MULTIGRID FOR H(CURL) AND H(DIV) PROBLEMS

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Date: August 6, 2011.

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1. Introduction

In this lecture, we give a brief introduction on the multigrid methods and multilevel preconditioners for solving finite element discretization of $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ systems

$$\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } \Omega, \tag{1}$$

$$-\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2}$$

with homogeneous Dirichlet or Newumann boundary condition. Here $\Omega \subset \mathbb{R}^3$ is homotopy equivalent to a ball. Finite element discretization of (1) and (2) is to restrict the variational forms to appropriate edge or face element spaces based on a a shape regular tetrahedra triangulation $\mathcal{T}$ of Ω.

Standard multigrid methods developed for H^1 system i.e.

$$-\Delta u + u = -\,\mathrm{div}\,\mathrm{grad}\,u + u = f$$

cannot be transfered to the $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ systems directly. The reason is that for vector fields, the operator $\mathrm{curl}\,\mathrm{curl}$ and $-\,\mathrm{grad}\,\mathrm{div}$ is only half of the Laplace operator. Indeed we have the identity

$$-\boldsymbol{\Delta} = \mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}\,.$$

Therefore in the divergence free space, $\mathrm{curl}\,\mathrm{curl} + \boldsymbol{I}$ behaves like $-\boldsymbol{\Delta} + \boldsymbol{I}$, while in the kernel space of curl operator it is like $\boldsymbol{I}$. Considering the fact $\ker(\mathrm{curl}) = \mathrm{range}(\mathrm{grad})$, the differential operator on the null space is the scalar Laplacian $-\Delta$. Similar fact holds for $-\,\mathrm{grad}\,\mathrm{div} + \boldsymbol{I}$ operators. The efficient solvers should take care of the different behaviors in these two subspaces. Especially we need to include the smoother in the kernel space of curl or div, respectively. We note that for grad operator, the kernel space is a one dimensional (constant) space, while for curl and div operator, the kernel space is very big.

The smoothers in corresponding multigrid methods should satisfy certain properties to take care of the nontrivial null spaces. One approach is to perform a smoothing in the kernel space which can be expressed explicitly using exact sequences property between finite element spaces of $H^1, \boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ systems. This is used by Hiptmair to obtain the first results for multigrid of $\boldsymbol{H}(\mathrm{div})$ [14] and $\boldsymbol{H}(\mathrm{curl})$ [16] systems in three dimensions. See also Hiptmair and Toselli [18] for a unified and simplified treatment. Another important approach taken by Arnold, Falk and Winther in [2, 3] is to perform the smoothing on patches of vertex which contains a basis of the kernel space of curl and div operator.

The above methods can be classified as geometric multigrid methods for H(curl) and H(div) problems. To develop algebraic multigrid methods, we shall introduce the H-X preconditioner using the auxiliary space method. Then we can use well-developed AMG for H^1 systems for the Poisson solver to obtain robust AMG methods for $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ systems.

To isolate and focus on the essential difficulties, we do not include variable coefficients for (1) or (2). Most methods discussed in this lecture are robust to the coefficients although theoretical justification is only given for some special cases.

We shall use notation $x \lesssim y$ to stand for $x \leq Cy$. We also use $x \eqsim y$ to mean $x \lesssim y$ and $y \lesssim x$. All constants hidden in this notation are independent of problem size N and functions $v \in \mathbb{V}$.

2. Finite Element Spaces

In this section, we shall introduce the $H(\mathrm{grad})$, $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ Sobolev spaces and their finite element spaces. We also survey several improtant properties for later usage: exact sequences, interpolation operators, and commuting diagram.

