# FINITE ELEMENT METHODS FOR STOKES EQUATIONS

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In this notes, we shall prove the inf-sup condition for Stokes equation and present several inf-sup stable finite element spaces. We use Fortin operator to verify the discrete inf-sup condition.

## 1. Stokes Equations

In this section, we study the well-posedness of the weak formulation of the steady-state Stokes equations

$$-\mu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega \tag{1}$$

$$-\operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega \tag{2}$$

where $\boldsymbol{u}$ can be interpreted as the velocity field of an incompressible fluid motion, $p$ is then the associated pressure, and the positive constant $\mu$ is the viscosity coefficient of the fluid. For simplicity, we consider homogenous Dirichlet boundary condition for the velocity, i.e. $\boldsymbol{u}|_{\partial\Omega} = 0$.

Multiplying test function $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ to the momentum equation (1) and $q \in L^2(\Omega)$ to the mass equation (2), and applying integration by part for the momentum equation, we obtain the weak formulation of the Stokes equations: Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and a pressure $p \in L_0^2(\Omega)$ such that

$$(\mu\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{3}$$

$$-(\operatorname{div} \boldsymbol{u}, q) = 0 \qquad \text{for all } q \in L_0^2(\Omega). \tag{4}$$

The conditions for the well-posedness of a saddle point system is known as inf-sup conditions or Ladyzhenskaya-Babuška-Breezi (LBB) condition; see *Inf-sup conditions for operator equations* for details.

The setting for the Stokes equations is:

- Spaces:

$$\mathbb{V} = \boldsymbol{H}_0^1(\Omega) \text{ with norm } |\boldsymbol{v}|_1 = \|\nabla \boldsymbol{v}\|,$$

$$\mathbb{P} = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0\} \text{ with norm } \|p\|,$$

$$\mathbb{Z} = \mathbb{V} \cap \ker(\operatorname{div}).$$

- Bilinear forms:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = -\int_\Omega (\operatorname{div}\boldsymbol{v})\, q \,\mathrm{d}\boldsymbol{x}.$$

- Operators:

$$A = -\Delta : \boldsymbol{H}_0^1(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), \qquad \langle Au, v\rangle = a(u, v) = \mu(\nabla u, \nabla v),$$

$$B = -\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \mapsto (L_0^2(\Omega))' = L_0^2(\Omega), \qquad \langle Bv, q\rangle = b(v, q) = -(\operatorname{div} v, q),$$

$$B' = \operatorname{grad} : L_0^2(\Omega) \mapsto \boldsymbol{H}^{-1}(\Omega), \qquad \langle \operatorname{grad} q, v\rangle = b(v, q) = -(\operatorname{div} v, q).$$

**Remark 1.1.** A natural choice of the pressure space is $L^2(\Omega)$. Note that

$$\int_\Omega \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n} \, \mathrm{d}S = 0$$

due to the boundary condition. Thus div operator will map $\boldsymbol{H}_0^1(\Omega)$ into the subspace $L_0^2(\Omega)$, in which the pressure solving the Stokes equations is unique. In $L^2(\Omega)$, it is unique only up to a constant.

**Remark 1.2.** By the same reason, for Stokes equations with non-homogenous Dirichlet boundary condition $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$, the data $\boldsymbol{g}$ should satisfy the compatible condition

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} \, \mathrm{d}S = \int_{\partial\Omega} \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = 0.$$

Recall that we need to verify the following assumptions

(A)

$$\inf_{u\in\mathbb{Z}} \sup_{v\in\mathbb{Z}} \frac{a(u,v)}{|u|_1|v|_1} = \inf_{v\in\mathbb{Z}} \sup_{u\in\mathbb{Z}} \frac{a(u,v)}{|u|_1|v|_1} = \alpha > 0.$$

(B)

$$\inf_{q\in\mathbb{P}} \sup_{v\in\mathbb{V}} \frac{b(v,q)}{|v|_1\|q\|} = \beta > 0$$

(C)

$$\begin{aligned} a(u,v) &\leq C_a|u|_1|v|_1, \quad \text{for all } u,v\in\mathbb{V},\\ b(v,q) &\leq C_b|v|_1\|q\|, \quad \text{for all } v\in\mathbb{V}, q\in\mathbb{P}. \end{aligned}$$

Conditions (A) and (C) are relatively easy to verify (the readers are encouraged to do that). The key is the inf-sup condition (B) which is equivalent to either

- div : $\boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ is surjective, or
- grad : $L_0^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is injective and bounded below.

