction with respect to the short tree strategy of $\mathcal{N}_\xi$), then we can still carry on the construction until it stops (perhaps prematurely) because we can still use iterability above δ to prove solidity and universality for the models of the construction. If this were the case, then there will be no (relevant) extenders E such that $cr(E) = \delta^{\mathcal{P}}$ or $cr(E) \leq \delta < \mathrm{lh(E)}$ being added during the course of the construction.*

Remark 12.2.7 *(I) The extender sequence of $\mathcal{N}_\xi$ utilizes two indexing schemes: the cutpoint indexing scheme (for extenders with critical point $\delta^{\mathcal{P}}$) and the Jensen indexing scheme (for extenders with critical point $> \delta^{\mathcal{P}}$). This follows from the definition of correctly backgrounded extenders for relevant extenders; if E is non-relevant (so E is on the sequence of a "level of Lp"), our convention* *is E is indexed according to the Jensen indexing scheme. The Jensen indexing scheme could be replaced by the Mitchell-Steel indexing scheme, but we choose not to do so out of convenience; we want to quote direct results from [5] and [1] as well as using results of Chapter 9.*

(II) *If $\gamma_\xi = o(\mathcal{N}_\xi)$, then the construction above (as dictated by the theory in Chapters 2, 3) does not immediately activate the strategy for $\mathcal{N}_\xi$. Instead, it constructs an E-active $\mathcal{N}_\epsilon$, where $\epsilon > \xi$ is the least such that $\mathcal{N}_\epsilon$ is E-active before activating the strategy for $\mathcal{N}_\epsilon$. It is easy to see that $cr(E^{\mathcal{N}_\epsilon}) > \gamma_\epsilon$.*

(III) *If $\mathcal{N}_\xi$ is not of lsa type, then (3) follows from (2) and (1) by what has been proved in Chapters 3 and 9.*

(IV) *Suppose Σ_ξ^Y is as in (2) and $\mathcal{N}_\xi^Y$ is not of lsa type. Then Σ_ξ^Y is positional and commuting by results of Section 4.5.*

Suppose the construction does not stop prematurely. Let $\mathcal{N} = \mathcal{N}_\Upsilon$. So $o(\mathcal{N}) = \kappa^{+++}$ by Lemma 12.4.2. Let $\delta^{\mathcal{N}} > \delta^{\mathcal{P}}$ be the unique γ such that $\mathcal{N} \vDash$ "$\delta^{\mathcal{P}}$ is strong to γ and γ is Woodin" if it exists; otherwise we let $\delta^{\mathcal{N}} = 0$. Note that by the remarks above, which is a consequence of our smallness assumption (†), $\delta^{\mathcal{N}}$ is a strong cutpoint of $\mathcal{N}$. Following [5], we define the following *stack of hod mice above* $\mathcal{N}$. The following definition takes place in $V[G]$ but it is easily seen that $S(\mathcal{N}) \in V$.

Definition 12.2.8 *Let $S(\mathcal{N})$ be the stack of sound hod premice $\mathcal{M}$ if $\delta = 0$ or else $\Sigma^{sts}_{\mathcal{M}^+(\mathcal{N}|\delta)}$-mice extending $\mathcal{N}$ such that $\rho_\omega(\mathcal{M}) = o(\mathcal{N})$ and for every countable $\mathcal{M}^*$ embeddable into $\mathcal{M}$ via $\pi_{\mathcal{M}^*}$ such that $X \cup \{X, \mathcal{P}^-, \mathcal{P}, \mathcal{N}\} \subset \mathrm{rng}(\pi_{\mathcal{M}^*})$, $\mathrm{rng}(\pi_{\mathcal{M}^*})$ is an honest extension of X, $\mathcal{M}^*$ is (ω_1+1)-iterable above δ via a strategy $\Lambda_{\mathcal{M}^*}$ such that if $\delta^{\mathcal{N}} = 0$, then $\Lambda_{\mathcal{M}^*} \restriction \mathrm{HC} \in \Omega$, Λ is locally Ω-fullness preserving and has local strong branch condensation. Furthermore, if E is on the $\mathcal{M}$-sequence such that $cr(E) = \delta^{\mathcal{P}}$ and $lh(E) \geq o(\mathcal{N})$, then for every such $\mathcal{M}^*$ as above such that $E \in \mathrm{rng}(\pi_{\mathcal{M}^*})$, letting ν be the length of $\pi^{-1}_{\mathcal{M}^*}(E)$, then for any $a \in [\nu]^{<\omega}$, $A \in \wp(\delta^{\mathcal{P}})^{|a|} \cap \mathcal{P}$ such that $(a, A) \in E \cap \mathrm{rng}(\pi_{\mathcal{M}^*})$, then $\pi^{\Lambda_{\mathcal{M}^*}}_{\mathcal{M}^*||\nu,\infty}(\pi^{-1}_{\mathcal{M}^*}(a)) \in A$.*

The following facts about $S(\mathcal{N})$ more or less follow immediately from results in [5].

Lemma 12.2.9 *Suppose $\Upsilon = \kappa^{+++}$ and $\mathcal{N} = \mathcal{N}_\Upsilon$.*

(i) *For $\mathcal{M}_0, \mathcal{M}_1 \in S(\mathcal{N})$, either $\mathcal{M}_0 \trianglelefteq \mathcal{M}_1$ or $\mathcal{M}_1 \trianglelefteq \mathcal{M}_0$. In other words, $S(\mathcal{N})$ is a hod premouse (and is ω-small).*

(ii) For all $\mathcal{M} \trianglelefteq S(\mathcal{N})$, there is some $\mathcal{R} \triangleleft S(\mathcal{N})$ such that $\mathcal{M} \triangleleft \mathcal{R}$. In particular, $S(\mathcal{N}) \vDash \mathsf{ZFC}^-$.

(iii) $cof(o(S(\mathcal{N}))) \geq \kappa^{+++}$.

Proof. (i) and (ii) are analogs of [5, Lemma 3.1] and [5, Lemma 3.3] respectively and follow straightforwardly from the condensation lemma, Theorem 11.1.2. The point is that if $\delta^{\mathcal{N}} = 0$, then the theory developed above allows us to perform comparisons (and shows that no strategy disagreement can occur); otherwise, the construction above δ is with respect to a fixed (short-tree) strategy predicate, so the comaprison is again an extender comparison. (iii) follows from the proof of [5, Theorem 3.4], noting that $\kappa^{+++} \geq \aleph_3$, is countably closed and $2^{\kappa^{++}} = \kappa^{+++}$ in $V[G]$. □

12.3 Iterability of lsa-small, non-lsa type levels

Now we inductively prove (1)-(3) hold for all $\xi \leq \Upsilon$. First, we verify that (1)-(3) holds for $\xi = 0$. By Lemma 12.1.1, no $\mathcal{P}^- \triangleleft \mathcal{M} \triangleleft \mathcal{P}$ projects across $\delta^{\mathcal{P}}$; also $\mathcal{P} \vDash \mathsf{ZFC}^-$, and hence $\rho_\omega(\mathcal{P}) = o(\mathcal{P})$. By definition, $\mathcal{P} = \mathcal{N}_0^*$.

Lemma 12.3.1 *(1)-(3) hold for $\xi = 0$. Hence $\mathcal{N}_0 = \mathcal{N}_0^* = \mathcal{P}$.*

Proof. Fix Y as in the statement of (1). Let $\delta_Y = \delta^{\mathcal{N}_0^Y}$. By definition, Σ_0^Y has branch condensation as it is the join of strategies with those properties. Furthermore, note that Σ_0^Y acts on $\mathcal{N}_0^Y$ in the following way. Let $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{N}_0^Y, \Sigma_0^Y)$ and let $i : \mathcal{N}_0^Y \to \mathcal{Q}$ be the iteration map and $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ be the $\vec{\mathcal{T}}$-tail of Σ_0^Y.

Suppose $x \in \mathcal{Q}$, then there is some $f \in \mathcal{N}_0^Y$ and $a \in i(\delta_Y)^{<\omega}$ such that

$$x = i(f)(a).$$

Let $k : \mathcal{Q} \to \mathcal{N}_0$ be defined as follows:

$$k(i(f)(a) = \pi_Y(f)(\pi^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q}(i(\delta_Y)),\infty}(a)),$$

for any $f \in \mathcal{N}_0^Y$ and any $a \in i(\delta_Y)^{<\omega}$. Note that since X is a condensing set and $i \circ \pi_{X,Y} \restriction \delta_X$ is according to Σ_0^X, $\mathrm{rng}(k)$ is an honest extension of X. By Lemma 12.1.3, k is well-defined, Σ_1-elementary (and cofinal), $k \circ i = \pi_Y$, and $k \restriction \delta^{\mathcal{Q}} = \pi^{\Sigma_{\vec{\mathcal{T}},\mathcal{Q}}}_{\mathcal{Q}|\delta^{\mathcal{Q}},\infty} \restriction \delta^{\mathcal{Q}}$. It is clear that this is the only way to define k; the uniqueness of Σ_0^Y also follows.

We remark that local strong branch condensation is just branch condensation in this case. Now to see that Σ_0^Y is Ω-fullness preserving, it suffices to show $\mathcal{Q}$ is Ω-full.

But this follows from the definition of condensing sets and the fact that Y and rng(k) are honest extensions of X. Also, we get local strong Ω-fullness preservation.

We have shown (1). (2) holds by the remark immediately before the lemma and (3) follows from (2) and (1) by Remark 12.2.7. $\square$

Verifying that (1)-(3) hold at limit ξ is easy; we leave it to the reader. Next, we verify that (1)-(3) hold for ξ^* implies (1)-(3) hold for $\xi = \xi^* + 1$. This is the main case.

Let $Y \in V$ be an honest extension of X; we assume also $Y = Y^* \cap \mathcal{N}^*_\xi$ for some $Y^* \prec H^V_{\kappa^{+++}}$. We assume $\mathcal{N}^*_\xi$ (and hence $\mathcal{N}^Y_\xi$) is *lsa-small and is not of lsa type*; more precisely, we assume that letting γ_ξ be the supremum of indices of extenders E on the $\mathcal{N}^*_\xi$ sequence such that $\mathrm{cr}(E) = \delta^{\mathcal{N}_\xi}$, then some $\mathcal{M} \trianglelefteq \mathcal{N}^*_\xi$ defines the failure of Woodinness of γ_ξ. We now define the strategy Σ^Y_ξ for $\mathcal{N}^Y_\xi$.[11] We write x^Y for $\pi_Y^{-1}(x)$ for $x \in \mathcal{N}^*_\xi \cap \mathrm{rng}(\pi_{\mathrm{Y}})$.

Definition 12.3.2 (Normal form) *An iteration $((\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta)$ on $\mathcal{P}_0 = \mathcal{N}^Y_\xi$ is said to be in **normal form** if the following hold:*

(i) $\vec{\mathcal{T}}_\alpha$ is a stack of normal trees with base model $\mathcal{P}_\alpha$ and last model $\mathcal{P}_{\alpha+1}$.

(ii) If $\lambda \leq \eta$ is limit, $\mathcal{P}_\lambda = lim_{\alpha<\lambda}\mathcal{P}_\alpha$.

(iii) Either $\vec{\mathcal{T}}_\alpha$ uses no extenders in the top block of $\mathcal{P}_\alpha$ or its images or $\mathcal{P}_{\alpha+1} = Ult(\mathcal{P}_\alpha, E)$ for some extender E on the $\mathcal{P}_\alpha$-sequence with $cr(E) = \delta^{\mathcal{P}_\alpha}$ or else $\vec{\mathcal{T}}_\alpha$ is completely above $\delta^{\mathcal{P}_\alpha}$.

(iv) If $\eta = \alpha + 1$ for some α, then for all $\beta < \alpha$, $\vec{\mathcal{T}}_\beta$ does not drop.