2.1. **Sobolev spaces.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which is homomorphism to a ball. We define the following Sobolev spaces

$$
\begin{aligned}
H(\mathrm{grad};\Omega) &= \{v \in L^2(\Omega) : \mathrm{grad}\, v \in \boldsymbol{L}^2(\Omega)\},\\
\boldsymbol{H}(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},\\
\boldsymbol{H}(\mathrm{div};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},\\
\text{and}\quad \boldsymbol{H}^1(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \nabla \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}.
\end{aligned}
$$

Through out this paper we use boldface letter to denote vectors and vector spaces. We shall use a generic notation $\boldsymbol{H}(\mathcal{D},\Omega)$ to refer to $\boldsymbol{H}(\mathrm{curl};\Omega)$ or $\boldsymbol{H}(\mathrm{div};\Omega)$, where $\mathcal{D} = \mathrm{curl}$ or div represents differential operators according to the context. For a vector function $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), v_3(\boldsymbol{x}))$, $\nabla \boldsymbol{v} := (\nabla v_1, \nabla v_2, \nabla v_3)$. Since $\mathrm{curl}\, \boldsymbol{v}$ and $\mathrm{div}\, \boldsymbol{v}$ are special combinations of components of $\nabla \boldsymbol{v}$, in general $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{H}(\mathcal{D},\Omega)$. We also note that $H(\mathrm{grad};\Omega) = H^1(\Omega)$.

Let $(\cdot,\cdot)$ denote the inner product for $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$. As subspaces of $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$, $H(\mathrm{grad};\Omega)$, $\boldsymbol{H}(\mathrm{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$ are endowed with $(\cdot,\cdot)$ as their default inner product. We assign new inner products using differential operator D to these spaces.

$$
\begin{aligned}
H(\mathrm{grad};\Omega):&\quad (u,v)_{A^g} := (u,v) + (\mathrm{grad}\, u, \mathrm{grad}\, v),\\
\boldsymbol{H}(\mathrm{curl};\Omega):&\quad (\boldsymbol{u},\boldsymbol{v})_{A^c} := (\boldsymbol{u},\boldsymbol{v}) + (\mathrm{curl}\, \boldsymbol{u}, \mathrm{curl}\, \boldsymbol{v}),\\
\boldsymbol{H}(\mathrm{div};\Omega):&\quad (\boldsymbol{u},\boldsymbol{v})_{A^d} := (\boldsymbol{u},\boldsymbol{v}) + (\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}).
\end{aligned}
$$

The corresponding norm will be denoted by $\|\cdot\|_{A^g}$, $\|\cdot\|_{A^c}$ and $\|\cdot\|_{A^d}$, respectively.

These inner products introduce corresponding symmetric positive definite (SPD) operators (with respect to the default $(\cdot,\cdot)$ inner product).

$$
\begin{aligned}
A^g : H(\mathrm{grad};\Omega) \to L^2(\Omega) &\qquad (A^g\, u, v) := (u,v)_{A^g},\\
A^c : \boldsymbol{H}(\mathrm{curl};\Omega) \to \boldsymbol{L}^2(\Omega) &\qquad (A^c\, u, v) := (u,v)_{A^c},\\
A^d : \boldsymbol{H}(\mathrm{div};\Omega) \to \boldsymbol{L}^2(\Omega) &\qquad (A^d\, u, v) := (u,v)_{A^d}.
\end{aligned}
$$

We shall mainly focus on the $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ systems, namely the operator equations for a given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$:

$$A^c \boldsymbol{u} = \boldsymbol{f}, \tag{3}$$

$$A^d \boldsymbol{u} = \boldsymbol{f}. \tag{4}$$

These are the operator formulation of problems

$$
\begin{aligned}
\mathrm{curl}\,\mathrm{curl}\, \boldsymbol{u} + \boldsymbol{u} &= \boldsymbol{f},\\
-\,\mathrm{grad}\,\mathrm{div}\, \boldsymbol{u} + \boldsymbol{u} &= \boldsymbol{f},
\end{aligned}
$$

with homogeneous Neumann boundary condition.

To deal with the Dirichlet boundary condition, we introduce the space

$$\begin{aligned} H_0^1(\Omega) &= \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}, \\ \boldsymbol{H}_0(\mathrm{div};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}|_{\partial\Omega} \cdot n = 0\}, \\ \boldsymbol{H}_0(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega) : \boldsymbol{v}|_{\partial\Omega} \times n = 0\}, \\ L_0^2(\Omega) &= \{v \in L^2(\Omega) : \int_\Omega v = 0\}, \end{aligned}$$

where $\boldsymbol{n}$ is the outwards normal of $\partial\Omega$.