We shall verify the inf-sup condition (B) in both ways.

**Lemma 1.3.** *For any $q \in L_0^2(\Omega)$, there exists a $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that*

$$\operatorname{div} \boldsymbol{v} = q, \quad \textit{and} \ \|\boldsymbol{v}\|_1 \lesssim \|q\|_0.$$

*Consequently the inf-sup condition (B) holds.*

*Proof.* We consider a simpler case when $\Omega$ is smooth and in two dimensions. We first solve the Poisson equation with homogenous Neumann boundary condition

$$\Delta\psi = q \ \text{ in } \ \Omega, \quad \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \partial\Omega. \tag{5}$$

The Neumann problem (5) is well posed since $q \in L_0^2(\Omega)$. If we set $\boldsymbol{v} = \nabla\psi$, then $\operatorname{div}\boldsymbol{v} = \Delta\psi = q$, and $\|\boldsymbol{v}\|_1 = \|\psi\|_2 \lesssim \|p\|_0$ by the $H^2$-regularity result of Poisson equation.

The remaining part is to verify the boundary condition. First $\boldsymbol{v}\cdot\boldsymbol{n} = \nabla\psi\cdot\boldsymbol{n} = 0$ by the construction. To take care of the tangential component $\boldsymbol{v}\cdot\boldsymbol{t}$, we invoke the trace theorem for $H^2(\Omega)$ to conclude that: there exist $\phi \in H^2(\Omega)$ such that $\phi|_{\partial\Omega} = 0$ and $\nabla\phi\cdot\boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{t}$ and $\|\phi\|_2 \lesssim \|\boldsymbol{v}\|_1$. Let $\tilde{\boldsymbol{v}} = \operatorname{curl}\phi$. We have

$$\begin{aligned} &\operatorname{div}\tilde{\boldsymbol{v}} = 0,\\ &\tilde{\boldsymbol{v}}\cdot\boldsymbol{n} = \operatorname{curl}\phi\cdot n = \operatorname{grad}\phi\cdot\boldsymbol{t} = 0,\\ \text{and} \ \ &\tilde{\boldsymbol{v}}\cdot\boldsymbol{t} = -\operatorname{grad}\psi\cdot\boldsymbol{n} = -\boldsymbol{v}\cdot\boldsymbol{t}. \end{aligned}$$

Then we set $\boldsymbol{v}_q = \boldsymbol{v} + \tilde{\boldsymbol{v}}$ to obtain the desired result.

If the domain is non-smooth, we can still construct such $\psi$; see [2, 9, 5]. □

If we introduce the Sobolev space

$$\boldsymbol{H}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega), \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)\},$$
$$\boldsymbol{H}_0(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega), \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\},$$

then the first step in the proof of Lemma 1.3 verifies the weaker inf-sup condition $\mathrm{div} : \boldsymbol{H}_0(\mathrm{div}, \Omega) \to L_0^2(\Omega)$ is surjective. Note that $\boldsymbol{H}_0^1(\Omega) \subset \boldsymbol{H}_0(\mathrm{div}, \Omega)$ is strict subspace. Controlling the tangential trace of a vector in $\boldsymbol{H}_0(\mathrm{div}, \Omega)$ is tricky.

**Exercise 1.4.** Prove

$$-\Delta = -\mathrm{grad}\ \mathrm{div} + \mathrm{curl}\,\mathrm{curl}$$

holds as an operator from $\boldsymbol{H}_0^1(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$. Namely for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}) + (\mathrm{curl}\, \boldsymbol{u}, \mathrm{curl}\, \boldsymbol{v}).$$

Therefore

$$\|\, \mathrm{div}\, \boldsymbol{u}\| \le \|\nabla \boldsymbol{u}\| \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega).$$

**Remark 1.5.** Since

$$(\mathrm{div}\, \boldsymbol{v}, q) \le \|\mathrm{div}\, \boldsymbol{v}\| \|q\| \le \|\nabla \boldsymbol{v}\| \|q\|,$$

we have a upper bound on the inf-sup constant

$$\beta = \inf_{q \in \mathbb{P}} \sup_{\boldsymbol{v} \in \mathbb{V}} \frac{(\mathrm{div} \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\| \|q\|} \le 1.$$

We now sketch another approach to prove the operator $\mathrm{grad}$ is injective and bounded below which can formulated as the generalized Poincaré inequality

$$\|\mathrm{grad}\, p\|_{-1} \ge \beta \|p\| \quad \text{for any } p \in L_0^2(\Omega). \tag{6}$$

The natural domain of the gradient operator is $H^1(\Omega)$, i.e., $\mathrm{grad} : H^1(\Omega) \to L^2(\Omega)$. We can continuously extend the domain of the gradient operator from $H^1(\Omega)$ to $L^2(\Omega)$, i.e., $\mathrm{grad} : L^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ and prove the range $\mathrm{grad}\,(L^2)$ is a closed subspace of $\boldsymbol{H}^{-1}$. The difficulty comes from the dual norm is not computable in general. In the following, we present a simple case when an exact formulae on $H^{-1}$-norm is available.