We define Σ^Y_ξ for stacks in normal form. We say that a stack $((\mathcal{P}_\alpha, \vec{\mathcal{T}}_\alpha) \mid \alpha < \eta)$ in normal form, where $\mathcal{P}_0 = \mathcal{N}^Y_\xi$, is according to Σ^Y_ξ if: letting $\tau_0 = \pi_Y \restriction \mathcal{P}_0$, $i_{\gamma,\tau} : \mathcal{P}_\gamma \to \mathcal{P}_\tau$ be iteration maps, and $\kappa^{\mathcal{P}_\gamma}$ be the cutpoint cardinal that begins the top block of $\mathcal{P}_\gamma$ [12]

(A) there are maps $\tau_\alpha : \mathcal{P}_\alpha \to \mathcal{N}^*_\xi$ for all $\alpha < \eta$;

[11] Technically, since we construct Σ^Y_ξ in $V[G]$, we should denote it $\Sigma^{Y,G}_\xi$. But in fact $\Sigma^{Y,G}_\xi \cap V \in V$ and $\Sigma^{Y,G}_\xi$ does not depend on the choice of G. This will be clear from the construction of $\Sigma^{Y,G}_\xi$. So in effect, we are constructing an invariant name $\dot{\Sigma}^Y_\xi$ in V whose interpretation in $V[G]$ is $\Sigma^{Y,G}_\xi$ for any G. For notational simplicity, we will simply write Σ^Y_ξ without the "G".

[12] $\kappa^{\mathcal{P}_\gamma} = i_{0,\gamma}(\delta^{\mathcal{P}})$.

(B) for all $\gamma \leq \alpha < \eta$, $\tau_\gamma = \tau_\alpha \circ i_{\gamma,\alpha}$;

(C) for all $\alpha < \eta$, the $\mathcal{P}_\alpha$-tail of $\Sigma^Y_{\xi^*}$, Λ_α, is the τ_α-pullback strategy and $\pi^{\Lambda_\alpha}_{\mathcal{P}_\alpha|\kappa^{\mathcal{P}_\alpha},\infty} = \tau_\alpha \restriction \mathcal{P}_\alpha|\kappa^{\mathcal{P}_\alpha}$;

(D) if $\eta = \alpha + 1$ and $\vec{\mathcal{T}}_\alpha$ drops and say $\vec{\mathcal{T}}_\alpha$ is k-maximal, then either $\vec{\mathcal{T}}_\alpha$ is "Lp-based", that is there is some $\mathcal{Q} \trianglelefteq \mathcal{P}_\alpha$ such that $\vec{\mathcal{T}}_\alpha$ is based on $\mathrm{Lp}^{\Lambda_{\alpha,\mathcal{Q}},\Omega}(\mathcal{Q})$ (or $\mathrm{Lp}^{\Lambda^{sts}_{\alpha,\mathcal{Q}},\Omega}(\mathcal{Q})$) and is above $o(\mathcal{Q})$ or else there is a (unique) branch b of $\vec{\mathcal{T}}_\alpha$, some $\xi' < \xi$, and a weak-deg(b)-embedding $\tau_\eta : \mathcal{M}^{\vec{\mathcal{T}}_\alpha}_b \to \mathcal{N}_{\xi'}$. Otherwise, there is a (unique) branch b and map $\tau_\eta : \mathcal{M}^{\vec{\mathcal{T}}_\alpha}_b \to \mathcal{N}_\xi$ such that $\tau_\alpha = \tau_\eta \circ i_{\alpha,\eta}$.

It is clear how to extend Σ^Y_ξ to all stacks of normal trees. This is because all stacks of normal trees on $\mathcal{N}^Y_\xi$ can be decomposed into stacks in normal form. We will need to define maps τ_α in the definition of Σ^Y_ξ in such a way that makes Σ^Y_ξ a π_Y-realization strategy.

The next lemma shows that if $\mathcal{R}$ is a Σ^Y_ξ-iterate of $\mathcal{N}^Y_\xi$ via map i, letting $\delta^{\mathcal{R}} = i(\delta^{\mathcal{P}^Y})$ and F is an extender on the $\mathcal{R}$-sequence with critical point $\delta^{\mathcal{R}}$, then F is certified. The lemma proves something a bit more general.

Lemma 12.3.3 *Suppose Y is a countable, elementary in $\mathcal{N}_\xi$ and is an honest extension of X. Suppose $i : \mathcal{N}^Y_\xi \to \mathcal{R}$ and $\sigma : \mathcal{R} \to \mathcal{N}_\xi$ are such that $\pi_Y = \sigma \circ i$, and letting Z be rng(σ), then Z is an honest extension of Y. Let Λ be the σ-pullback strategy on $\mathcal{R}$. Then:*

(a) If $j : \mathcal{R} \to \mathcal{S}$ is a Λ-iteration based on $\mathcal{R}^b$ and suppose $\delta^{\mathcal{S}} = \sup j[\delta^{\mathcal{R}}] = j(\delta^{\mathcal{R}})$, then letting $\tau : \mathcal{S} \to \mathcal{N}_\xi$ be the map: $\tau(j(f)(a)) = \sigma(f)(\pi^{\Lambda_{\mathcal{S}}}_{\mathcal{S}|\kappa^{\mathcal{S}},\infty}(a))$, where $f \in \mathcal{R}$, $a \in [\kappa^{\mathcal{S}}]^{<\omega}$, and $\Lambda_{\mathcal{S}}$ is the tail of Λ. Then τ is well-defined, elementary, and $\pi^{\Lambda_{\mathcal{S}}}_{\mathcal{S}|\kappa^{\mathcal{S}},\infty} = \tau \restriction (\mathcal{S}|\kappa^{\mathcal{S}})$.

(b) Suppose F is an extender on the $\mathcal{R}$-sequence with $cr(F) = \kappa^{\mathcal{R}} = i(\delta^{\mathcal{P}})$. Then F is σ-certified over $(\mathcal{R}||lh(G), \Lambda_{\mathcal{R}||lh(G)})$.

Proof. (a) follows from Lemma 12.1.3 and the fact that the iteration map j is continuous at $\delta^{\mathcal{R}}$.

For (b), first, note that i is continuous at $(\delta^+)^{\mathcal{N}^Y_\xi}$ and is cofinal in $((\kappa^{\mathcal{R}})^+)^{\mathcal{R}}$. This is because π_Y is continuous at $(\delta^+)^{\mathcal{N}^Y_\xi}$ and is cofinal in $(\delta^+)^{\mathcal{P}}$. Finally, F is total over $\mathcal{R}$; this follows from the continuity of i.

Now, let $\mathcal{S} = \mathrm{Ult}(\mathcal{R}, F)$, i_F be the ultrapower map. Let Z be countable, honest extension of X such that $Y \prec Z$ and $\mathrm{rng}(\sigma) \subseteq \mathrm{rng}(\pi_Z)$. Let $\sigma_Z = \pi^{-1}_Z \circ \sigma$. Let

$$\begin{tikzcd}
 & \mathcal{N}_\xi & \\
\mathcal{N}_\xi^Y \arrow[ur, "\pi_Y"] \arrow[r, "\pi_{Y,Z}"] \arrow[d, "i"'] & \mathcal{N}_\xi^Z \arrow[u, "\pi_Z"] \arrow[r, "i_G"] & \mathcal{W} \arrow[ul, "\psi"'] \\
\mathcal{R} \arrow[ur, "\sigma_Z"] \arrow[r, "i_F"] & \mathcal{S} \arrow[ur, "\tau_Z"'] \arrow[uu, dashed, bend right, "\tau"'] &
\end{tikzcd}$$

Figure 12.3.1: Hypothesis of Lemma 12.3.3

$G = \sigma_Z(F)$ [13] and $i_G : \mathcal{N}_\xi^Z \to \mathrm{Ult}(\mathcal{N}_\xi^Z, G) =_{\mathrm{def}} \mathcal{W}$ be the ultrapower map. Let $\tau_Z : \mathcal{S} \to \mathcal{W}$ be the copy map and $\psi : \mathrm{Ult}(\mathcal{N}_\xi^Z, G) \to \mathcal{N}_\xi$ be the map

$$\psi(i_G(f)(a)) = \pi_Z(f)(\pi^{\Sigma_\xi^Z}_{\mathcal{N}_\xi^Z,\infty}(a)).$$

Since G is π_Z-certified over $(\mathcal{N}_\xi^Z||lh(G), (\Sigma_\xi^Z)_{\mathcal{N}_\xi^Z||lh(G)})$, ψ is well-defined, elementary, and $\psi \circ i_G = \pi_Z$. Now,

$$\sigma = \psi \circ \tau_Z \circ i_F,$$

so letting $\Lambda_{\mathcal{S}}$ be the $\psi \circ \tau_Z$-pullback strategy for $\mathcal{S}$, then by strategy coherence for hod mice, $\Lambda_{\mathcal{S}}$ agrees with $\Lambda_{\mathcal{R}}$ on $\mathcal{R}||lh(F)$. Now let $\tau : \mathcal{S} \to \mathcal{N}_\xi$ be defined as follows: for all $a \in [lh(F)]^{<\omega}$ and $f \in \mathcal{R}$,

$$\tau(i_F^{\mathcal{R}}(f)(a)) = \sigma(f)(\pi^{\Lambda_{\mathcal{R}}}_{\mathcal{R}||lh(F),\infty}(a)).$$

By Lemma 12.1.3, τ is well-defined, elementary, and agrees with $\pi^{\Lambda_{\mathcal{S}}}_{\mathcal{S},\infty}$ up to $\delta^{\mathcal{S}}$ and hence with σ up to $\mathcal{R}||lh(F)$. This proves part (b). $\square$

The following remarks summarize how we can inductively define maps τ_α and hence define Σ_ξ^Y on stacks in normal form.

Remark 12.3.4 *(i) If $\vec{\mathcal{T}}_\alpha = \langle E \rangle$ for $cr(E) = \delta^{\mathcal{P}_\alpha}$, then*

$$\tau_{\alpha+1}(i_E^{\mathcal{P}_\alpha}(f)(a)) = \tau_\alpha(f)(\pi^{\Lambda_\alpha}_{\mathcal{P}_\alpha||lh(E),\infty}(a)).$$

Lemma 12.3.3(b) shows that $\tau_{\alpha+1}$ is well-defined, elementary,[14]*, agrees with τ_α up to $lh(E)$, and $\pi^{\Lambda_{\alpha+1}}_{\mathcal{P}_\alpha||lh(E),\infty} = \tau_{\alpha+1} \restriction \mathcal{P}_\alpha||lh(E), \infty$.*

[13] If F is the top extender of $\mathcal{R}$, then by $\sigma_Z(F)$, we mean $\sigma_Z[F]$.

[14] If τ_α is a weak k-embedding for some k, as is typical of realization maps, then so is $\tau_{\alpha+1}$.

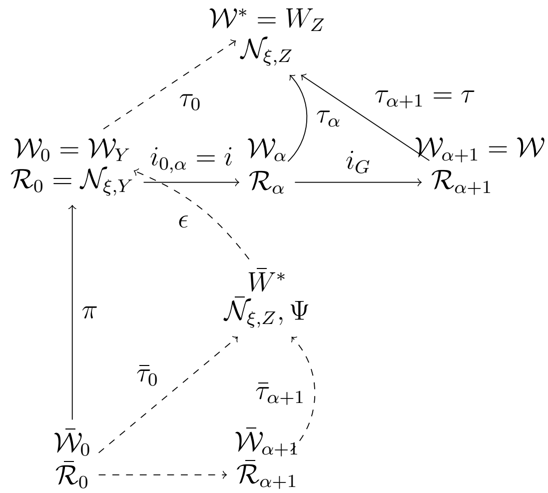

Figure 12.3.2: Sketch of Remark 12.3.4(ii)

