HOMEWORK 3 OF MATH 226 A: FALL 2011

LONG CHEN

1. Error Estimate with Numerical Quadrature

We shall consider the effect of the numerical quadrature to the convergence rate of linear finite element method for solving Poisson equation.

The weak formulation of Poisson equation with homogenous Dirichlet boundary condition is: given a $f \in H^{-1}(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle f, v \rangle, \qquad \forall v \in H_0^1(\Omega).$$

Given a triangulation $\mathcal{T}_h$, let $\mathbb{V}_h$ be the linear finite element space based on $\mathcal{T}_h$. The linear finite element approximation to Poisson equation with numerical quadrature is to find $u_h \in \mathbb{V}_h \cap H_0^1(\Omega)$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle_h, \qquad \forall v_h \in \mathbb{V}_h \cap H_0^1(\Omega),$$

where $\langle f, v_h \rangle_h$ is an approximation of $\langle f, v \rangle$.

Recall the Strang's first lemma

$$(1) \qquad |u - u_h|_{1,\Omega} \leq C \left(\inf_{v_h \in \mathbb{V}_h \cap H_0^1(\Omega)} |u - v_h|_{1,\Omega} + \sup_{w_h \in \mathbb{V}_h} \frac{|\langle f, w_h \rangle - \langle f, w_h \rangle_h|}{|w_h|_{1,\Omega}} \right).$$

We denote the error function of numerical quadrature in one element as

$$E_\tau(g) = \int_\tau g(x)\, dx - \sum_{i=1}^{k} \omega_i g(p_i).$$

The numerical quadrature is of order k, if $E(g) = 0$ for any $g \in \mathcal{P}_k(\tau)$, where $\mathcal{P}_k(\tau)$ is the polynomial space in τ with degree k. Prove the following theorem.

Theorem 1.1. *Suppose the numerical quadrature is of order* 0*, i.e., it is exact for constant function. Then for any* $f \in W^{1,q}(\tau), v \in \mathcal{P}_1(\tau)$ *with* $1 - n/q > 0$*, we have*

$$|E_\tau(fv)| \leq c h_\tau |\tau|^{1/2-1/q} \|f\|_{1,q,\tau} \|v\|_{1,\tau}.$$

Hint: the requirement $1 - n/q > 0$ is to ensure $W^{1,q}(\tau)$ is embedded into $C(\tau)$ such that the point value $f(p_i)$ make sense.

Using the Theorem 1.1 and 1.2, prove the error estimate with numerical quadrature.

Theorem 1.2. *Suppose the solution of Poisson equation* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *and the right hand side* $f \in W^{1,q}(\Omega)$*. Furthermore the numerical quadrature scheme is of order* 0*. Then we have optimal convergent rate*

$$|u - u_h|_{1,\Omega} \leq h \left(|u|_{2,\Omega} + \|f\|_{1,q,\Omega} \right).$$

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2. LOWER BOUND OF THE RESIDUAL TYPE A POSTERIORI ERROR ESTIMATE

Let u be the solution of Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary condition and u_h be the linear finite element approximation of u based on a shape regular and conforming triangulation $\mathcal{T}_h$.

(1) For a triangle τ, we denote $V_\tau = \{f_\tau \in L_2(\tau) \mid f_\tau = \text{constant}\}$ equipped with L^2 inner product. Let $\lambda_i(x), i = 1, 2, 3$ be the barycenter coordinates of $x \in \tau$, and let $b_\tau = \lambda_1\lambda_2\lambda_3$ be the bubble function on τ. We define $B_\tau f_\tau = f_\tau b_\tau$.

Prove that $B_\tau : V_\tau \mapsto V = H_0^1(\Omega)$ is bounded in L^2 and H^1 norm:

$$\|B_\tau f_\tau\|_{0,\tau} = C\|f_\tau\|_{0,\tau}, \text{ and } \|\nabla(B_\tau f_\tau)\|_{0,\tau} \lesssim h_\tau^{-1}\|f_\tau\|_{0,\tau}.$$

(2) Using (1) to prove that

$$\|hf_\tau\|_{0,\tau} \lesssim |u - u_h|_{1,\tau} + \|h(f - f_\tau)\|_{0,\tau}.$$

(3) For an interior edge e, we define $V_e = \{g_e \in L^2(E) \mid g_e = \text{constant}\}$. Suppose e has end points x_i, and x_j, we define $b_e = \lambda_i\lambda_j$ and $B_e : V_e \mapsto V$ by $B_e g_e = g_e b_e$.

Let ω_e denote two triangles sharing e. Prove that

(a) $\|g_e\|_{0,e} = C\|B_e g_e\|_{0,e}$,

(b) $\|B_e g_e\|_{0,\omega_e} \lesssim h_e^{1/2}\|g_e\|_{0,e}$ and,

(c) $\|\nabla(B_e g_e)\|_{0,\omega_e} \lesssim h_e^{-1/2}\|g_e\|_{0,e}$.

(4) Using (3) to prove that

$$\|h^{1/2}[\nabla u_h \cdot n_e]\|_{0,e} \lesssim \|hf\|_{0,\omega_e} + |u - u_h|_{1,\omega_e}.$$

(5) Using (1) and (4) to prove the lower bound of the error estimator. There exists a constant C_2 depending only on the shape regularity of the triangulation such that for any piecewise constant approximation f_τ of $f \in L^2$,

$$C_2\eta^2(u_h, \mathcal{T}_h) \le |u - u_h|_{1,\Omega}^2 + \sum_{\tau\in\mathcal{T}_h} \|h(f - f_\tau)\|_\tau^2.$$