

SOME USES OF SET THEORY IN ALGEBRA

Stanford Logic Seminar
February 10, 2009

Plan

- I. The Whitehead Problem – early history
- II. Compactness and Incompactness
- III. Deconstruction

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P. Eklof and A. Mekler, **Almost Free Modules**, North-Holland (1990);

Preface begins:

“The modern era in set-theoretic methods in algebra can be said to have begun on July 11, 1973 when Saharon Shelah borrowed László Fuchs’ *Infinite Abelian Groups* from the Hebrew University library. Soon thereafter, he showed that Whitehead’s Problem — to which many talented mathematicians had devoted much creative energy — was not solvable in ordinary set theory (ZFC).”

“One day I have come and see the second volume of László; its colour was attractive green. I take it and ask myself isn’t everything known on [abelian groups]... I start to read each linearly; after reading about two thirds of the first volume I move to the second volume and read the first third. I mark the problems (I think six) which attract me—combination of being stressed by László, seem to me I have a chance, and how nice the problem look”

“I have thought the most important is to build indecomposable abelian groups in every cardinality. I thought the independence of Whitehead’s problem will be looked on suspiciously. As you know abelian group theorists thought differently.” [communication from Shelah]

I. The Whitehead Problem

“module” means left R -module, where R is a ring

“group” means abelian group, i.e. \mathbb{Z} -module

A module is **free** if it has a basis

or, equivalently, is isomorphic to a direct sum of copies of R .

Fact: A subgroup of a free group is free.

(This is not true for modules in general.)

The Whitehead Problem

Is every Whitehead group (of arbitrary cardinality) free?
(Ehrenfeucht 1955)

Fact: Every free group is a Whitehead group.

Classic result: Every countable Whitehead group is free. (Stein 1951; Ehrenfeucht 1955)

Fact: A subgroup of a Whitehead group is a Whitehead group.

Hence, if A is a Whitehead group of cardinality \aleph_1 , then every countable subgroup of A is free, i.e., A is \aleph_1 -free.

S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math **18** (1974)

Theorem (Shelah)

- (1) Assuming $V = L$, every Whitehead group of cardinality \aleph_1 is free;
- (2) Assuming $MA + \neg CH$, there is a Whitehead group of cardinality \aleph_1 which is not free.

The proof of (1) requires $\diamond_{\omega_1}(S)$ for every stationary subset S of \aleph_1 .

In Math. Reviews, a review of L. Fuchs' **Infinite Abelian Groups, vol II** states:

Since Volume II was written, S. Shelah [Israel J. Math. 18 (1974), 243–256] has shown that the statement “Every W -group of cardinal \aleph_1 is free” is independent of ZFC (Paul Hill has shown that GCH implies that every W -group of cardinality \aleph_Ω is free and J. Rotman has an easy proof that CH implies that every W -group is free).

The mathematical assertions in parenthesis are wrong, as Shelah later showed:

Devlin-Shelah (c. 1977): weak CH ($2^{\aleph_0} < 2^{\aleph_1}$) implies “weak diamond” implies that Whitehead groups are $L_{\infty\omega_1}$ -free.

But

Shelah (c. 1977): It is consistent with ZFC + GCH that there are Whitehead groups of cardinality \aleph_1 which are not free.

Ila. Compactness

Shelah's original paper proves:

Theorem. Assuming $V = L$, if κ is a regular uncountable cardinal and every Whitehead group of cardinality $< \kappa$ is free, then every Whitehead group of cardinality κ is free.

(The proof uses $\diamond_{\kappa}(S)$ for every stationary subset S of κ .)

An inductive argument shows that every Whitehead group of cardinality \aleph_n is free for all $n \in \omega$.

To go further one needs a result for singular cardinals.

Shelah's Singular Compactness Theorem

Let λ be a singular cardinal and A a group [abelian or non-abelian] of cardinality λ such that every subgroup of cardinality $< \lambda$ is free. Then A is free.

Hence: assuming $V = L$, **every** Whitehead group is free.

Shelah's Singular Compactness Theorem was much more general.

Let \mathcal{F} be a class of modules; the members of \mathcal{F} will be called " \mathcal{F} -free".

Shelah's Singular Compactness Theorem—more general version

Suppose λ is a singular cardinal and M is a module of cardinality λ such that for all $\kappa < \lambda$, *enough* submodules of cardinality κ are \mathcal{F} -free; then M is \mathcal{F} -free.

This is a template for a theorem. A specific example will be given later.

11b. Incompactness

Question: For which regular uncountable cardinals κ is there an abelian group of cardinality κ which is not free, but such that every subgroup of cardinality $< \kappa$ is free? Such a κ will be called **incompact**.