(ii) *With the exact same situation as i and suppose* $cof^V(o(\mathcal{N}_\xi^*)) \leq \kappa$,[15] *we claim that the* $\mathcal{S} =_{def} \mathcal{P}_{\alpha+1}$*-tail of* $\Psi =_{def} \Sigma_{\xi-1}^Y$ *is* $\Psi_\mathcal{S}$*, the* $\tau_{\alpha+1}$*-pullback strategy of* $\mathcal{S}$*. This is strategy coherence at* $\alpha + 1$*. Suppose not. Write* τ *for* $\tau_{\alpha+1}$ *and* i *for* $i_{0,\alpha}$*. This is basically the proof of Theorem 2.7.6 in [10] (see Figure 12.3.2). We briefly sketch it here. Let* $Y \prec Z$ *and* $Z \in V$ *is countable (in* $V[G]$*), honest extension of* X*. Let* $\mathcal{W}_Y$ *be a* Ψ*-hod mouse with* ω *many Woodin cardinals and* $\mathcal{W}_Z = Ult(\mathcal{W}_Y, E)$*, where* E *is the* $(crit(\pi_Y), o(\mathcal{N}_\xi^Z))$*-extender derived from* $\pi_{Y,Z}$*. So letting* $j = i_G \circ i$*,* j *extends to* $j^+ : \mathcal{W}_Y \to \mathcal{W}$ *and* τ *extends to* $\tau^+ : \mathcal{W} \to \mathcal{W}_Z$*, where* $\mathcal{N}_\xi^Z \lhd \mathcal{W}_Z$ *(this is because* $o(\mathcal{N}_\xi^Y)$ *is a cardinal cutpoint in* $\mathcal{W}_Y$ *and* $\pi_{Y,Z}$ *is cofinal in* $o(\mathcal{N}_{\xi,Z})$*). Let* $\pi : M \to H_{\kappa^{+4}}^V$ *be the inverse of the transitive collapse of some countable elementary substructure of* $H_{\kappa^{+4}}^V$ *in* V *containing all relevant objects. For any* $a \in H_{\kappa^{+4}}^V \cap ran(\pi)$*, let* $\bar{a} = \pi^{-1}(a)$*. Let* $\bar{g} \subseteq Col(\omega, \bar{\kappa})$ *be* M*-generic with* $g \in V$ *and* $\bar{\mathcal{S}}$*,* $\bar{j}$*,* $\bar{\tau}$ *be the objects in* $M[\bar{g}]$ *witnessing the failure of the claim in* $M[\bar{g}]$*. Since* M *is countable, there is a map* $\epsilon : \bar{W}_Z \to W_Y$ *such that* $\pi \restriction \bar{\mathcal{W}}_Y = \epsilon \circ \bar{\pi_Y} \restriction \bar{\mathcal{W}}_Y$*. Let* Φ *be the* ϵ*-pullback of* Ψ*. By the proof of Theorem 2.7.6 in [10], working in* $M[\bar{g}]$*, the uB code for* $\bar{\Psi}$ *gets moved to the uB code for its* $\bar{\mathcal{S}}$*-tail and also to the uB code for the* $\bar{\tau}$*-pullback of* Φ*. This is a contradiction.*

(iii) *If* $\vec{\mathcal{T}}_\alpha$ *is below* $\delta^{\mathcal{P}_\alpha}$ *then it is according to* Λ_α *and so* $\tau_{\alpha+1}$ *is given by the inductive*

[15] Cofinally many ξ has the property that $\mathrm{cof}^V(o(\mathcal{N}_\xi^*)) \leq \kappa$. In our case, $\xi = \xi^* + 1$, this holds because $\mathcal{N}_{\xi^*} \vDash \forall\kappa\Box_\kappa$ by Chapter 9.

assumption on Λ_α. Strategy coherence at $\alpha+1$ is maintained here. See Lemma 12.3.3(a).

(iv) *If $\vec{\mathcal{T}}_\alpha$ is above $\delta^{\mathcal{P}_\alpha}$ and is not Lp-based, then the map $\tau_{\alpha+1}$ is given by the K^c-construction theorem (cf. [1, Theorem 3.2]) and our smallness assumption on the hod mice that we are constructing. If $\vec{\mathcal{T}}_\alpha$ is above $\delta^{\mathcal{P}_\alpha}$ and is Lp-based, then $\vec{\mathcal{T}}_\alpha$ drops and we can't undo the drop, so no more realizations are needed.*

(v) *Suppose $\lambda < \eta$ is limit. Let for $\alpha < \lambda$*

$$\tau_\lambda(i_{\alpha,\lambda}(x)) = \tau_\alpha(x).$$

So we get $\tau_\lambda : \mathcal{P}_\lambda \to \mathcal{N}_\xi^Y$ is such that for all $\alpha < \lambda$, $\tau_\alpha = \tau_\lambda \circ i_{\alpha,\lambda}$. Using the above argument, we get strategy coherence at λ. Finally, we verify that letting $\pi : \mathcal{P}_\lambda|\delta^{\mathcal{P}_\lambda} \to \mathcal{N}_\xi^*$ be the iteration maps by the τ_λ-pullback strategy Λ_λ, $\pi = \tau_\lambda \restriction \delta^{\mathcal{P}_\lambda}$. Let $\nu < \delta^{\mathcal{P}_\lambda}$. We note that Λ_λ is the Λ_α-tail by strategy coherence at λ. Let $i_{\alpha,\lambda}(\nu^*) = \nu$ for some $\alpha < \lambda$ and $\nu^* < \delta^{\mathcal{P}_\alpha}$. Then

$$\tau_\lambda(\nu) = \tau_\alpha(i_{\alpha,\lambda}(\nu^*)) = \pi_{\mathcal{P}_\alpha|\kappa^{\mathcal{P}_\alpha},\infty}(\nu^*) = \pi(i_{\alpha,\lambda}(\nu^*)) = \pi(\nu).$$

The following lemma gives some useful consequences regarding uniqueness of strategies, whose proof is essentially the proof of Lemma 10.3.6.

Lemma 12.3.5 *(i) Suppose $\pi : \mathcal{Q} \to \mathcal{P}$ is elementary such that $X \subset \mathrm{rng}(\pi)$. Suppose $i : \mathcal{Q} \to \mathcal{R}$ is such that $i \restriction \delta^{\mathcal{P}}$ is according to the π-pullback strategy and $\tau_0, \tau_1 : \mathcal{R} \to \mathcal{P}$ are such that $\tau_0 \circ i = \tau_1 \circ i = \pi$. Then the τ_0-pullback strategy is the same as the τ_1-pullback strategy.*

(ii) *Suppose Y is countable, elementary in $\mathcal{N}_\xi$ and is an honest extension of X. Suppose κ is a cardinal of $\mathcal{N}_\xi$ and $\omega\rho^{n+1}_{\mathcal{N}_\xi^Y} \leq \kappa < \omega\rho^n_{\mathcal{N}_\xi^Y}$. Let $\Psi = \Sigma_\xi^Y$.*

(a) *Suppose $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{N}_\xi^Y, \Psi)$ is such that $\pi^{\vec{\mathcal{T}},b}$ exists and $\tau : \mathcal{N}_\xi^Y \to \mathcal{S}$ is $\Sigma_0^{(n)}$ and cardinal preserving and $\mathcal{S} \trianglelefteq \mathcal{R}$. Suppose $(\vec{\mathcal{U}}, \mathcal{Q}) \in I(\mathcal{N}_\xi^Y, \Psi)$ is such that $\pi^{\vec{\mathcal{U}},b}$ exists, $\mathcal{Q}^b = \mathcal{S}^b$, and $\tau \restriction \mathcal{P} = \pi^{\vec{\mathcal{U}}} \restriction \mathcal{P}$, then $\Psi^\tau_{\vec{\mathcal{T}},\mathcal{S}} = \Psi$.*

(b) *Suppose $(\vec{\mathcal{T}}, \mathcal{R})$ is such that $\pi^{\vec{\mathcal{T}},b}$ exists and is according to Ψ. Suppose $\mathcal{U}$ is a normal tree of limit length on $\mathcal{R}(\beta)$ according to $\Psi_{\vec{\mathcal{T}},\mathcal{R}}$, where $\beta < \lambda^{\mathcal{R}} - 1$. Suppose c are cofinal branches of $\mathcal{U}$ (considered as a tree on $\mathcal{R}$) and there is a map $\tau_c : \mathcal{M}_c^{\mathcal{U}} \to \mathcal{N}_\xi$ such that $\pi_Y \restriction \mathcal{P} = \tau_c \circ \pi_c^{\mathcal{U}} \circ \pi^{\vec{\mathcal{T}},b}$. Then $c = \Psi_{\vec{\mathcal{T}},\mathcal{R}}(\mathcal{U})$.*

$$
\begin{tikzcd}
 & & \mathcal{N}_\xi & \\
\mathcal{S}^b \arrow[urr, "\sigma"] & \mathcal{M}_b^{\mathcal{W}^*} \arrow[ur, "\tau_b"'] & & \mathcal{M}_c^{\mathcal{W}^*} \arrow[ul, "\tau_c"'] \\
\mathcal{N}_\xi^{Y,b} \arrow[u, "\tau"'] \arrow[u, bend left, "\pi^{\vec{\mathcal{U}},b}"] \arrow[rr, "\vec{\mathcal{W}}"'] & & \mathcal{R}_0 \arrow[ul, "\pi_b^{\mathcal{W}^*}"] \arrow[ur, "\pi_c^{\mathcal{W}^*}"'] &
\end{tikzcd}
$$

Figure 12.3.3: Lemma 12.3.5 (ii)(a)

Proof. (i) follows straightforwardly from Lemma 12.1.3 (iv). The main point is that, letting Λ_i be the τ_i-pullback strategy (for $i = 0, 1$), then letting $\sigma_i : \mathcal{R} \to \mathcal{P}$ be

$$\sigma(i(f)(a)) = \pi(f)(\pi^{\Lambda_i}_{\mathcal{R},\infty}(a))$$

for $f \in \mathcal{Q}$ and $a \in \delta^{\mathcal{R}}$. Then $\sigma_i[\mathcal{R}]$ is an honest extension of X.

(ii)(b) follows easily from (i) and Remark 12.3.4(ii). For (ii)(a) (see Figure 12.3.3), suppose $\Psi^{\tau}_{\vec{\mathcal{T}},\mathcal{S}} \neq \Psi$, then by results of Section 4.4.1, there is a low-level disagreement, that is there is $(\vec{\mathcal{W}}, \mathcal{R}_0, \mathcal{W}^*)$ such that:

- $\vec{\mathcal{W}}$ is according to both strategies.
- $\mathcal{R}_0$ is the last model of $\vec{\mathcal{W}}$.
- $\mathcal{W}^*$ is a tree of limit length on $\mathcal{R}_0(\beta)$ for some $\beta \leq \lambda^{\mathcal{R}_0} - 1$.

Let $b = \Psi(\vec{\mathcal{W}}^\frown \mathcal{W}^*)$ and $c = \Psi^{\tau}_{\vec{\mathcal{T}},\mathcal{S}}(\vec{\mathcal{W}}^\frown \mathcal{W}^*)$. Let $\sigma : \mathcal{Q} \to \mathcal{N}_\xi$ be the realization map; hence

$$\pi_Y \restriction \mathcal{P} = \sigma \circ \pi^{\vec{\mathcal{U}},b} = \sigma \circ \tau \restriction \mathcal{P}. \tag{12.2}$$

Note that there are embeddings $\tau_b : \mathcal{M}_b^{\mathcal{W}^*,b} \to \mathcal{P}$ and $\tau_c : \mathcal{M}_c^{\mathcal{W}^*,b} \to \mathcal{P}$ such that:

$$\sigma \circ \tau \restriction \mathcal{P} = \tau_c \circ \pi^{\vec{\mathcal{W}}} \tag{12.3}$$

and

$$\pi_Y \restriction \mathcal{P} = \tau_b \circ \pi^{\vec{\mathcal{W}}}. \tag{12.4}$$

By (i) and Equations 12.2, 12.3, 12.4, $b = c$. Contradiction. $\square$

$$\begin{tikzcd}
 & & \mathcal{N}_\xi & & \\
 & \mathcal{Q} \arrow[ur, "\sigma"] & & & \\
 & & \mathcal{M}_b^{\mathcal{U}} \arrow[ul, "\pi"'] & & \mathcal{M}_{b^*}^{\vec{\mathcal{U}}^*} \arrow[uull, "\tau_{b^*}"'] \\
\mathcal{N} \arrow[uur, "\vec{\mathcal{W}}"'] \arrow[r, "\vec{\mathcal{T}}"'] \arrow[ddrr, "\pi_Y"] \arrow[uuurr, bend left, "\pi_Y"] & \mathcal{R} \arrow[ur, "{\mathcal{U}, b}"] \arrow[dr, "{\mathcal{U}, c}"'] & \mathcal{M}^+(\mathcal{U}) \arrow[r, "\vec{\mathcal{W}}^*"] & \mathcal{Y} \arrow[ur, "{\vec{\mathcal{U}}^*, b^*}"] \arrow[dr, "{\vec{\mathcal{U}}^*, c^*}"'] & \\
 & & \mathcal{M}_c^{\mathcal{U}} & & \mathcal{M}_{c^*}^{\vec{\mathcal{U}}^*} \arrow[dll, "\tau_{c^*}"] \\
 & & \mathcal{N}_\xi & &
\end{tikzcd}$$

Figure 12.3.4: Branch condensation

Lemma 12.3.6 *Σ_ξ^Y has locally strong branch condensation, and is Ω-fullness preserving.*

Proof. Ω-fullness preservation follows from the construction of Σ_ξ^Y (see Lemma 12.3.3 and the subsequent remarks). We first prove branch condensation (see Figure 12.3.4). Suppose not. Let $\mathcal{N} = \mathcal{N}_\xi^Y$ and $\Psi = \Sigma_\xi^Y$ and suppose the following hold: there are stacks $\vec{\mathcal{T}}^\frown\mathcal{U}$ and $\vec{\mathcal{W}}$ on $\mathcal{N}$ such that

- $\vec{\mathcal{T}}$ is via Ψ with end model $\mathcal{R}$.
- $\vec{\mathcal{U}}$ is according to Ψ and $i = i^{\vec{\mathcal{W}}} : \mathcal{P} \to \mathcal{Q}$ is the iteration map.
- There are cofinal branches b, c of $\mathcal{U}$ and $\pi : \mathcal{M}_b^{\mathcal{U}} \to \mathcal{Q}$ such that

  1. $i = \pi \circ i_b^{\mathcal{U}} \circ i^{\vec{\mathcal{T}}}$.
  2. $c = \Psi(\vec{\mathcal{T}}^\frown\mathcal{U})$.
  3. $b \neq c$.

