HOMEWORK 1 OF MATH 226 B, WINTER 2009

(1) Prove the following properties of the matrix A formed in the finite difference methods for Poisson equation with Dirichlet boundary condition:
   (a) it is symmetric: $a_{ij} = a_{ji}$;
   (b) it is diagonally dominant: $a_{ii} \geq \sum_{j=1, j\neq i}^{N} a_{ij}$;
   (c) it is positive definite: $u^t A u \geq 0$ for any $u \in \mathbb{R}^N$ and $u^t A u = 0$ if and only if $u = 0$.

(2) Let us consider the finite difference discritization of Poisson equation with Neumann boundary condition.
   (a) Write out the 9×9 matrix A for $h = 1/2$.
   (b) Prove that in general the matrix corresponding to Neumann boundary condition is only semi-positive definite.
   (c) Show that the kernel of A consists of constant vectors: $Au = 0$ if and only if $u = c$.

(3) Given a triangulation, we choose barycenter of triangles to construct the dual mesh. Let u_h^{FE} and u_h^{FV} be the approximation of Poisson equation $-\Delta u = f$ using linear finite element and linear finite volume methods, respectively. Prove that if $f \in H^1(\Omega)$, then

$$|u_h^{FE} - u_h^{FV}|_1 \leq Ch^2 \|f\|_1.$$

Note that both $|u - u_h^{FE}|_1$ and $|u - u_h^{FV}|_1$ are only first order. The above estimate says that u_h^{FE} and u_h^{FV} are superclose.