**Theorem 1.6.** *Let $X(\Omega) = \{v \,|\, v \in H^{-1}(\Omega), \mathrm{grad}\, v \in (H^{-1}(\Omega))^n\}$ endowed with the norm $\|v\|_X^2 = \|v\|_{-1}^2 + \|\mathrm{grad}\, v\|_{-1}^2$. Then for Lipschitz domains, $X(\Omega) = L^2(\Omega)$.*

*Proof.* A proof $\|v\|_X \lesssim \|v\|$, consequently $L^2(\Omega) \subseteq X(\Omega)$, is trivial (using the definition of the dual norm). The non-trival part is to prove the inequality

$$\|v\|^2 \lesssim \|v\|_{-1}^2 + \|\mathrm{grad}\, v\|_{-1}^2 = \|v\|_{-1}^2 + \sum_{i=1}^d \|\frac{\partial v}{\partial x_i}\|_{-1}^2. \tag{7}$$

The difficulty is associated to the non-computable dual norm. We only present a special case $\Omega = \mathbb{R}^n$ and refer to [10, 4] for general cases.

We use the characterization of $H^{-1}$ norm using Fourier transform. Let $\hat{u}(\xi) = \mathscr{F}(u)$ be the Fourier transform of $u$. Then

$$\|u\|_{\mathbb{R}^n}^2 = \|\hat{u}\|_{\mathbb{R}^n}^2 = \left\|1/(\sqrt{1+|\xi|^2})\hat{u}\right\|_{\mathbb{R}^n}^2 + \sum_{i=1}^d \left\|\xi_i/(\sqrt{1+|\xi|^2})\hat{u}\right\|_{\mathbb{R}^n}^2 = \|u\|_X^2.$$

For half space $\Omega = \mathbb{R}_+^n$, one needs to extend a functional in $H^{-1}(\Omega)$ to $H^{-1}(\mathbb{R}^n)$ continuously. □

**Exercise 1.7.** Use the fact $L^2$ is compactly embedded into $H^{-1}$ and inequality (7) to prove the Poincaré inequality (6).

**Exercise 1.8.** For Stokes equations, we can solve $\boldsymbol{u} = A^{-1}(f - B'p)$ and substitute into the second equation to get the Schur complement equation

$$BA^{-1}B'p = BA^{-1}f - g. \tag{8}$$

Define a bilinear form on $\mathbb{P} \times \mathbb{P}$ as

$$s(p, q) = \langle A^{-1}B'p, B'q \rangle.$$

Prove the well-posedness of (8) by showing:

- the continuity of $s(\cdot,\cdot)$ on $L_0^2 \times L_0^2$;
- the coercivity $s(p,p) \geq c\|p\|^2$ for any $p \in L_0^2$.
- relate the constants in the continuity and coercivity of $s(\cdot,\cdot)$ to the inf-sup condition of $A$ and $B$.

In summary, we have established the well-posedness of Stokes equations.

**Theorem 1.9.** *For a given $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, there exists a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ to the weak formulation of the Stokes equations* (3)-(4) *and*

$$\|\boldsymbol{u}\|_1 + \|p\| \lesssim \|\boldsymbol{f}\|_{-1}.$$

## 2. Fortin Operators

When considering a discretization of Stokes equations, verification of the discrete inf-sup condition for the bilinear form $a(\cdot,\cdot)$ is relatively easy. Again the difficult part is the verification of the inf-sup condition for the bilinear form $b(\cdot,\cdot)$ or simply called *div-stability* for Stokes equations.

Note that the inf-sup condition (B) in the continuous level implies: for any $q_h \in \mathbb{P}_h$, there exists $v \in \mathbb{V}$ such that $b(v, q_h) \geq \beta \|v\|_{\mathbb{V}} \|q_h\|_{\mathbb{P}}$ and $\|v\| \leq C\|q_h\|$. For the discrete inf-sup condition, we need a $v_h \in \mathbb{V}_h$ satisfying such property. One approach is to use the so-called Fortin operator [11] to get such a $v_h$ from $v$.