Let Ψ_0 be the π-pullback strategy of $\Psi_{\vec{\mathcal{W}},\mathcal{Q}}$ and Ψ_1 be $\Psi_{\vec{\mathcal{T}}^\frown\mathcal{U}^\frown c}$. Recall $\mathcal{M}^+(\mathcal{U}) = \mathcal{M}(\mathcal{U})^\sharp$. We may assume:

$$\Lambda_0 =_{\text{def}} (\Psi_0)^{sts}_{\mathcal{M}^+(\mathcal{U})} = (\Psi_1)^{sts}_{\mathcal{M}^+(\mathcal{U})} =_{\text{def}} \Lambda_1. \tag{12.5}$$

In the case there is $\mathcal{Q} \trianglelefteq \mathcal{M}^+(\mathcal{U})$ which is a Q-structure for $\delta(\mathcal{U})$ then $(\Psi_0)^{sts}_{\mathcal{M}^+(\mathcal{U})} = (\Psi_0)_{\mathcal{M}^+(\mathcal{U})}$ and similarly for Ψ_1. We assume this is not the case; otherwise, the argument is similar and simpler.

Let $\sigma : \mathcal{Q}^b \to \mathcal{P}$ be the π_Y-realization map, so that

$$\pi_Y \upharpoonright \mathcal{P} = \sigma \circ \pi^{\vec{\mathcal{T}}^\frown \mathcal{U}, b}.$$

In the above, we note that $\pi^{\vec{\mathcal{T}}^\frown \mathcal{U}, b}$ exists and is the same as $\pi^{\vec{\mathcal{T}}^\frown \mathcal{U}^\frown c}$ and this map does not depend on the choice of the cofinal branch; i.e. $\pi^{\vec{\mathcal{T}}^\frown \mathcal{U}, b} = \pi^{\vec{\mathcal{T}}^\frown \mathcal{U}^\frown b, b} = \pi^{\mathcal{U}, b}_b \circ \pi^{\vec{\mathcal{T}}}$ (even though b may drop).

By results of Section 4.4.1, if 12.5 fails, then there is a minimal disagreement $(\vec{\mathcal{W}}^*, \mathcal{Y}) \in B(\mathcal{M}^+(\mathcal{U}), \Lambda_0) \cap B(\mathcal{M}^+(\mathcal{U}), \Lambda_1)$ in the sense of Definition 4.4.2. Note that $\mathcal{Y}$ is of successor type and $(\Lambda_0)_{\vec{\mathcal{W}}^*, \mathcal{Y}(\alpha)} = (\Lambda_1)_{\vec{\mathcal{W}}^*, \mathcal{Y}(\alpha)}$ for all $\alpha < \lambda^{\mathcal{Y}} - 1$. Furthermore, there is a stack $\vec{\mathcal{U}}^*$ on $\mathcal{Y}$ such that there are distinct branches $b^* = (\Lambda_0)_{\vec{\mathcal{W}}^*, \mathcal{Y}} \neq c^* = (\Lambda_1)_{\vec{\mathcal{W}}^*, \mathcal{Y}}$. Note that

$$\pi^{\mathcal{U}, b}_b \circ \pi^{\vec{\mathcal{T}}} \upharpoonright \mathcal{P} = \pi^{\mathcal{U}}_c \circ \pi^{\vec{\mathcal{T}}} \upharpoonright \mathcal{P}.$$

Note further that there are $\tau_{b^*} : \mathcal{M}^{\vec{\mathcal{U}}^*, b}_{b^*} \to \mathcal{P}$ and $\tau_{c^*} : \mathcal{M}^{\vec{\mathcal{U}}^*, b}_{c^*} \to \mathcal{P}$ such that

$$\pi_Y \upharpoonright \mathcal{P} = \tau_{b^*} \circ \pi^{\vec{\mathcal{U}}^*}_{b^*} \circ \pi^{\vec{\mathcal{W}}^*, b} \circ \pi^{\mathcal{U}, b}_b \circ \pi^{\vec{\mathcal{T}}} \upharpoonright \mathcal{P}, \tag{12.6}$$

and

$$\pi_Y \upharpoonright \mathcal{P} = \sigma \circ \pi^{\vec{\mathcal{U}}, b} = \tau_{c^*} \circ \pi^{\vec{\mathcal{U}}^*}_{c^*} \circ \pi^{\vec{\mathcal{W}}^*, b} \circ \pi^{\mathcal{U}, b}_c \circ \pi^{\vec{\mathcal{T}}} \upharpoonright \mathcal{P}. \tag{12.7}$$

This is because

$$\pi^{\vec{\mathcal{U}}^*}_{b^*} \circ \pi^{\vec{\mathcal{W}}^*, b} = \pi^{\vec{\mathcal{U}}^*}_{c^*} \circ \pi^{\vec{\mathcal{W}}^*, b}.$$

Equations 12.6 and 12.7 contradict Lemma 12.3.5 (which implies that $b^* = c^*$).

So 12.5 holds. By our assumption, $\mathcal{Q}(c, \mathcal{U}) \trianglelefteq \mathcal{M}^{\mathcal{U}}_c$ and is a Λ_1-mouse and $\mathcal{Q}(b, \mathcal{U}) \trianglelefteq \mathcal{M}^{\mathcal{U}}_b$ and is a Λ_0-mouse. Results of Chapter 6 imply that $\mathcal{Q}(b, \mathcal{U}) = \mathcal{Q}(c, \mathcal{U})$ (by comparisons) and hence $b = c$. Contradiction.

The argument above shows branch condensation. The other clause of strong branch condensation follows from a very similar argument, so we leave it to the reader. $\square$

Lemma 12.3.7 *Σ^Y_ξ is locally strongly Ω-fullness preserving.*

$$\begin{tikzcd}
 & & \mathcal{P} \\
 & \mathcal{W}^b \arrow[ur, "\sigma"] \arrow[r, "\pi^{\tau\vec{\mathcal{U}},b}"'] & (\mathcal{W}^*)^b \arrow[u, "\sigma^*"] \\
\mathcal{N}^b \arrow[ur, "\pi^{\vec{\mathcal{T}},b}"] \arrow[uurr, bend left, "\pi_Y"] & \mathcal{R}^b \arrow[u, "\tau"] \arrow[r, "\pi^{\vec{\mathcal{U}},b}"] & (\mathcal{R}^*)^b \arrow[u, "\psi"']
\end{tikzcd}$$

Figure 12.3.5: Strong Ω-fullness preservation

Proof. Ω-fullness preservation follows from the previous lemma. We now prove the other clause of locally strongly Ω-fullness preservation (see Figure 12.3.5). Let $\mathcal{N} = \mathcal{N}_\xi^Y$ and $\Psi = \Sigma_\xi^Y$. Suppose $(\vec{\mathcal{T}}, \mathcal{S}) \in I(\mathcal{N}, \Psi)$ is such that $\pi^{\vec{\mathcal{T}},b}$ exists. Suppose $\mathcal{S}^b \lhd \mathcal{W} \unlhd \mathcal{S}$ is such that for some n and some cardinal κ of $\mathcal{W}$,

$$o(\mathcal{S}^b) \leq \omega\rho_{\mathcal{W}}^{n+1} \leq \kappa < \omega\rho_{\mathcal{W}}^n.$$

Suppose $\tau : \mathcal{R} \to \mathcal{W}$ is cardinal preserving, is $\Sigma_0^{(n)}$, and $\omega\rho_{\mathcal{R}}^n > \mathrm{cr}(\tau) \geq \omega\rho_{\mathcal{R}}^{n+1} = \omega\rho_{\mathcal{W}}^{n+1}$. We want to show the τ-pullback of the strategy $\Sigma_{\vec{\mathcal{T}},\mathcal{W}}$ is Ω-fullness preserving.

Note that $\tau \restriction \mathcal{R}^b = \mathrm{id}$ and $\mathcal{R}^b = \mathcal{W}^b$. This implies $\mathrm{rng}(\pi^{\vec{\mathcal{T}},b}) \subseteq \mathrm{rng}(\tau)$. Let $\sigma : \mathcal{W}^b = \mathcal{S}^b \to \mathcal{P}$ be the π_Y-realization map, so that $\pi_Y \restriction \mathcal{N}^b = \sigma \circ \pi^{\vec{\mathcal{T}},b}$. Since $X \subset \mathrm{rng}(\sigma)$ and $\mathrm{rng}(\sigma)$ is an honest extension of X.

We now show $\Sigma_{\vec{\mathcal{T}},\mathcal{W}}^\tau$ is Ω-fullness preserving. To see this, let $(\mathcal{W}^*, \vec{\mathcal{U}}) \in I(\mathcal{W}, \Sigma_{\vec{\mathcal{T}},\mathcal{W}}^\tau)$ be such that $\pi^{\vec{\mathcal{U}},b} : \mathcal{R}^b \to (\mathcal{R}^*)^b$ exists and let $\tau\vec{\mathcal{U}}$ be the copy tree on $\mathcal{W}$ with last model $\mathcal{W}^*$. So $\pi^{\tau\vec{\mathcal{U}},b} : \mathcal{W}^b \to (\mathcal{W}^*)^b$ exists. Let $\psi : (\mathcal{R}^*)^b \to (\mathcal{W}^*)^b$ be the copy map and $\sigma^* : (\mathcal{W}^*)^b \to \mathcal{P}$ be given by the construction of Ψ, so $\sigma^* \circ \pi^{\tau\vec{\mathcal{U}},b} = \sigma$ and $\sigma^* \circ \pi^{\tau\vec{U},b} \circ \sigma = \pi_Y \restriction \mathcal{N}^b$.

Note that $\psi = \mathrm{id}$ and $\mathrm{rng}(\sigma^*)$ is an honest extension of X. So $(\mathcal{W}^*)^b$ is Ω-full. This is our desired conclusion. $\square$

An easy corollary of the above Lemmata is the following.

Corollary 12.3.8 *Suppose $Y \prec Z \prec \mathcal{N}_\xi$ are countable (in $V[G]$), honest extensions of X, $Y, Z \in V$, and $Y = Y^* \cap \mathcal{N}_\xi$, $Z = Z^* \cap \mathcal{N}_\xi$ for some $Y^* \prec Z^* \prec H_{\kappa^{+++}}^V$. Let $\pi_{Y,Z} = \pi_Z^{-1} \circ \pi_Y$. Then $\Sigma_\xi^Y = (\Sigma_\xi^Z)^{\pi_{Y,Z}}$.*

Proof. Let $\delta_Y = \pi_Y^{-1}(\delta^{\mathcal{P}})$ and $\delta_Z = \pi_Z^{-1}(\delta^{\mathcal{P}})$. By our assumption on Y and Z, we have:

$$\pi_Z \restriction \delta_Z = \pi^{\Sigma_\xi^Z}_{\mathcal{N}_\xi^Z,\infty} \restriction \delta_Z,$$

and

$$\pi_Y \restriction \delta_Y = \pi^{\Sigma_\xi^Y}_{\mathcal{N}_\xi^Y,\infty} \restriction \delta_Y = \pi_Z \circ \pi_{Y,Z} \restriction \delta_Y.$$

Using the above equations, Lemma 12.3.5, and the proof of Lemma 12.3.7 (especially the idea that if two strategies disagree, then there is a lower-level disagreement), we obtain the desired conclusion.