2.2. **Finite element spaces.** Given a shape regular triangulation $\mathcal{T}$ of Ω and integer $k \geq 1$, we define the following finite element spaces:

$$\begin{aligned} \mathbb{V}(\mathrm{grad}, \mathcal{P}_k, \mathcal{T}) &:= \{v \in H(\mathrm{grad};\Omega) : v|_\tau \in \mathcal{P}_k(\tau),\ \forall \tau \in \mathcal{T}\}, \\ \mathbb{V}(\mathrm{curl}, \mathcal{P}_k^-, \mathcal{T}) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega) : \boldsymbol{v}|_\tau \in \boldsymbol{P}_{k-1}(\tau) + \boldsymbol{P}_{k-1} \times \boldsymbol{x},\ \forall \tau \in \mathcal{T}\}, \\ \mathbb{V}(\mathrm{curl}, \mathcal{P}_k, \mathcal{T}) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl};\Omega) : \boldsymbol{v}|_\tau \in \boldsymbol{P}_k(\tau),\ \forall \tau \in \mathcal{T}\}, \\ \mathbb{V}(\mathrm{div}, \mathcal{P}_k^-, \mathcal{T}) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}|_\tau \in \boldsymbol{P}_{k-1}(\tau) + \mathcal{P}_{k-1}(\tau)\boldsymbol{x},\ \forall \tau \in \mathcal{T}\} \\ \mathbb{V}(\mathrm{div}, \mathcal{P}_k, \mathcal{T}) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}|_\tau \in \boldsymbol{P}_k(\tau),\ \forall \tau \in \mathcal{T}\} \\ \mathbb{V}(L^2, \mathcal{P}_{k-1}, \mathcal{T}) &:= \{v \in L^2(\Omega) : v|_\tau \in \mathcal{P}_{k-1}(\tau)\}. \end{aligned}$$

We follow [4] to use notation $\mathcal{P}_k^-$ to indicate the local polynomial space is incomplete. When we do not emphasis the type of finite element spaces, we shall use a generic notation $\mathbb{V}(\mathcal{D}, \mathcal{T})$ or more precisely $\mathbb{V}(\mathrm{grad}, \mathcal{T})$, $\mathbb{V}(\mathrm{curl}, \mathcal{T})$, $\mathbb{V}(\mathrm{div}, \mathcal{T})$ and $\mathbb{V}(L^2, \mathcal{T})$ to denote these finite element spaces. In particular we simply denote by $\mathbb{V} = \mathbb{V}(\mathrm{grad}, \mathcal{P}_1, \mathcal{T})$, the continuous piecewise linear finite element space.

Finite element spaces $\mathbb{V}(\mathcal{D}, \mathcal{T})$ are conforming in the sense that $\mathbb{V}(\mathcal{D}, \mathcal{T}) \subset \boldsymbol{H}(\mathcal{D}, \Omega)$. This requires certain continuity on different sub-simplex. For example, $\mathbb{V}(\mathrm{grad}, \mathcal{T}) \subset H^1(\Omega)$ requires the continuity at node points; $\mathbb{V}(\mathrm{curl}, \mathcal{T}) \subset \boldsymbol{H}(\mathrm{curl};\Omega)$ requries the continuity of the mometem on edges and $\mathbb{V}(\mathrm{div}, \mathcal{T}) \subset \boldsymbol{H}(\mathrm{div};\Omega)$ requires the continuity of mometem on faces. Note that $\mathbb{V}(L^2, \mathcal{T}) \subset L^2(\Omega)$ do not impose any continuity condition.

The degree of freedom and uni-solvent of these finite element spaces are not easy to sketch here. We refer to [1, 4, 15, 17] for a unified presentation using differential forms. Here we use the following figure to illustrate the elements of the lowest order.

Since $\mathbb{V}(\mathcal{D}, \mathcal{T}) \subset \boldsymbol{H}(\mathcal{D}, \Omega)$, the operator equations (3) or (4) can be restricted to the finite element spaces $\mathbb{V}(\mathrm{curl}, \mathcal{T})$ or $\mathbb{V}(\mathrm{div}, \mathcal{T})$. Our task is to develop fast solvers of these linear algebraic systems for finite element grids $\mathcal{T}$.