**Definition 2.1** (Fortin operator)**.** *A linear operator $\Pi_h : \mathbb{V} \to \mathbb{V}_h$ is called a* Fortin operator *if*

*(1)* $b(\Pi_h v, q_h) = b(v, q_h)$ *for all* $q_h \in \mathbb{P}_h$
*(2)* $\|\Pi_h v\|_{\mathbb{V}} \leq C\|v\|_{\mathbb{V}}$.

Namely the following commutating diagram holds

$$\begin{tikzcd} \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h"] & \mathbb{P} \arrow[d, "Q_h"] \\ \mathbb{V}_h \arrow[r, "\mathrm{div}_h"] & \mathbb{P}_h \end{tikzcd}$$

with a stable projection $\Pi_h$. Condition (1) is relatively easy to meet but the stability in the correct norm $\|\cdot\|_{\mathbb{V}}$ is sometimes tricky.

**Theorem 2.2.** *Assume the continuous inf-sup condition (B) holds and there exists a Fortin operator $\Pi_h$, then the discrete inf-sup condition ($B_h$) holds.*

*Proof.* The inf-sup condition (B) in the continuous level implies: for any $q_h \in \mathbb{P}_h$, there exists $v \in \mathbb{V}$ such that $b(v, q_h) \geq \beta \|v\|_{\mathbb{V}} \|q_h\|$ and $\|v\|_{\mathbb{V}} \leq C\|q_h\|$. We choose $v_h = \Pi_h v$.

By the definition of Fortin operator

$$b(v_h, q_h) = b(v, q_h) \geq \beta \|v\|_{\mathbb{V}} \|q_h\|_{\mathbb{P}} \geq \beta C \|v_h\|_{\mathbb{V}} \|q_h\|_{\mathbb{P}}.$$

The discrete inf-sup condition then follows. □

In the application to Stokes equations, $\mathbb{P} = L_0^2(\Omega)$ endowed with $L^2$-norm $\|\cdot\|$ and $\mathbb{V} = \boldsymbol{H}_0^1(\Omega)$ with norm $|v|_1 := \|\nabla v\|$. In the definition of Fortin operator, we require the operator is stable in $|\cdot|_1$-norm and call it the $H^1$-stability of the operator $\Pi_h$.

When velocity spaces containing the linear finite element space, it suffices to construct a Fortin operator stable in a weaker norm. Let us define a mesh dependent norm

$$\|v\|_h = \|v\| + h|v|_1.$$

For $v \in \mathbb{V}_h$, by the inverse inequality $\|v\|_h \eqsim \|v\|$. The idea is to apply a weaker stable Fortin operator to the high frequency part. For high frequency functions, a weaker stability will imply the stronger $H^1$ stability.

**Theorem 2.3.** *Suppose the velocity space $\mathbb{V}_h$ contains the piecewise linear and continuous function space. Suppose there exists a Fortin operator $\Pi_B : \boldsymbol{H}_0^1(\Omega) \to \mathbb{V}_h$ and stable in $\|\cdot\|_h$ norm which is equivalent to*

$$\|\Pi_B u\| \lesssim \|u\| + h|u|_1, \quad \text{for all } u \in \boldsymbol{H}_0^1(\Omega), \tag{9}$$

*then there exists a Fortin operator $\Pi_h : \boldsymbol{H}_0^1(\Omega) \to \mathbb{V}_h$ and stable in $H^1$ norm.*

*Proof.* Let $\mathcal{P}_1 \subset \boldsymbol{H}_0^1(\Omega)$ be the linear finite element space, and let $\Pi_1 : \boldsymbol{H}_0^1(\Omega) \to \mathcal{P}_1$ be a quasi-interpolation which satisfies

$$|\Pi_1 u|_1 + h^{-1}\|u - \Pi_1 u\| \lesssim |u|_1. \tag{10}$$

See *Convergence Theory of Adaptive Finite Element Methods* for such a quasi-interpolation. We define the Fortin operator as

$$\Pi_h u = \Pi_1 u + \Pi_B(u - \Pi_1 u).$$

Then $b(u - \Pi_h u, q_h) = 0$ for all $q_h \in \mathbb{P}_h$ by definition.