□

Corollary 12.3.9 *Σ_ξ^Y is positional and commuting.*

Proof. This follows from Lemmata 12.3.6, 12.3.7, and results of Section 4.7. □

Now, we discuss how to lift $\Psi = \Sigma_\xi^Y$ to a (necessarily unique) $(\kappa^{+4}, \kappa^{+4})$-strategy Ψ^+ with branch condensation and show $Code(\Sigma_\xi^Y) \in \Omega$.

Recall Ψ is an (ω_1, ω_1)-strategy for $\mathcal{N}_\xi^Y$ with branch condensation, is positional and Ω-fullness preserving. Furthermore, $\Psi \cap V \in V$ and is independent of the choice of generic G. By arguments in [31], $\Psi_{\xi+1,X} \cap V$ can be uniquely extended to an $(\kappa^{+4}, \kappa^{+4})$ strategy with branch condensation and is positional. We also call this extension Ψ. We briefly give a sketch as to how to obtain a $(\kappa^{+4}, \kappa^{+4})$-strategy Ψ^+ extending Ψ with branch condensation and is positional in $V[G]$.

In $V[G]$, suppose $\mathcal{T}$ is of limit length $< \kappa^{+4}$ and is according to Ψ^+. We show how to define $\Psi^+(\mathcal{T})$ (stacks of normal trees can be handled similarly). In V, let $A \subseteq \kappa^{+++}$ code $H_{\kappa^{+++}}$ and a (nice) $Col(\omega, \kappa)$-name $\dot{\mathcal{T}} \in H_{\kappa^{+4}}$ for $\mathcal{T}$ (here we use our cardinal arithmetic assumption 12.1). Let

$$M_A = L^\Lambda_{\kappa^{+4}}[A, \mathcal{M}_2^{\Psi,\sharp}]$$

where Λ is the unique $(\kappa^{+4}, \kappa^{+4})$-strategy for $\mathfrak{M} =_{\text{def}} \mathcal{M}_2^{\Psi,\sharp}$, the minimal E-active Ψ-mouse with two Woodin cardinals. We note that the existence of $\mathcal{M}_2^{\Psi,\sharp}$ follows from [31, Section 3.2]. By $\neg\square(\kappa^{+4})$,

$$M_A \vDash \text{there are no largest cardinals.}$$

In particular $(((\kappa^{+++})^V)^+)^{M_A} < \kappa^{+4}$, so in M_A, which is closed under Λ, we can use Λ to perform a generic genericity iteration to make A-generically generic (see [10] or [20] for more on generic genericity iterations). Let $\mathcal{Q} \in M_A$ be the result of such an iteration. There is a $\mathcal{Q}$-generic $h \subseteq Col(\omega, \delta_0^{\mathcal{Q}})$ such that $H_{\kappa^{+++}}, G, \dot{\mathcal{T}} \in \mathcal{Q}[h]$, where $\delta_0^{\mathcal{Q}}$ is the first Woodin cardinal of $\mathcal{Q}$. Since $\mathcal{Q}$ is closed under Ψ; we can generically interpret Ψ on any generic extensions of $\mathcal{Q}$ (as done in [20] or in Chapter 6).[16] This allows us to define $\Psi^+(\mathcal{T})$ as the branch chosen by the interpretation of Ψ applied to $\mathcal{T}$ in $\mathcal{Q}[h]$. The well-definition and uniqueness of Ψ^+ follow from hull arguments in [31, Section 3.2].[17]

Using Ψ^+, now suppose Ψ is a strategy, we can define the stack of Θ-g-organized mice over $\mathbb{R}$, $\mathrm{Lp}^{\mathsf{G}\Psi^+}(\mathbb{R}, Code(\Psi))$, in $V[G]$ (cf. [20, Definition 4.23]),[18] and show that there is a maximal initial segment $\mathcal{M} \trianglelefteq \mathrm{Lp}^{\mathsf{G}\Psi^+}(\mathbb{R}, Code(\Psi))$ such that $\mathcal{M}$ is constructibly closed and $\mathcal{M} \vDash \mathsf{AD}^+ + \mathsf{SMC} + \Theta = \theta_\Psi$. This implies $Code(\Psi) \in \Omega$.

Remark 12.3.10 *If Ψ is a short-tree strategy, we hold off on showing that $\Psi \in \Omega$. The idea is that we'll wait until we reach a level $\mathcal{N}_\gamma$ (if exists) extending $\mathcal{N}_\xi$ such that some $\mathcal{Q} \trianglelefteq \mathcal{N}_\gamma$ is a $\mathcal{Q}$-structure for $\delta^{\mathcal{N}_\xi}$ and then we can show $(\Sigma_\xi^Y)_{\mathcal{Q}} \in \Omega$ by the above discussion (for Y as above). If we never reach such a level $\mathcal{N}_\gamma$, then we'll see in the next section that the construction stops prematurely. This will allow us to conclude that a model of* LSA *exists.*

Corollary 12.3.11 *Let $\mathcal{N} = \mathcal{N}_\xi^Y$. Then $\rho_\omega(\mathcal{N}) \geq o(\mathcal{N}^b) = o(\mathcal{P})$ and $\mathcal{N}$ is n-solid and n-universal for all $n \in \omega$.*

Proof. By induction, we prove for all $n < \omega$, $\rho_n(\mathcal{N}) \geq o(\mathcal{N}^b)$ and $\mathcal{N}$ is n-solid and n-universal. For $n = 0$, clearly $\mathcal{N}$ is 0-sound. We just prove this for the case $n = 1$; the case $n > 1$ is similar (one just has to work with the $n-1$-reduct).

Without loss of generality, we assume that Ψ is a strategy; otherwise, $\rho_\omega(\mathcal{N}) \geq \delta^{\mathcal{N}} > o(\mathcal{N}^b)$ and there is nothing to prove.

[16] If Ψ is a strategy, we could have simply let $\mathfrak{M} = \mathcal{M}_1^{\Psi,\sharp}$; but if Ψ is a short-tree strategy, then one seems to need $\mathcal{M}_2^{\Psi,\sharp}$ to apply results in Chapter 6. Relevant results in [20] can be applied to $\mathcal{M}_2^{\Psi,\sharp}$ as well.

[17] Let M, M^* be such that $\mathcal{T} \in M \cap M^*$; let τ, τ^* be nice $\mathrm{Col}(\omega,\kappa)$-terms for M, M^* respectively. In $V[G]$, let $X[G]$ contain all relevant objects and $X \prec H_{\kappa^{+4}}$ is good. Let $\bar{a} = \pi_X^{-1}(a)$ for all $a \in X[G]$. Then letting b_0, b_1 be the branches of $\bar{\mathcal{U}}$ given by applying [20, Lemma 4.8] in $L^{\Lambda^*}[tr.cl.(\bar{\tau}), <_1, \mathfrak{M}], L^{\Lambda^*}[tr.cl.(\bar{\tau^*}), <_2, \mathfrak{M}]$ (built inside $M_X[G]$), where $<_1$ is a well-ordering of $\bar{\tau}$ and $<_2$ is a well-ordering of $\bar{\tau^*}$. Then $b_0 = b_1$ as both are according to Λ, since $(\mathfrak{M}, \Lambda^*)$ generically interprets Λ in $V[G]$.

[18] [31] shows that $Code(\Psi)$ is self-scaled in the sense of [20, Definition 4.22] if Ψ is a strategy.

Claim 12.3.12 *$\rho_1(\mathcal{N}) \geq o(\mathcal{N}^b)$.*

Proof. Suppose not. Let $\delta_Y = \pi_Y^{-1}(\delta^{\mathcal{P}})$, $\Psi = \Sigma_\xi^Y$. Let $\mathcal{Q} = \mathrm{Ult}_0(\mathcal{N}, \nu)$ where ν is the order 0 total measure with critical point δ_Y. Let $q = i_\nu(p)$ where $p = p_1(\mathcal{M})$. Hence

(i) $\mathcal{N}^b$ is a cutpoint initial segment of $\mathcal{Q}$ and $o(\mathcal{N}^b)$ is the cardinal successor of δ_Y in $\mathcal{Q}$.

(ii) We can regard $\mathcal{Q}$ as a hod premouse over $(\mathcal{N}^b, \Psi_{\mathcal{N}^b})$ with strategy $\Sigma_{\mathcal{Q}} \in \Omega$ that is commuting and is Ω-fullness preserving.[19]

(iii) There is some $A \subseteq \delta_Y$ such that A is Σ_1-definable over $\mathcal{Q}$ from q and $A \notin \mathcal{N}^b$.

We say that a triple $(\mathcal{Q}, \Sigma_{\mathcal{Q}}, q)$ satisfying (i)-(iii) is *minimal* if there is no $\Sigma_{\mathcal{Q}}$ iteration $\vec{\mathcal{T}}$ with iteration map $i : \mathcal{Q} \to \mathcal{R}$ and some $r < i(q)$ (in the reverse lexicographic order) such that $(\mathcal{R}, \Sigma_{\mathcal{R},\vec{\mathcal{T}}}, i(\mathcal{N}^b), r)$ satisfies (i)-(iii).

Fix two minimal triples $(\mathcal{R}, \Sigma_{\mathcal{R}}, r)$ and $(\mathcal{S}, \Sigma_{\mathcal{S}}, s)$. We can then compare them above $\mathcal{N}^b$. Letting $i : \mathcal{R} \to \mathcal{W}$ and $j : \mathcal{S} \to \mathcal{W}$ be iteration maps. Note that $i(r) = j(s)$ and so

$$\mathrm{Th}^{\mathcal{R}}_{\Sigma_1}(\delta_Y \cup \{r\}) = \mathrm{Th}^{\mathcal{S}}_{\Sigma_1}(\delta_Y \cup \{s\}).$$

This means $\mathrm{Th}^{\mathcal{R}}_{\Sigma_1}(\delta_Y \cup \{r\})$ is $OD^{\Omega}_{\mathcal{N}^b,\Psi_{\mathcal{N}^b}}$ for any minimal $(\mathcal{R}, \Sigma_{\mathcal{R}}, r)$. By $\mathsf{MC}(\Psi_{\mathcal{N}^b})$,

$$\mathrm{Th}^{\mathcal{R}}_{\Sigma_1}(\delta_Y \cup \{r\}) \in \mathcal{N}^b.$$[20]

This contradicts (iii).

$\square$

The claim and Theorem 11.1.2 (which is built on the results of Section 4.9) imply that $\mathcal{N}$ is 1-solid and 1-universal. By similar arguments, we get the conclusion for all $n \in \omega$.

Now if Ψ is a short-tree strategy, then the first conclusion holds as $\rho_\omega(\mathcal{N}) \geq \delta^{\mathcal{N}} > o(\mathcal{N}^b)$ discussed above. Furthermore, $\mathcal{N}$ is iterable above $\rho_\omega(\mathcal{N})$ and hence the proof that $\mathcal{N}$ is n-solid and n-universal for all $n \in \omega$ is as usual.

$\square$

Finally, we show that there are (enough) extenders with critical point $\delta^{\mathcal{P}}$ being put on the extender sequence of the $\mathcal{N}_\xi$'s during the course of the construction.

[19] We can take $\Sigma_{\mathcal{Q}}$ be the $\mathcal{Q}$-tail of Ψ. By Lemma 12.3.7, $\Sigma_{\mathcal{Q}}$ is Ω-fullness preserving. By Corollary 12.3.9 and results of Section 4.7, $\Sigma_{\mathcal{Q}}$ is positional and commuting.

[20] Note that we take Y so that $\mathcal{N}^b = \mathrm{Lp}^{\Psi_{\mathcal{N}|\delta_Y},\Omega}(\mathcal{N}|\delta_Y)$.