2.3. **Exact sequence.** The following exact sequence, called de Rham differential complex, plays an important role in the error anlysis of finite element approximations of (3) and (4) as well as the iteration methods for solving the agebraic systems:

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\mathrm{grad}} \boldsymbol{H}(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0. \tag{5}$$

Namely we have the following important properties

$$\ker(\mathrm{grad}) = \mathbb{R},\ \ker(\mathrm{curl}) = \mathrm{img}(\mathrm{grad}),\ \text{and}\ \ \ker(\mathrm{div}) = \mathrm{img}(\mathrm{curl}). \tag{6}$$

Similar exact sequence holds for spaces with zero traces

$$H_0^1(\Omega) \xrightarrow{\mathrm{grad}} \boldsymbol{H}_0(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}_0(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \to 0. \tag{7}$$

For the finite element spaces $\mathbb{V}(\mathcal{D}, \mathcal{T})$, we have the following form of exact sequences:

$$\mathbb{R} \hookrightarrow \mathbb{V}(\mathrm{grad}, \mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbb{V}(\mathrm{curl}, \mathcal{T}) \xrightarrow{\mathrm{curl}} \mathbb{V}(\mathrm{div}, \mathcal{T}) \xrightarrow{\mathrm{div}} \mathbb{V}(L^2, \mathcal{T}), \tag{8}$$

where the typ of each space is different and will be discussed below.

The starting finite element space $\mathbb{V}(\mathrm{grad}, \mathcal{T})$ and the ending space $\mathbb{V}(L^2, \mathcal{T})$ are continuous or discontinuous complete polynomial spaces. For the two spaces in the middle, each one has two choices: 1st or 2nd kind. Therefore we have 4 exact sequences in $\mathbb{R}^3$ and these are all possible exact sequences in $\mathbb{R}^3$ [4]. For the completeness we list these exact sequences below:

Exact sequences (ES)

$$\mathbb{R} \hookrightarrow \mathbb{V}(\mathrm{grad}, \mathcal{P}_k, \mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbb{V}(\mathrm{curl}, \mathcal{P}_{k-1}, \mathcal{T}) \xrightarrow{\mathrm{curl}} \mathbb{V}(\mathrm{div}, \mathcal{P}_{k-2}, \mathcal{T}) \xrightarrow{\mathrm{div}} \mathbb{V}(L^2, \mathcal{P}_{k-3}, \mathcal{T})$$

$$\mathbb{R} \hookrightarrow \mathbb{V}(\mathrm{grad}, \mathcal{P}_k, \mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbb{V}(\mathrm{curl}, \mathcal{P}_{k-1}, \mathcal{T}) \xrightarrow{\mathrm{curl}} \mathbb{V}(\mathrm{div}, \mathcal{P}_{k-1}^-, \mathcal{T}) \xrightarrow{\mathrm{div}} \mathbb{V}(L^2, \mathcal{P}_{k-2}, \mathcal{T})$$

$$\mathbb{R} \hookrightarrow \mathbb{V}(\mathrm{grad}, \mathcal{P}_k, \mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbb{V}(\mathrm{curl}, \mathcal{P}_k^-, \mathcal{T}) \xrightarrow{\mathrm{curl}} \mathbb{V}(\mathrm{div}, \mathcal{P}_{k-1}, \mathcal{T}) \xrightarrow{\mathrm{div}} \mathbb{V}(L^2, \mathcal{P}_{k-2}, \mathcal{T})$$

$$\mathbb{R} \hookrightarrow \mathbb{V}(\mathrm{grad}, \mathcal{P}_k, \mathcal{T}) \xrightarrow{\mathrm{grad}} \mathbb{V}(\mathrm{curl}, \mathcal{P}_k^-, \mathcal{T}) \xrightarrow{\mathrm{curl}} \mathbb{V}(\mathrm{div}, \mathcal{P}_k^-, \mathcal{T}) \xrightarrow{\mathrm{div}} \mathbb{V}(L^2, \mathcal{P}_{k-1}, \mathcal{T}).$$