Next we prove the $H^1$-stability of $\Pi_h$. By the triangle inequality, inverse inequality, stability of $\Pi_B$, and the property (10) of $\Pi_1$, we get the desired inequality

$$|\Pi_h u|_1 \leq |\Pi_1 u|_1 + |\Pi_B(u - \Pi_1 u)|_1 \lesssim |\Pi_1 u|_1 + h^{-1}\|\Pi_B(u - \Pi_1 u)\| \lesssim |u|_1.$$

□

## 3. Finite Element Spaces for Stokes Equations

Given a triangulation $\mathcal{T}$ of the domain $\Omega$, we shall use the following piecewise polynomial spaces

$$\mathcal{P}_k(\mathcal{T}) = \{v \in C(\Omega) : v|_\tau \in \mathcal{P}_k, \text{for all } \tau \in \mathcal{T}\}, \quad \text{for } k \geq 1$$
$$\mathcal{P}_k^{-1}(\mathcal{T}) = \{v \in L^2(\Omega) : v|_\tau \in \mathcal{P}_k, \text{for all } \tau \in \mathcal{T}\}, \quad \text{for } k \geq 0.$$

Here the superscript $^{-1}$ means the space is discontinuous. Finite element spaces will be chosen as $\mathbb{V}_h = (\mathcal{P}_k(\mathcal{T}))^n \cap \boldsymbol{H}_0^1(\Omega)$ and $\mathbb{P}_h = \mathcal{P}_l(\mathcal{T}) \cap L_0^2(\Omega)$ or $P_l^{-1}(\mathcal{T}) \cap L_0^2(\Omega)$ for careful chosen integers $k$ and $l$. To simplify the notation, we simply write the space as $(\mathcal{P}_k, \mathcal{P}_l^{-1})$ for discontinuous pressure spaces or $(\mathcal{P}_k, \mathcal{P}_l)$ for continuous pressure spaces.

Here is a list of stable pairs for Stokes equations with brief comments.

- $(\mathcal{P}_2, \mathcal{P}_0)$: A simple element with element-wise mass conservation.
- $(\mathcal{P}_1^{\rm CR}, \mathcal{P}_0)$: A simple element with element-wise divergence free property. Velocity is linear but non-conforming.
- $(\mathcal{P}_{1,h/2}, \mathcal{P}_{0,h})$ and $(\mathcal{P}_{1,h/2}, \mathcal{P}_{1,h}^0)$: Linear velocity in the refined mesh. Easy to code using linear elements.
- $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ Scott-Vogelius element: stable if $k \geq 4$ in $\mathbb{R}^2$ and for meshes without singular-vertex. Exact divergence free. Not easy to code due to the high degree.
- $(\mathcal{P}_k, \mathcal{P}_{k-1})$ Taylor-Hood element $k \geq 2$: Optimal convergent rate.
- $(\mathcal{P}_1 + \mathcal{B}_3, \mathcal{P}_1)$ Mini element: Most economic element.
- $(\mathcal{P}_k + \mathcal{B}_{k+1}, \mathcal{P}_{k-1}^{-1}), k \geq 2$: stabilization using bubble functions. Discontinuous pressure space.

Before we discuss these pairs in detail, we emphasize several considerations when design stable finite element pairs.

- Since the inf-sup condition for Stokes equations holds in the continuous level, for a fixed pressure space, the velocity space can be enlarged to get the discrete inf-sup condition. The enlargement can be done by either increasing the polynomial order or refining the mesh.
- The equation $\operatorname{div} \boldsymbol{u}_h = 0$ holds in a weak topology and in general $\operatorname{div} \boldsymbol{u}_h \neq 0$ point-wise. To enforce $\operatorname{div} \boldsymbol{u}_h = 0$ pointwise, it is better to use $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ since $\operatorname{div} \mathcal{P}_k \subseteq \mathcal{P}_{k-1}^{-1}$. But inf-sup condition will be an issue.
- Due to the coupling of $\boldsymbol{u}_h$ and $p_h$, it is efficient to equilibrate the rates of convergence. Note the error measured in $H^1$ norm is usually one order lower than that in $L^2$ norm. To balance the approximation order, it is better to use $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ or $(\mathcal{P}_k, \mathcal{P}_{k-1})$.
- The trade-off between the increased accuracy of high-order elements and the increased complexity should be also taken into account. Piecewise linear or constant function spaces will be much easier to code in practice.
- We shall construct Fortin operator approach to verify the div stability. This approach is relatively simple but has its own limitation. There are other methods to verify the inf-sup condition for Stokes equations: Verfürth [15], Boland and Nicolaides [3], and Stenberg [14].