Definition 12.3.13 (Extender-ready levels) *We say that $\mathcal{N}_\xi$ is **extender-ready** if for a V-club $\mathcal{C}_\xi$ of $Y \prec \mathcal{N}_\xi$ which is an honest extension of X and $Y \in V$ is countable in $V[G]$, letting $\mathcal{N}_\xi^Y = \pi_Y^{-1}(\mathcal{N}_\xi)$, $\Psi = \Sigma_\xi^Y$ and γ_ξ^Y be the supremum of the indices of extenders on the $\mathcal{N}_\xi^Y$-sequence with critical point $\delta_Y =_{\mathrm{def}} \pi_Y^{-1}(\delta^{\mathcal{P}})$ (we let $\gamma_\xi^Y = ((\delta_Y)^+)^{\mathcal{N}_\xi^Y}$ if $\mathcal{N}_\xi^Y$ has no such extenders on its sequence), we have that no sound $\mathcal{M} \lhd Lp^{\Psi,\Omega}(\mathcal{N}_\xi^Y)$ projects across γ_ξ^Y and every $\mathcal{M} \lhd Lp^{\Psi,\Omega}(\mathcal{N}_\xi^Y)$ is sound.*

Remark 12.3.14 *Extender-ready levels are those $\mathcal{N}_\xi$'s that are eligible to be extended to a hod premouse $(\mathcal{N}_\xi, F)$ where F has critical point $\delta^{\mathcal{P}}$. Let $Y, \mathcal{M}$ be as in the above definition, it is easy to see that $\mathcal{M}$ also does not project across $o(\mathcal{N}_\xi^Y)$.*

The lemma below shows that the collection of correctly-backgrounded extenders with critical point $\delta^{\mathcal{P}}$ is sufficiently rich. For instance, if $\mathcal{P}_Y = \pi_Y^{-1}(\mathcal{P})$, and $\mathcal{N}_\xi^Y = \mathrm{Lp}^{\Sigma_{\mathcal{P}_Y},\Omega}(\mathcal{P}_Y)$, then $\mathcal{N}_\xi^Y$ is extender-ready (Corollary 12.3.11 shows that no level of $\mathcal{N}_\xi^Y$ projects below $o(\mathcal{P}_Y)$ and Theorem 11.1.2 and Corollary 12.3.11 show that every level of $\mathrm{Lp}^{\Sigma_{\mathcal{P}_Y},\Omega}(\mathcal{N}_\xi^Y)$ is sound). Lemma 12.3.15 shows that if $\mathcal{N}_\xi$ is extender-ready then for every $Y \in \mathcal{C}_\xi$, there is an a correctly backgrounded extender E with critical point δ_Y such that $(\mathcal{N}_\xi^Y, E)$ is a hod premouse.

Lemma 12.3.15 *Suppose $\mathcal{N}_\xi$ is extender-ready. Fix $Y \prec \mathcal{N}_\xi$ in $\mathcal{C}_\xi$. Let $\mathcal{N} = \mathcal{N}_\xi^Y$,$\delta_Y = \pi_Y^{-1}(\delta^{\mathcal{P}})$, and $\Psi = \Sigma_\xi^Y$ be the π_Y-realization strategy for $\mathcal{N}$. Then there is an extender E_Y with $cr(E_Y) = \delta^{\mathcal{P}}$ such that E_Y is π_Y-certified over $(\mathcal{N}, \Psi)$.*

Proof. Let $\gamma = o(\mathcal{N})$. Let $E = E_Y$ be the following extender over $\mathcal{N}$: for $a \in [\gamma]^{<\omega}$ and $A \in \wp(\delta_Y)^{|a|} \cap \mathcal{N}$,

$$(a, A) \in E \Leftrightarrow \pi^{\Psi}_{\mathcal{N},\infty}(a) \in \pi_Y(A).$$

Fix a $Y \prec Z \in \mathcal{C}_\xi$ such that $Z = Z' \cap H^V_{\kappa^{+++}}$ and $Z' \prec H^V_{\kappa^{+++}}$ contains all relevant objects. Naturally, $M_{Z'}[G] \prec H_{\kappa^{+++}}[G]$ and $\pi_{Z'}$ extends to act on all of $M_{Z'}[G]$. Let $\pi = \pi^{\Psi}_{\mathcal{N},\infty}$ and $\pi' = (\pi^{\Psi}_{\mathcal{N},\infty})^{M_{Z'}}$. Let $\pi_{Z'} : M_{Z'} \to Z'$ be the uncollapse map (we also denote the extension map $\pi_{Z'}$). Recall that Ψ is Ω-fullness preserving, commuting, and has branch condensation; furthermore, $\pi \restriction \mathcal{N}^b = \pi_Y \restriction \mathcal{N}^b$ and $\pi' \restriction \mathcal{N}^b = \pi_{Y,Z} \restriction \mathcal{N}^b$.

It is easy to see that E is the extender E' defined as follows: for $a \in \gamma^{<\omega}$ and $A \in \wp(\delta_Y)^{|a|} \cap \mathcal{N}$,

$$(a, A) \in E' \Leftrightarrow \pi'(a) \in \pi_{Y,Z}(A).$$

We need to see that $(\mathcal{N}, E)$ is a hod premouse.

Amenability: Let $\eta < \gamma$ and $\xi < o(\mathcal{N}^b)$, we show: $E \cap (\eta^{<\omega} \times \mathcal{N}|\xi) \in \mathcal{N}$. Let $\mathcal{A} = (A_\alpha \mid \alpha < \delta_Y)$ enumerate $\mathcal{N}|\xi$. Let

$$B = \pi_Y(A) \cap (\pi(\eta) \times \pi(\eta)).$$

Then $B \in \mathcal{N}_\xi|\delta^{\mathcal{P}}$ and so is OD^Ω. Now for all $a \in \eta^{<\omega}$, for all $\alpha < \delta_Y$,

$$(a, A_\alpha) \in E \quad \Leftrightarrow \quad \pi(a) \in B_{\pi(\alpha)}$$

This shows $E \cap (\eta^{<\omega} \times \mathcal{N}|\xi)$ is OD^Ω_Ψ. By SMC and the fact that $\mathcal{N}$ is extender-ready, $E \cap (\eta^{<\omega} \times \mathcal{N}|\xi) \in \mathcal{N}$.

Θ_X-completeness: Let $\epsilon < \delta_Y$, $c \in \gamma^{<\omega}$, $\mathcal{A} = (A_\alpha \mid \alpha < \epsilon) \in \mathcal{N}$ be such that $A_\alpha \in E_c$ for all $\alpha < \epsilon$. We need to show: $\pi(c) \in \pi_Y(\bigcap_{\alpha<\epsilon} A_\alpha) = \bigcap_{\alpha<\pi_Y(\epsilon)} \pi_Y(\mathcal{A})_\alpha$.

Since $\pi_Y(\epsilon) = \pi(\epsilon)$, let $\alpha < \pi_Y(\epsilon)$ and $\mathcal{M}$ be a Ψ-iterate of $\mathcal{N}$ such that letting $i : \mathcal{N} \to \mathcal{M}$ be the iteration map and $\Psi_\mathcal{M}$ be the $\mathcal{M}$-tail of Ψ, then there is some $\alpha^* < i(\epsilon)$ such that $\pi^{\Psi_\mathcal{M}}_{\mathcal{M},\infty}(\alpha^*) = \alpha$. Now,

$$\forall \nu < \epsilon(\pi(c) \in \pi_Y(\mathcal{A})_{\pi(\nu)}).$$

Note that letting $\tau : \mathcal{M} \to \pi_Y(\mathcal{N})$ be given by the construction of Ψ then $\Psi_\mathcal{M}$ is the τ-pullback strategy, by Lemma 12.1.3, we get

$$\forall \nu < i(\epsilon)(\pi^{\Psi_\mathcal{M}}_{\mathcal{M},\infty}(i(c)) \in \pi_Y(\mathcal{A})_{\pi^{\Psi_\mathcal{M}}_{\mathcal{M},\infty}(\nu)}).$$

In particular,

$$\pi^{\Psi_\mathcal{M}}_{\mathcal{M},\infty}(i(c)) = \pi(c) \in \pi_Y(\mathcal{A})_\alpha.$$

Since α is arbitrary, we're done.

Normality: Let $c \in \gamma^{<\omega}$, $f : [\delta_Y]^{|c|} \to \delta_Y$ be such that $f \in \mathcal{N}^b$ and $\forall^*_{E_c} u\ f(u) < \max(u)$ or equivalently $\pi_Y(f)(\pi(c)) < \max(\pi(c))$. We want to find a $\xi < \max(c)$ such that

$$\pi_Y(f)(\pi(c)) = \pi(\xi) = \pi_Y(c_\xi)(\pi(c)),$$

where c_ξ is the constant function with range $\{\xi\}$.

Let $\mathcal{M}$ be a Ψ-iterate of $\mathcal{N}$ such that $\pi^\Psi_{\mathcal{N},\mathcal{M}} = \pi_{\mathcal{N},\mathcal{M}}$ exists, and let, $\Psi_\mathcal{M}$ be the $\mathcal{M}$-tail strategy of $\mathcal{M}$ induced by Ψ, and $\tau_\mathcal{M} : \mathcal{M} \to \mathcal{N}_\xi$ be the π_Y-realization map given by the definition of Ψ. Let $E_\mathcal{M}$ be the extender that is $\tau_\mathcal{M}$-certified over $(\mathcal{M}, \Psi_\mathcal{M})$, that is:

$$(a, A) \in E_{\mathcal{M}} \Leftrightarrow \pi^{\Psi_{\mathcal{M}}}_{\mathcal{M},\infty}(a) \in \tau_{\mathcal{M}}(A).$$

It is easy to see that $\pi_{\mathcal{N},\mathcal{M}}[E_{\mathcal{N}}] \subseteq E_{\mathcal{M}}$.

Let $\mathcal{M}_\infty = \mathcal{M}_\infty(\mathcal{N}, \Psi)$ be the direct limit of all non-dropping Ψ-iterates and $\pi : \mathcal{N} \to \mathcal{M}_\infty$ be the direct limit map. Let $g = \pi(f)$. Then by the construction of Ψ and Lemma 12.1.3, the natural map $k : \mathcal{M}_\infty \to \mathcal{N}_\xi$ has critical point $\pi(\delta_Y)$ (and $\pi_Y \restriction \mathcal{N} = k \circ \pi$). So $E_{\mathcal{M}_\infty}$ is defined as:

$$(a, A) \in E_{\mathcal{M}_\infty} \Leftrightarrow a \in k(A).$$

In particular,

$$k(g)(\pi(c)) = \pi_Y(f)(\pi(c)) < \max(\pi(c)).$$

Since $\mathrm{cr}(k) = \pi(\delta_Y)$, it is easy to see that $E_{\mathcal{M}_\infty}$ is normal. By normality and amenability of $E_{\mathcal{M}_\infty}$, there is ξ' such that

$$\pi_{E_\infty}(g)(\xi') = \pi(c).$$

So by elementarity, the desired ξ exists and $\xi' = \pi(\xi)$.

Coherence: We now show:

1. $\mathrm{Ult}_0(\mathcal{N}, E)|\gamma = \mathcal{N}$.

2. Let $\nu = \max\{(\delta_Y^+)^{\mathcal{N}}, \gamma_\xi\}$. Then ν is a cutpoint of $\mathrm{Ult}_0(\mathcal{N}, E)$ and $\gamma = ((\nu)^+)^{\mathrm{Ult}_0(\mathcal{N},E)}$.

For 1), let $\tau : \mathrm{Ult}_0(\mathcal{N}, E) \to \mathcal{N}_\xi$ be the natural map:

$$\tau(i_E(f)(a)) = \pi_Y(f)(\pi^{\Psi}_{\mathcal{N},\infty}(a))$$

for all $f \in \mathcal{H}^+_X$ and $a \in \gamma^{<\omega}$. It's clear that $\tau \restriction \gamma = \pi^{\Psi}_{\mathcal{N},\infty} \restriction \gamma$. This implies $\mathrm{Ult}_0(\mathcal{N}, E)|\gamma$ is isomorphic to $\pi_Y[\mathcal{N}]$ and hence isomorphic $\mathcal{N}$.