2.4. **Interpolation operators and commuting diagram.** There exist a sequence of operators

$$\Pi^{\mathcal{D}} : \boldsymbol{H}(\mathcal{D}, \Omega) \cap \mathrm{dom}(\Pi^{\mathcal{D}}) \to \mathbb{V}(\mathcal{D}, \mathcal{T})$$

to connect the Sobolev spaces $\boldsymbol{H}(\mathcal{D}, \Omega)$ with their finite element spaces $\mathbb{V}(\mathcal{D}, \mathcal{T})$. These operators enjoy the following important commuting diagram:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(\Omega) & \xrightarrow{\mathrm{grad}} & C^\infty(\Omega) & \xrightarrow{\mathrm{curl}} & C^\infty(\Omega) & \xrightarrow{\mathrm{div}} & C^\infty(\Omega) \\
\downarrow & & \Pi^{\mathrm{grad}}\downarrow & & \Pi^{\mathrm{curl}}\downarrow & & \Pi^{\mathrm{div}}\downarrow & & \Pi^{L^2}\downarrow \\
\mathbb{R} & \longrightarrow & \mathbb{V}(\mathrm{grad}, \mathcal{T}) & \xrightarrow{\mathrm{grad}} & \mathbb{V}(\mathrm{curl}, \mathcal{T}) & \xrightarrow{\mathrm{curl}} & \mathbb{V}(\mathrm{div}, \mathcal{T}) & \xrightarrow{\mathrm{div}} & \mathbb{V}(L^2, \mathcal{T}),
\end{array}$$

where for simplicity, we replace $\boldsymbol{H}(\mathcal{D}, \Omega) \cap \mathrm{dom}(\Pi^{\mathcal{D}})$ by its subspace $C^\infty(\Omega)$. More precisely we have the following important relation

$$\mathrm{grad}\, \Pi^{\mathrm{grad}} = \Pi^{\mathrm{curl}}\, \mathrm{grad}, \tag{9}$$

$$\mathrm{curl}\, \Pi^{\mathrm{curl}} = \Pi^{\mathrm{div}}\, \mathrm{curl}, \tag{10}$$

$$\mathrm{div}\, \Pi^{\mathrm{div}} = \Pi^{L^2}\, \mathrm{div}\, . \tag{11}$$

The sequence in the bottom should be one of the 4 exact sequences in (**ES**). The operator $\Pi^{\mathcal{D}}$, of course, also depends on the specific choice of $\mathbb{V}(\mathcal{D}, \mathcal{T})$. We refer to [4, 15, 17] for the construction of such canonical interpolation operators. Here we only list properties we are going to use. For proof, see [17] (Section 3.6 and Lemma 4.6).

Through out this lecture, we shall understand $h \in L^\infty(\Omega)$ as a piecewise constant mesh-size function, i.e. $h_\tau = \mathrm{diam}(\tau)$ in each simplex $\tau \in \mathcal{T}$.

Lemma 2.1. *$\Pi^{\mathcal{D}}$ is idenity restricted to $\mathbb{V}(\mathcal{D}, \mathcal{T})$. Namely*

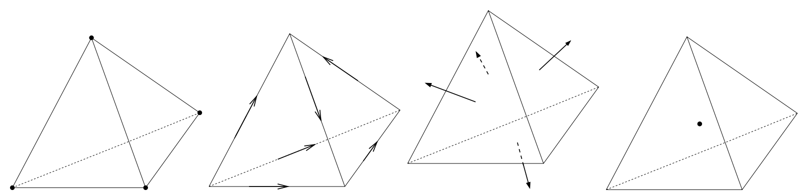
$$\Pi^{\mathcal{D}} \boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathbb{V}(\mathcal{D}, \mathcal{T}). \tag{12}$$

Lemma 2.2. *The interpolation Π^{curl} are bounded on $\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \mathrm{curl}\, \boldsymbol{v} \in \mathbb{V}(\mathrm{div}, \mathcal{T})\}$ and, with constants only depending the shape regularity of $\mathcal{T}$, they satisfy*

$$\|h^{-1}(I - \Pi^{\mathrm{curl}})\boldsymbol{v}\| \lesssim \|\boldsymbol{v}\|_1 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \mathrm{curl}\, \boldsymbol{v} \in \mathbb{V}(\mathrm{div}, \mathcal{T}). \tag{13}$$