3.1. $(\mathcal{P}_1, \mathcal{P}_0)$**.** The simplest and most straightforward pair is $(\mathcal{P}_1, \mathcal{P}_0)$, i.e., piecewise linear and continuous space for velocity and piecewise constant space for pressure. The continuity of the velocity space is due to the requirement $\mathbb{V}_h \subset \boldsymbol{H}_0^1(\Omega)$. Recall that a piecewise smooth function to be in $H^1(\Omega)$ is equivalent to be globally continuous. The space for pressure is not necessary continuous since only $L^2$ integrable is required.

Unfortunately this simple pair is not suitable for the Stokes equations. The discrete inf-sup condition cannot be true. The rectangular matrix representation $B$ of the divergence operator is of dimension $NT \times 2N$, where $N$ is the number of interior nodes and $NT$ is the number of triangles. Counting the angles nodal-wise and element-wise, we obtain the inequality $2\pi N < \pi NT$. Note that the inf-sup condition for $B$ implies $B$ is surjective. So $\operatorname{rank}(B) = NT$ which is impossible since $2N < NT$.

In other words, the discrete gradient operator $B'$ contains kernel more than a global constant function. For the stable pair, $B'p = 0$ implies $p$ =constant. For $(\mathcal{P}_1, \mathcal{P}_0)$ pair, there exists non-constant pressure $p$ s.t. $B'p = 0$ which is called spurious pressure modes.

One way to stabilize the $(\mathcal{P}_1, \mathcal{P}_0)$ pair is to remove those spurious pressure modes if they can be identified. This process is highly mesh dependent.

3.2. $(\mathcal{P}_2, \mathcal{P}_0)$**.** We enlarge the space of velocity to quadratic polynomials to get a stable pair. We prove the discrete inf-sup condition by constructing a Fortin operator. Apply the integration by parts element by element, we obtain

$$\sum_{\tau\in\mathcal{T}}\int_\tau \operatorname{div}(\boldsymbol{v}-\Pi_B\boldsymbol{v})q_h = \sum_{\tau\in\mathcal{T}}\int_{\partial\tau}(\boldsymbol{v}-\Pi_B\boldsymbol{v})\cdot\boldsymbol{n}\, q_h.$$

Since $q_h$ is piecewise constant, it is sufficient to construct a stable operator $\Pi_h\boldsymbol{v}$

$$\int_e \boldsymbol{v}\,\mathrm{d}s = \int_e \Pi_B\boldsymbol{v}\,\mathrm{d}s \quad \text{for all edges } e \text{ of } \mathcal{T}_h, \tag{11}$$

and verify the stability $\|\Pi_B\boldsymbol{v}\| \leq \|\boldsymbol{v}\|_h$.

Let us write $\mathcal{P}_2 = \mathcal{P}_1 \bigoplus \mathcal{B}_E$, where $\mathcal{B}_E$ is the quadratic bubble functions associated to edges. Then (11) is indeed define a function in $\mathcal{B}_E$. More specifically, let $e$ be an edge with vertices $v_i, v_j$. Denoted by $b_e = 6\phi_i\phi_j/|e|$ where $\phi_i$ is standard hat basis for $\mathcal{P}_1$. By Simpson rule, the integral $\int_e b_e = 1$. Then the operator

$$\Pi_B\boldsymbol{v} := \sum_{e\in E}\left(\int_e \boldsymbol{v}\ \mathrm{d}s\right) b_e$$

satisfies (11). Now we check the stability. For bubble function spaces, since $b_e$ are finite overlapping,

$$\|\Pi_B\boldsymbol{v}\|^2 \lesssim \sum_{e\in E}\left(\int_e \boldsymbol{v}\ \mathrm{d}t\right)^2 \|b_e\|^2 \lesssim \sum_T\left(\int_T |\boldsymbol{v}|^2 + h^2|\nabla\boldsymbol{v}|^2\,\mathrm{d}x\right) = \|\boldsymbol{v}\|^2 + h^2\|\nabla\boldsymbol{v}\|^2.$$

In the second step, we have used Cauchy-Schwarz inequality and the scaled trace theorem: for any function $g \in H^1(T)$

$$\|g\|_e^2 \leq C\left(h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2\right). \tag{12}$$

The drawback of this stable pair is that:

- $\mathbb{Z}^h = \ker(\operatorname{div}_h) \not\subset \mathbb{Z} = \ker(\operatorname{div})$ since $\operatorname{div}\mathcal{P}_2 \subset \mathcal{P}_1^{-1}$ contains more than piecewise constant functions. The velocity approximation $u_h$ is thus not pointwise divergence free. Nevertheless the mass conservation holds element-wise as $\int_{\partial T}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_T \operatorname{div}\boldsymbol{u}\,\mathrm{d}x = 0$ by choosing the characteristic function of $T$.
- the approximation is only first order since $\|p - p_h\| \leq Ch$ although the velocity space could provide one order higher approximation.