For 2), suppose not. Using the fact that $\mathcal{N}$ is extender-ready, we first observe that,

$$\mathcal{N} \vDash \forall \nu \leq \alpha < \gamma \ (|\alpha| \leq \nu). \tag{12.8}$$

Let F be on the sequence of $\mathrm{Ult}_0(\mathcal{N}, E)$ such that

(i) $\mathrm{crit}(F) = \delta_Y$.

(ii) $\mathrm{lh}(F) \geq \nu$.

(iii) $\mathrm{lh}(F)$ is the least such that (i) and (ii) hold.

$$\begin{tikzcd}
 & & \mathcal{N}_\xi & \\
\mathcal{N} \arrow[urr, "\pi_Y"] \arrow[r, "\pi_{Y,Z}"] \arrow[dr, "i"'] & \mathcal{N}_\xi^Z \arrow[ur, "\pi_Z"] & \tau(F) \arrow[r, "u"] & \mathrm{Ult}(\mathcal{N}_\xi^Z, \tau(F)) \arrow[ul, "\sigma"'] \\
 & \mathcal{M} \arrow[u, "\tau"] & F \arrow[u, dashed] \arrow[r, "t"] & \mathrm{Ult}(\mathcal{M}, F) \arrow[u, "k"']
\end{tikzcd}$$

Figure 12.3.6: Coherence

We have then that $\mathrm{lh}(F) \geq \gamma$ by the definition of ν and the fact that $\mathrm{Ult}_0(\mathcal{N}, E)|\gamma = \mathcal{N}$.[21]

Let τ and Z be defined as above. Let $\mathcal{M} = \mathrm{Ult}_0(\mathcal{N}, E)$, i be the corresponding ultrapower map. Let $t : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, F)$ be the ultrapower map by F and $u : \mathcal{N}_\xi^Z \to \mathrm{Ult}(\mathcal{N}_\xi^Z, \tau(F))$ be the ultrapower map by $\tau(F)$. Let $k : \mathrm{Ult}(\mathcal{M}, F) \to \mathrm{Ult}(\mathcal{N}_\xi^Z, \tau(F))$ be the natural map and $\sigma : \mathrm{Ult}(\mathcal{N}_\xi^Z, \tau(F)) \to \mathcal{N}_\xi$ be the realization map. The existence of σ comes from the fact that $\tau(F)$ is π_Z-certified over $(\mathcal{N}_\xi^Z||\mathrm{lh}(\tau(F)), (\Sigma_\xi^Z)_{\mathcal{N}_\xi^Z||\mathrm{lh}(\tau(F))})$.

Claim 12.3.16 *$lh(F) = \gamma$.*

Proof. Note that ξ is a cutpoint in $\mathrm{Ult}(\mathcal{M}, F)$ and is the least such $> \delta_Y$. So by 12.8,

$$\mathrm{lh}(F) = (\nu^+)^{\mathrm{Ult}(\mathcal{M},F)}$$

Suppose $\mathrm{lh}(F) > \gamma$. Let $\mathcal{Q} \lhd \mathcal{M}||\mathrm{lh}(F)$ be least such that

$$\mathcal{N} \lhd \mathcal{Q} \wedge \mathcal{Q} \vDash |\gamma| = \nu.$$

Note that $\mathcal{W}$ is a level of $Lp^{\Psi,\Omega}(\mathcal{N})$. This is by by SMC and the fact that

$$\pi_Z \circ \tau \restriction \mathcal{N} = \pi^{\Psi}_{\mathcal{N},\infty} .$$

This contradicts the assumption that $(\mathcal{N}, \Psi)$ is extender-ready. $\square$

Now we show F is π_Y-certified over $(\mathcal{N}, \Psi)$. This would give $E = F \in \mathrm{Ult}(\mathcal{N}, E)$. Contradiction.

Let $\Lambda = \Sigma_\xi^Z$. First note that

[21] If ν is not a cutpoint of $\mathrm{Ult}_0(\mathcal{N}, E)$, then there is some extender H on the sequence of $\mathrm{Ult}_0(\mathcal{N}, E)$ such that $\mathrm{cr}(H) \leq \nu < \mathrm{lh}(H)$. This easily implies that there is some extender F on the sequence of $\mathrm{Ult}_0(\mathcal{N}, E)$ such that $\mathrm{cr}(F) = \delta_Y$ and $\mathrm{lh}(E) \geq \nu$.

- $\pi^{\Psi}_{\mathcal{N},\infty} = \pi^{\Lambda_{\tau(\mathcal{N})}}_{\tau(\mathcal{N}),\infty} \circ \pi^{\Psi}_{\mathcal{N},\mathcal{N}^Z_\xi}$.[22]
- $\tau \restriction \mathcal{N} = \pi^{\Psi}_{\mathcal{N},\mathcal{N}^Z_\xi}$.
- $\sigma \restriction \tau(\mathcal{N}) = \pi^{\Lambda_{\tau(\mathcal{N})}}_{\tau(\mathcal{N}),\infty}$.

Let $c \in [o(\mathcal{N})]^{<\omega}$, $A \in \mathcal{P}_Y$, we have:

$$\begin{aligned} A \in F_c &\Leftrightarrow c \in t(A) \\ &\Leftrightarrow c \in t(i(A)) \quad \text{(because } i(A) \cap \delta_Y = A) \\ &\Leftrightarrow \tau(c) \in u \circ \tau(i(A)) \quad \text{(Corollary 9.2.9)} \\ &\Leftrightarrow \tau(c) \in u(\pi_{Y,Z}(A)) \\ &\Leftrightarrow \sigma(\tau(c)) \in \pi_Y(A) \quad \text{(because } \pi_Y(A) = \sigma(u(\pi_{Y,Z}(A)))) \\ &\Leftrightarrow \pi^{\Psi}_{\mathcal{N},\infty}(c) \in \pi_Y(A). \end{aligned}$$

This finishes the proof of the lemma. □

Lemma 12.3.15 implies that if $\mathcal{N}_\xi$ is extender-ready then $\mathcal{N}^*_{\xi+1} = (\mathcal{N}_\xi, E)$ where using the notation of Lemma 12.3.15

$$(a, A) \in E \Leftrightarrow \forall Y \in \mathcal{C}_\xi((a, A) \in Y \to \pi_Y^{-1}(a, A) \in E_Y).$$

We continue by proving another condensation lemma for relevant extenders with critical point $\delta^{\mathcal{P}}$. This condensation property does not seem to follow from Theorem 11.1.2.

Lemma 12.3.17 *Suppose $\mathcal{N}_\xi$ is E-active, say $\mathcal{N}_\xi$ is of the form $(\mathcal{N}^-_\xi, F_\xi)$, where $cr(F_\xi) = \delta^{\mathcal{P}}$. Suppose $\pi : \mathcal{M} = (\mathcal{M}^-, \tilde{F}) \to \mathcal{N}_\xi$ is Σ_0 and cofinal, or Σ_2, with $cr(\pi) > o(\mathcal{P})$ and suppose further that $\mathcal{M}^- \trianglelefteq \mathcal{N}^-_\xi$. Then $\tilde{F}$ is on the sequence of $\mathcal{N}_\xi$.*

Furthermore, let Y be a good hull that contains all relevant objects, let $\pi_Y : M_Y[G] \to H_{\kappa^{+++}}[G]$ be the uncollapse map, and let $x^Y = \pi_Y^{-1}(x)$ for all x in the range of π_Y. Let Ψ be the π_Y-pullback strategy for $\mathcal{M}^Y$ and suppose that $\Psi_{\mathcal{M}^{-,Y}} = (\Sigma^Y_\xi)_{\mathcal{M}^{-,Y}} = (\Sigma^Y_\xi)^{\pi^Y}_{\mathcal{M}^{-,Y}}$, then $\Psi = (\Sigma^Y_\xi)_{\mathcal{M}^Y} = (\Sigma^Y_\xi)^{\pi^Y}_{\mathcal{M}^Y}$, where Σ^Y_ξ is the strategy for $\mathcal{N}^Y_\xi$ defined above.

[22] $\mathcal{N}^Z_\xi$ is not literally a Ψ-iterate of $\mathcal{N}$, but $\mathcal{N}$ iterates into a hod initial segment of $\mathcal{N}^Z_\xi$. By $\pi^{\Psi}_{\mathcal{N},\mathcal{N}^Z_\xi}$, we mean $(\pi^{\Psi}_{\mathcal{N},\infty})^{M_{Z'}}$.

Proof. The preservation of π guarantees that $\mathcal{M}$ is a hod premouse. Recall that $o(\mathcal{P})$ is the cardinal successor of $\delta^{\mathcal{P}}$ in both $\mathcal{N}_\xi$ and $\mathcal{M}$ and the models agree up to $\mathcal{P}$.

We work with $\mathcal{M}^Y$ and $\mathcal{N}_\xi^Y$ and first show that $\tilde{F}^Y$ is on the sequence of $\mathcal{N}_\xi^Y$. Let $\Lambda = \Psi_{\mathcal{M}^{-,Y}} = (\Sigma_\xi^Y)_{\mathcal{M}^{-,Y}}$.

Claim 12.3.18 *For $A \in \wp(\delta^{\mathcal{P}}) \cap \mathcal{P}$ and $a \in [o(\mathcal{M}^Y)]^{<\omega}$, $(a, A) \in \tilde{F}^Y$ if and only if $\pi^\Lambda_{\mathcal{M}^{-,Y},\infty}(a) \in \pi_Y(A)$.*

Proof. First, note that $\tilde{F}^Y$ is total over $\mathcal{N}_\xi^Y$ and hence it makes sense to apply $\tilde{F}^Y$ to $\mathcal{N}_\xi^Y$. Also, $\mathrm{Ult}(\mathcal{N}_\xi^Y, \tilde{F}^Y)$ embeds into $\mathrm{Ult}(\mathcal{N}_\xi^Y, F_\xi^Y)$ via the natural map τ:

$$\tau(i_{\tilde{F}^Y}(f)(b)) = i_{F_\xi^Y}(f)(\pi^Y(b)),$$

and

$$\tau \restriction \mathcal{M}^Y||lh(\tilde{F}^Y) = \pi^Y \restriction \mathcal{M}^Y||lh(\tilde{F}^Y).$$

Now,

$$\begin{aligned}
(a, A) \in \tilde{F}^Y \quad &\Leftrightarrow \quad (\tau(a) = \pi^Y(a), A) \in F_\xi^Y \quad (\pi^Y(A) = \tau(A) = A)\\
&\Leftrightarrow \quad \pi^\Lambda_{\mathcal{M}^{-,Y},\infty}(\pi^Y(a)) \in \pi_Y(A) \quad (\text{definition of } F_\xi^Y)\\
&\Leftrightarrow \quad \pi^\Lambda_{\mathcal{M}^{-,Y},\infty}(a) \in \pi_Y(A). \quad (\text{Corollary 9.2.9})
\end{aligned}$$

This finishes the proof of the claim. $\square$

The claim and Lemma 12.3.15 imply that $\tilde{F}^Y$ is on the $\mathcal{N}_\xi^Y$-sequence. By elementarity, $\tilde{F}$ is on the $\mathcal{N}_\xi$-sequence.

$\Psi = (\Sigma_\xi^Y)_{\mathcal{M}^Y} = (\Sigma_\xi^Y)^{\pi^Y}_{\mathcal{M}^Y}$ follows from Lemma 9.1.9 and the proof of Lemma 12.3.6 (the main point is the fact that if the strategies disagree then we can find a lower-level disagreement).

$\square$

Now suppose $\mathcal{N}_\xi^*$ is a sts hod premouse, that is there is some δ such that $\mathcal{M}^+(\mathcal{N}_\xi^*|\delta) \lhd \mathcal{N}_\xi^*$ and $\mathcal{N}_\xi^* \vDash \delta$ is Woodin and $\delta^{\mathcal{P}}$ is the least $<\delta$-strong cardinal. Let Y be as above. Suppose $\mathcal{N}_\xi^*$ defines a failure of Woodinness of δ, then $\mathcal{N}_\xi^Y$ is iterable via the π_Y-realization strategy and this is also the $\mathcal{Q}$-structure guided strategy for stacks above $\rho_\omega(\mathcal{N}_\xi^Y)$. Suppose $\mathcal{N}_\xi^*$ does not define a failure of Woodiness of δ then it is not clear that for Y as above, $\mathcal{N}_\xi^Y$ is iterable via a π_Y-realization strategy. However, in this case, $\rho_\omega(\mathcal{N}_\xi^Y) \geq \delta$ and $\mathcal{N}_\xi^Y$ is iterable above δ^Y via the π_Y-realization strategy. This is enough to show that $\mathcal{N}_\xi^Y$ is solid and universal, hence $\mathcal{N}_\xi^*$ is solid and universal. In either case, $\mathcal{N}_\xi$ is defined and sound. If $\xi < \Upsilon$, we can then define $\mathcal{N}_{\xi+1}^*$ and go on with our construction.