Lemma 2.3. *The interpolation $\Pi^{\rm div}$ are bounded on $\boldsymbol{H}^1(\Omega)$ and, with constants only depending the shape regularity of $\mathcal{T}$, they satisfy*

$$\|h^{-1}(I-\Pi^{\rm div})\boldsymbol{v}\| \lesssim \|\boldsymbol{v}\|_1 \qquad \forall\, \boldsymbol{v}\in \boldsymbol{H}^1(\Omega). \tag{14}$$



3. The Method of Subspace Corrections

Discretization of partial differential equations often leads to linear algebraic equations of the form

$$Au = f, \tag{15}$$

where $A\in\mathbb{R}^{N\times N}$ is a sparse matrix and $f\in\mathbb{R}^N$. In this section, we give some general and basic results that will be used in later sections to construct efficient multilevel iterative methods (such as multigrid methods) for (15) resulting from finite element discretizations of elliptic partial differential equations. The presentation in this section follows closely to Xu [24] with simplified analysis.

3.1. **Iterative Methods.**

A basic linear iterative method for $Au = f$ can be written in the following form

$$u^{k+1} = u^k + B(f - Au^k),$$

starting from an initial guess $u^0\in\mathbb{V}$; B is called *iterator*. If $A = (a_{ij})\in\mathbb{R}^{N\times N}$ is split into diagonal, lower and upper triangular parts, namely $A = D+L+U$, then two classical examples are the Jacobi method $B = D^{-1}$ and the Gauss-Seidel method $B = (D+L)^{-1}$.

The art of constructing *efficient* iterative methods lies on the design of B which captures the essential information of A^{-1} and its action is easily computable. In this context the notion of "efficiency" entails two essential requirements:

- One iteration requires a computational effort proportional to the number of unknowns.
- The rate of convergence is well below 1 and independent of the number of unknowns.

The approximate inverse B, when it is SPD, can be used as a preconditioner for Conjugate Gradient (CG) method. The resulting method, known as preconditioned conjugate gradient method (PCG), admits the following error estimate in terms of the condition number $\kappa(BA) = \lambda_{\max}(BA)/\lambda_{\min}(BA)$

$$\frac{\|u-u^k\|_A}{\|u-u^0\|_A} \le 2\left(\frac{\sqrt{\kappa(BA)}-1}{\sqrt{\kappa(BA)}+1}\right)^k \qquad (k\ge 1);$$

B is called *preconditioner*. A good preconditioner should have the properties that the action of B is easy to compute and that $\kappa(BA)$ is significantly smaller than $\kappa(A)$.

3.2. **Space Decomposition and Method of Subspace Corrections.** In the spirit of divide and conquer, we decompose the space $\mathbb{V} = \sum_{i=0}^{J} \mathbb{V}_i$ as the summation of subspaces $\mathbb{V}_i \subset \mathbb{V}$; $\{\mathbb{V}_i\}_{i=0}^{J}$ is called a *space decomposition* of $\mathbb{V}$. Since $\sum_{i=0}^{J} \mathbb{V}_i$ is not necessarily a direct sum, decompositions of $u \in \mathbb{V}$ of the form $u = \sum_{i=0}^{J} u_i$ are in general not unique. The original problem (15) can thus be split into sub-problems in each $\mathbb{V}_i$ with smaller size which are relatively easier to solve.

Throughout this paper, we use the following operators, for $i = 0, 1, \dots, J$:

- $Q_i : \mathbb{V} \to \mathbb{V}_i$ the projection in the inner product $(\cdot, \cdot)$;
- $I_i : \mathbb{V}_i \to \mathbb{V}$ the natural inclusion which is often called prolongation;
- $P_i : \mathbb{V} \to \mathbb{V}_i$ the projection in the inner product $(\cdot, \cdot)_A = (A\cdot, \cdot)$;
- $A_i : \mathbb{V}_i \to \mathbb{V}_i$ the restriction of A to the subspace $\mathbb{V}_i$;
- $R_i : \mathbb{V}_i \to \mathbb{V}_i$ an approximation of A_i^{-1} (often known as smoother);
- $T_i : \mathbb{V} \to \mathbb{V}_i$ $T_i = R_i Q_i A = R_i A_i P_i$.