**Remark 3.1.** To gain the stability, for an edge, only one edge bubble function $\boldsymbol{n}_e b_e$ is needed. In 3-D, adding one face bubble function in the normal direction is enough.

3.3. $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$**.** Scott and Vogelius [12] showed that the inf-sup condition holds for $(\mathcal{P}_k, \mathcal{P}_{k-1}^{-1})$ pairs in 2D if $k \geq 4$ provided the meshes are singular-vertex free. An internal vertex in 2D is said to be singular if edges meeting at the point fall into two straight lines. Note that one can perturb the singular vertex to easily get singular-vertex free triangulations. The stability of this type of pair in 3D is not clear and partial results can be found in [16].

The relation $\operatorname{div}\mathcal{P}_k \subseteq \mathcal{P}_{k-1}^{-1}$ implies that the pointwise divergence free for the approximated velocity $\boldsymbol{u}_h$ which is a desirable property (since the conservation of mass everywhere.) The convergent rate is optimal

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\| \lesssim h^k, \tag{13}$$

provided the solution $(\boldsymbol{u}, p)$ are smooth enough, say $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega), p \in H^k(\Omega)$ which is not likely to hold in practice.

The only drawback is the complication of programming. There are a lot of unknowns for high order polynomials for vector functions and for discontinuous polynomials. For example, for one triangle, the lowest order element $(\mathcal{P}_4, \mathcal{P}_3^{-1})$ contains 30 d.o.f for velocity and 10 for pressure. Globally the dimension of the velocity space is $2(N+3NE+3NT) \approx 32N$ and the dimension of the pressure space is $10NT \approx 20N$.

3.4. $(\mathcal{P}_k, \mathcal{P}_{k-1})$**.** If we use a continuous space for the pressure, then the degree of freedom for the pressure can be saved a lot. For example, the dimension of $\mathcal{P}_1^{-1}$ is $3NT$ which is almost 6 times larger than $N$, the dimension of $\mathcal{P}_1$.

Going from a discontinuous space to a continuous one, the dimension of pressure space is reduced. Then it is optimistic that the velocity space might become large enough to have the div-stability. Indeed one can show the pair $(\mathcal{P}_k, \mathcal{P}_{k-1})$ for $k \geq 2$ satisfy the div stability. This is known as Taylor-Hood (or Hood-Taylor) elements. Proof of the div stability for Taylor-Hood element is delicate. We shall skip it here and refer to [7] for a relatively simple proof on $(\mathcal{P}_2, \mathcal{P}_1)$ pair.

For Taylor-Hood elements, we still maintain the optimal convergent order; see (13). The pair is stable for $k \geq 2$. The simplest case $k = 2$ (not $k = 1$ since $(\mathcal{P}_1, \mathcal{P}_0)$ is unstable), $(\mathcal{P}_2, \mathcal{P}_1)$ is very popular. It uses less degree of freedom than the stable pair $(\mathcal{P}_2, \mathcal{P}_0)$ but provide one order higher approximation.

The drawback of Hood-Taylor elements is: First it is still not point-wise divergence free. Second since continuous pressure space is used, there is no element-wise mass conservation. A simple fix of the latter issue is adding the piecewise constant into the pressure space, i.e., $(\mathcal{P}_k, \mathcal{P}_{k-1} + \mathcal{P}_0)$. The div stability of the modified Hood-Taylor elements can be found in [7].

3.5. $(\mathcal{P}_1 \bigoplus \mathcal{B}_{\mathcal{T}}, \mathcal{P}_1)$**.** Start from Taylor-Hood element $(\mathcal{P}_2, \mathcal{P}_1)$, we can further reduce the degree of freedom of velocity space to get a stable pair. One well known element is the so-called mini-element developed by Arnold, Brezzi, and Fortin [1].