12.4 K^c breaks down and a model of LSA

We have shown in the previous section that the construction lasts κ^{+++} steps if every $\mathcal{N}_\xi$ is lsa-small and is not of lsa type (or more accurately speaking, every Woodin cardinal δ in $\mathcal{N}_\xi$ eventually fails to be Woodin with respect to the short tree strategy in $\mathcal{N}_\gamma$ for some $\gamma > \xi$). Suppose the construction lasts κ^{+++} steps; as in the previous subsection, let $\mathcal{N} = \mathcal{N}_{\kappa^{+++}}$. We also let $\mathcal{S} = \mathcal{S}(\mathcal{N})$.

Lemma 12.4.1 *cof*$(\mathcal{S}) < \kappa^{+++}$.

Proof. Let $\lambda = \kappa^{+++}$. As shown in Chapter 9, $\mathcal{S} \vDash \square_{\lambda,2}$. Also, $\mathcal{S} \in V$ by definition. Working in V, $\neg\square(3, \kappa^{+4})$ implies then that $\mathrm{o}(\mathcal{S}) < \kappa^{+4}$ and $\neg\square(3, \kappa^{+++})$ now implies that $\mathrm{cof}(o(\mathcal{S})) < \kappa^{+++}$ since otherwise, the canonical $\square_{\lambda,2}$-sequence $\vec{C}$ of $\mathcal{S}$ (as defined in Chapter 9) has a thread D. The thread D will produce a hod mouse $\mathcal{M}$ such that $\mathrm{o}(\mathcal{M}) \geq o(\mathcal{S})$ and $\rho_\omega(\mathcal{M}) \leq \lambda$. This contradicts (ii) of Lemma 12.2.9. $\square$

Lemma 12.4.1 contradicts (iii) of Lemma 12.2.9. Now we assume the construction stops prematurely. We obtain a model of LSA from this assumption. Recall in this case, Υ is the least such that $\mathcal{N}_\Upsilon$ is a sts hod premouse that satisfies:

(i) There is a unique Woodin cardinal $\delta_0 > \delta^{\mathcal{P}}$ such that $\delta^{\mathcal{P}}$ is the least $< \delta_0$-strong.

(ii) There are ω many Woodin cardinals above δ_0, say these Woodin cardinals are $(\delta_n : 1 \leq n < \omega)$.

(iii) There is an extender F with $\mathrm{crt}(F) > \sup_n \delta_n$ such that $\mathcal{N}_\Upsilon = (\mathcal{R}, F)$ for some $\mathcal{R}$.

(iv) $\mathcal{N}_\Upsilon$ is a sts hod premouse over $\mathcal{M}^+(\mathcal{N}_\Upsilon|\delta_0) =_{\mathrm{def}} (\mathcal{N}_\Upsilon|\delta_0)^\sharp$.

(v) For every countable $Y \prec \mathcal{N}_\Upsilon$ (Y is an honest extension of X), letting $\mathcal{Q} = (\mathcal{N}_\Upsilon|\delta)^\sharp$ and $\mathcal{Q}_Y = \pi_Y^{-1}(\mathcal{Q})$, then $\mathcal{N}_\xi^Y$ is $\omega_1 + 1$-iterable above $\mathcal{Q}_Y$ via the π_Y-realization strategy.

(i)-(iv) follow easily from the assumption that the construction stops prematurely. (v) follows from results in [1] (we tacitly assume that π_Y is minimal relative to some enumeration of $\mathcal{N}_\Upsilon^Y - \mathcal{Q}_Y$ of order type ω; otherwise, just replace π_Y with such an embedding that agrees with $\pi_Y \restriction \mathcal{Q}_Y$). Let $\lambda = \sup_n \delta_n$ and for every $\mathcal{Q} \lhd \mathcal{M} \trianglelefteq \mathcal{N}_\Upsilon$, let $\Sigma^{\mathcal{M}}$ be the internal sts strategy of $\mathcal{Q}$ as defined in $\mathcal{M}$.

Lemma 12.4.2 *Suppose the construction stops prematurely. Then* $\Upsilon < \kappa^{+++}$.

Proof. If the construction stops prematurely, then $\mathcal{N}_\Upsilon$ is E-active. This clearly implies that $\Upsilon < \kappa^{+++}$ because if $\Upsilon = \kappa^{+++}$, then $\mathcal{N}_\Upsilon$ is the lim inf of $\mathcal{N}_\alpha$ for $\alpha < \Upsilon$ and hence is passive. □

Now suppose there is some $\mathcal{M} \trianglelefteq \mathcal{N}_\Upsilon$ satisfying Definition 8.2.2, then the results of Section 8.2 show that the derived model of $\mathcal{M}$ (at the sup of its Woodin cardinals) satisfies LSA. Suppose this is not the case. We would like to produce an active ω Woodin lsa mouse as in Definition 8.2.2 from $\mathcal{N}_\Upsilon$.

Lemma 12.4.3 *Let $\mathcal{M}$ be the transitive collapse of $Hull^{\mathcal{N}_\Upsilon}(\mathcal{P} \cup p(\mathcal{N}_\Upsilon))$ and let π be the transitive collapse. Then there is a countable substructure of some $\mathcal{N} \trianglelefteq \mathcal{M}$ satisfying Definition 8.2.2.*

Proof. First, note that we have the following:

$$\rho_\omega(\mathcal{N}_\Upsilon) \geq o(\mathcal{P}) \text{ and } \rho_\omega(\mathcal{M}) \leq o(\mathcal{P}) \text{ (in fact, } \rho_\omega(\mathcal{M}) = o(\mathcal{P})).$$

Now we claim that

Claim 12.4.4 *(i) if $Y \prec \mathcal{M}$ is an honest extension of X, letting π_Y be the uncollapse map and $x^Y = \pi^{-1}(x)$ for x in range π_Y or $x = \mathcal{M}$, then $\mathcal{M}^Y$ is iterable via the $\pi \circ \pi_Y$-realization strategy.*

(ii) Suppose Y is as in (i) and $\tau : \mathcal{M}^* \to \mathcal{M}^Y$ is either Σ_0 cofinal or Σ_2 elementary and $cr(\tau) > o(\mathcal{P}^Y)$, then the comparison $(\mathcal{M}^Y, \mathcal{M}^*, cr(\tau))$ against $\mathcal{M}^Y$ does not use extenders with critical point $(\delta^{\mathcal{P}})^Y$.

(iii) $\mathcal{M}$ is ω-sound.

Proof. For (i), we just check that every iteration tree $\mathcal{T}$ on $\mathcal{M}^Y$ above $(\delta^{\mathcal{P}})^Y$ has a unique $\pi \circ \pi_Y$-realizable branch. Suppose without loss of generality that $\mathcal{T}$ is not Lp-based, so in this case $\mathcal{T}$ is above $\mathcal{P}^Y$. By [1], there is a maximal $\pi \circ \pi_Y$-realizable branch b for $\mathcal{T}$, but this branch is precisely the cofinal branch guided by the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$, i.e. b is the unique such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.[23] The case that $\mathcal{T}$ uses extenders $\leq (\delta^{\mathcal{P}})^Y$ is similar and has been treated in details in the previous section; the general case that $\vec{\mathcal{T}}$ is a stack can be treated as follows: decompose $\vec{\mathcal{T}}$ into a sequence of stacks $\langle \vec{\mathcal{T}}_i : i < \alpha \rangle$ where for each i, $\vec{\mathcal{T}}_i$ is either strictly below $(\delta^{\mathcal{P}})^Y$

[23]The theory developed about for sts hod mice shows that there cannot be another cofinal, $\pi \circ \pi_Y$-realizable branch $c \neq b$. This is because we can compare $\mathcal{Q}(c, \mathcal{T})$ against $\mathcal{Q}(b, \mathcal{T})$, the comparison does not involve strategy disagreement, and hence is successful. This implies $\mathcal{Q}(c, \mathcal{T}) = \mathcal{Q}(b, \mathcal{T})$ and hence $c = b$.

or its images, or only uses extenders with critical point $(\delta^{\mathcal{P}})^Y$ or its images, or else is strictly above $(\delta^{\mathcal{P}})^Y$ or its images; inductively on i, we construct $\pi \circ \pi_Y$-realization maps σ_i on $\mathcal{M}^{\vec{\mathcal{T}}_i}$ using the above discussion.

(ii) follows from Lemma 12.3.17. For (iii), the point is that the relevant phalanx comparisons in the proof of solidity and universality are successful and by (ii), no extenders with critical point $(\delta^{\mathcal{P}})^Y$ are used. $\square$

Suppose without loss of generality, no countable substructures of any $\mathcal{N} \lhd \mathcal{M}$ satisfies Definition 8.2.2. We claim that for Y as in (i) of the above claim, $\mathcal{M}_Y$ does. Again, let Y be as above and it suffices to show $\mathcal{M}^Y$ satisfies Definition 8.2.2. Everything is clear except, perhaps, for (1). So let Σ be the $\pi \circ \pi_Y$-realization strategy for $\mathcal{M}^Y$ and $\mathcal{Q} = \mathcal{M}^Y|((\delta_0^Y)^{+\omega})^{\mathcal{M}^Y}$. By the argument as in Claim 12.4.4 and Lemma 12.3.6, $\Sigma_{\mathcal{Q}}^{stc}$ has (locally) strong branch condensation. Similarly to 12.3.7, $\Sigma_{\mathcal{Q}}^{stc}$ is also (locally) strongly Ω-fullness preserving and hence is (locallly) strongly $\Gamma(\mathcal{Q}, \Sigma_{\mathcal{Q}}^{stc})$-fullness preserving.[24]

$\square$

Again, Lemma 12.4.3 and results in Section 8.2 show that the new derived model of $\mathcal{N}$ as in the conclusion of Lemma 12.4.3 (at the sup of its Woodin cardinals) satisfies LSA.

Now by boolean comparisons, there is some $(\mathcal{M}, \Sigma) \in V$ satisfying Definition 8.2.2. By taking a countable hull of $\mathcal{M}$ if necessary, we may assume $\mathcal{M}$ is countable (in V). Let $\mathcal{M}^-$ be the class model obtained by iterating the top extender of $\mathcal{M}$ OR many times and $\mathcal{M}_\infty$ be the result of an $\mathbb{R}$-genericity iteration of $\mathcal{M}^-$ via Σ. Then (new) derived model N of $\mathcal{M}_\infty$ satisfies LSA as shown by Section 8.2. By homogeneity of $Col(\omega, \kappa)$, there is in V a model M containing $\mathbb{R} \cup \mathrm{OR}$ such that $M \vDash \mathsf{LSA}$.

Proof.[Proof of Theorem 12.0.22] The arguments above prove the consistency of LSA from the hypothesis of Theorem 12.0.22 plus the simplifying assumption 12.1. To eliminate 12.1, simply note that the constructions above can be done in $V^{\mathbb{Q}}$, where $\mathbb{Q} = \mathbb{Q}_0 * \mathbb{Q}_1$ and $\mathbb{Q}_0 = Col(\kappa^+, \kappa^+) * Col(\kappa^{++}, \kappa^{++}) * Col(\kappa^{+++}, \kappa^{+++})$ and $\mathbb{Q}_1 = Col(\omega, \kappa)$. It's easy to see that in $V^{\mathbb{Q}_0}$, 12.1 holds and continues to hold in $V^{\mathbb{Q}}$. Furthermore, the models $\mathcal{N}_\xi$ (built by the construction above inside $V^{\mathbb{Q}}$) are in V. The arguments above then can be applied in $V^{\mathbb{Q}}$ to obtain the consistency of LSA. $\square$

[24] We don't need (ii) to prove (iii); we just need the phalanx comparisons are successful. (ii) gives that the comparison is above $(\delta^{\mathcal{P}})^Y$.

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