FACTS: (1) A weakly compact cardinal is not incompact.

(2) Assuming $V = L$, a regular uncountable κ is incompact only if it is not weakly compact.

(3) [Magidor-Shelah] Every regular uncountable cardinal $< \aleph_{\omega^2}$ is incompact. Moreover, if \aleph_α and \aleph_β are incompact, then so are $\aleph_{\alpha+1}$ and $\aleph_{\alpha+\aleph_\beta+1}$.

(4) [Magidor-Shelah] It is consistent with ZFC + GCH (assuming the consistency of certain large cardinals) that \aleph_{ω^2+1} is not incompact.

(5) [Shelah] An uncountable cardinal κ is incompact if and only if there is a family of size κ of countable sets which does not have a transversal (a one-one choice function) but every subfamily of size $< \kappa$ does.

III. Deconstructibility

Let \mathcal{A} be a class of modules.

Definitions

\mathcal{A} is **μ -deconstructible** if every module in \mathcal{A} is the union of a chain of submodules $\{A_\alpha : \alpha < \sigma\}$ which are members of \mathcal{A} such that:

- (1) $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ for all limit ordinals $\beta < \sigma$; and
- (2) for all $\alpha < \sigma$, $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ and has cardinality $< \mu$.

\mathcal{A} is **deconstructible** if it is μ -deconstructible for some μ .

We will restrict to classes of the form

$\mathcal{A} = {}^\perp \mathcal{B} = \{A \mid \text{Ext}^1(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ for some set or class \mathcal{B} .

Recall: $\text{Ext}^1(A, B) = 0$ iff every short exact sequence

$$0 \rightarrow B \rightarrow M \rightarrow A \rightarrow 0$$

splits, i.e., up to isomorphism the only one is

$$0 \rightarrow B \rightarrow B \oplus A \rightarrow A \rightarrow 0$$

If $\mathcal{B} = \{\mathbb{Z}\}$, then \mathcal{A} is the class of **Whitehead groups**.

Lemma.

Fix \mathcal{A} and μ . If \mathcal{F} is the class of μ -deconstructible members of \mathcal{A} , then the Singular Compactness Theorem applies to the notion of \mathcal{F} -free (for sufficiently large singular cardinals); i.e.,

If λ is a sufficiently large singular cardinal and $M \in \mathcal{A}$ has cardinality λ and enough submodules of cardinality $< \lambda$ are μ -deconstructible, then M is μ -deconstructible.

Theorem.

Assuming $V = L$, the class of Whitehead groups is \aleph_1 -deconstructible.

Corollary.

Assuming $V = L$, every Whitehead group is free

The proof uses the homological fact:

If $A = \bigcup_{\alpha < \sigma} A_\alpha$ and $A_0 = 0$ and $A_{\alpha+1}/A_\alpha$ are free for all $\alpha < \sigma$, then A is free.

Baer modules

Let R be an integral domain. Let \mathcal{T} be the class of torsion-free R -modules.

Let $\mathcal{A} = {}^\perp\mathcal{T} = \{A \mid \text{Ext}^1(A, B) = 0 \text{ for all } B \in \mathcal{T}\}$.

Say A is a **Baer module** if it belongs to \mathcal{A} .

Question (Kaplansky): are all Baer modules over an arbitrary ID projective (i.e., a direct summand of a free module)?

Theorem. (Eklof-Fuchs-Shelah 1990)

The class of Baer modules is \aleph_1 -deconstructible.

Question: Are the countably-generated Baer modules projective?

Theorem. (Angeleri Hugel-Bazzoni-Herbera 2005)

Every countably-generated Baer module over an arbitrary ID is projective.
Hence, every Baer module is projective.

Definition

A module T is **n -tilting** if:

- (1) T has proj. dim. $\leq n$;
- (2) $\text{Ext}^i(A, B) = 0$ for all $i \geq 2$ and all $A \in {}^\perp(\{T\}^\perp)$, $B \in \{T\}^\perp$; and
- (3) $\{T\}^\perp$ is closed under direct sums.

Tilting modules

A tilting module has *finite type* (resp. *countable type*) if there is a set S of finitely-presented modules (resp. countably-presented modules) such that $S^\perp = \{T\}^\perp$.

Theorems

1. (Bazzoni-E-Trlifaj 2003) All 1-tilting modules are of countable type.
2. (Bazzoni-Herbera 2005) All 1-tilting modules are of finite type.
3. (Šťovíček-Trlifaj 2005) All n -tilting modules are of countable type.
4. (Bazzoni-Šťovíček 2005) All n -tilting modules are of finite type.

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