The idea is adding element-wise bubble functions to the velocity space

$$\mathcal{B}_{\mathcal{T}} = \bigoplus_{\tau\in\mathcal{T}} \mathcal{B}_\tau, \quad \mathcal{B}_\tau = \operatorname{span}\{\lambda_1\lambda_2\lambda_3\},$$

to stabilize the unstable pair $(\mathcal{P}_1, \mathcal{P}_1)$.

To construct a Fortin operator $\Pi_h$, note that now the pressure is continuous, we have

$$\sum_{\tau\in\mathcal{T}} \int_\tau \operatorname{div}(\boldsymbol{v} - \Pi_B \boldsymbol{v}) q_h \,\mathrm{d}\boldsymbol{x} = -\sum_{\tau\in\mathcal{T}} \int_\tau (\boldsymbol{v} - \Pi_B \boldsymbol{v}) \cdot \nabla q_h \,\mathrm{d}\boldsymbol{x}.$$

Since $\nabla q_h$ is constant, it suffices to get a stable operator such that $\int_\tau \boldsymbol{v}\,\mathrm{d}x = \int_\tau \Pi_B \boldsymbol{v}\,\mathrm{d}x$ for all $\tau \in \mathcal{T}$. The element-wise bubble functions are introduced for this purpose. Let us define $\Pi_B v \in \mathcal{B}_{\mathcal{T}}$ by

$$\int_\tau \Pi_B v \,\mathrm{d}x = \int_\tau v \,\mathrm{d}x, \quad \text{for all } \tau \in \mathcal{T}.$$

It is trivial to show that $\Pi_B$ is stable in $L^2$ norm and thus a $H^1$-stable Fortin operator can be constructed using Theorem 2.3.

3.6. $(\mathcal{P}_1^{\rm CR}, \mathcal{P}_0)$**.** An easy fix of the div-stability is through the sacrifices of conformity of the velocity space. From the proof of the stability of $(\mathcal{P}_2, \mathcal{P}_0)$, the degree of freedom on edges is important. We then introduce the following piecewise linear finite element space

$$\mathcal{P}_1^{\rm CR} = \{v \in L^2(\Omega), v|_\tau \in \mathcal{P}_1(\tau), \int_e v\, {\rm d}s \text{ is continuous for all } e\}.$$

The superscript $^{\rm CR}$ is named after Crouzeix and Raviart who introduced this space in [8]. To impose the boundary condition, one can require $\int_e v\, {\rm d}s = 0$ for $e \in \partial\Omega$. That is the boundary condition is not imposed pointwise but in a weak sense. One can easily show functions in $\mathcal{P}_1^{\rm CR}$ is continuous at middle points of edges but not on vertices and thus $\mathcal{P}_1^{\rm CR} \not\subset H^1(\Omega)$.

Follow the proof of the stability of $(\mathcal{P}_2, \mathcal{P}_0)$, one can also prove the inf-sup stability of $(\mathcal{P}_1^{\rm CR}, \mathcal{P}_0^{-1})$. Note that although $\mathcal{P}_1^{\rm CR}$ contains the piecewise linear polynomial, the CR space is not $H^1$-nonconforming and the approach in Section 2 should be modified slightly.

This is probably the simplest stable element for Stokes equations and the velocity is element-wise divergence free. The sacrifice is that $\mathcal{P}_1^{\rm CR} \not\subset H^1(\Omega)$. One needs to show the violation is controlled by estimating the consistency error.

3.7. $(\mathcal{P}_{1,h/2}, \mathcal{P}_{0,h})$ $(\mathcal{P}_{1,h/2}, \mathcal{P}_{1,h})$**.** Another way to enrich the velocity space is through the mesh refinement. We denoted by $\mathcal{T}_{h/2}$ a fine triangulation obtained by regular uniform refinement of $\mathcal{T}_h$, i.e., each triangle in $\mathcal{T}_h$ is divided into 4 similar triangles by connecting middle points of edges. $\mathcal{P}_{1,h/2}$ is piecewise linear and continuous finite element space on $\mathcal{T}_{h/2}$. Comparing with $\mathcal{P}_{1,h}$, new degree of freedoms are created on edges. Then $\mathcal{P}_{1,h/2}$ can be used to replace $\mathcal{P}_2$ in the stable pair $(\mathcal{P}_2, \mathcal{P}_0)$ and $(\mathcal{P}_2, \mathcal{P}_1)$. The benefit of replacing a better approximation space $\mathcal{P}_2$ by a less accurate one $\mathcal{P}_{1,h/2}$ is the simplificity of programming of linear element.

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