It is easy to verify the relation $Q_i A = A_i P_i$ and $Q_i = I_i^t$ with $(I_i^t u, v_i) := (u, I_i v_i)$. The operator I_i^t is often called restriction. If $R_i = A_i^{-1}$, then we have an exact local solver and $R_i Q_i A = P_i$. With slightly abused notation, we still use T_i to denote the restriction $T_i|_{\mathbb{V}_i} : \mathbb{V}_i \to \mathbb{V}_i$ and $T_i^{-1} = (T_i|_{\mathbb{V}_i})^{-1} : \mathbb{V}_i \to \mathbb{V}_i$.

For a given residual $r \in \mathbb{V}$, we let $r_i = Q_i r = I_i^t r$ denote the restriction of the residual to the subspace $\mathbb{V}_i$ and solve the residual equation $A_i e_i = r_i$ in $\mathbb{V}_i$ approximately

$$\hat{e}_i = R_i r_i.$$

Subspace corrections $\hat{e}_i$ are assembled to yield a correction in the space $\mathbb{V}$, thereby giving rise to the so-called *method of subspace corrections*. There are two basic ways to assemble subspace corrections.

Parallel Subspace Correction (PSC). This method performs the correction on each subspace in parallel. In operator form, it reads

$$u^{k+1} = u^k + B(f - Au^k), \tag{16}$$

where

$$B = \sum_{i=0}^{J} I_i R_i I_i^t. \tag{17}$$

The subspace correction is $\hat{e}_i = I_i R_i I_i^t (f - Au^k)$, and the correction in $\mathbb{V}$ is $\hat{e} = \sum_{i=0}^{J} \hat{e}_i$. The error equation reads

$$u - u^{k+1} = \left[I - \Big(\sum_{i=0}^{J} I_i R_i I_i^t \Big) A \right] (u - u^k) = \left(I - \sum_{i=0}^{J} T_i \right) (u - u^k);$$

Successive Subspace Correction (SSC). This method performs the correction in a successive way. In operator form, it reads

$$v^0 = u^k, \quad v^{i+1} = v^i + I_i R_i I_i^t (f - Av^i), i = 0, \dots, J, \quad u^{k+1} = v^{J+1}, \tag{18}$$

and the corresponding error equation is

$$u - u^{k+1} = \left[\prod_{i=0}^{J} (I - I_i R_i I_i^t A) \right] (u - u^k) = \left[\prod_{i=0}^{J} (I - T_i) \right] (u - u^k);$$

in the notation $\prod_{i=0}^{J} a_i$, we assume there is a built-in ordering from $i = 0$ to J, i.e., $\prod_{i=0}^{J} a_i = a_0 a_1 \dots a_J$. Therefore, PSC is an *additive* method whereas SSC is a *multiplicative* method.

As a trivial example, we consider the space decomposition

$$\mathbb{R}^N = \sum_{i=1}^{N} \text{span}\{e_i\}.$$

In this case, if we use exact (one dimensional) subspace solvers, the resulting SSC is just the Gauss-Seidel method and the PSC is just the Jacobi method. More complicated and effective examples, including multigrid methods and multilevel preconditioners, will be discussed later on.

3.3. **Multigrid methods as SSC.** Recall that the formation of SSC methods is

for $i = J : -1 : 1$ **do**
$r = f - Av_{i+1};$; // form residual
$r_i = Q_i\, r$; // restrict to subspace
$e_i = R_i\, r_i$; // solution in subspace
$e = Q_i^t\, e_i$; // prolongate to big space
$v_i = v_{i+1} + e$; // correction
end

In this algorithm, operators $Q_i : \mathbb{V} \to \mathbb{V}_i$ and $Q_i^t : \mathbb{V}_i \to \mathbb{V}$ are related to the big space. When the subspaces are nested, i.e.

$$\mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset V_J,$$

we don't need to return to the big space every time. Suppose (r_i, e_i) in the subspace $\mathbb{V}_i$ is known, let us compute r_{i-1}:

$$\begin{aligned} r_{i-1} &= Q_{i-1}(f - Av_i) \\ &= Q_{i-1}Q_i(f - Av_{i+1} - Ae) \\ &= Q_{i-1}(r_i - Q_i A Q_i^t e_i) \\ &= Q_{i-1}(r_i - A_i e_i). \end{aligned}$$

The correction step can be also done accumulatively. We rewrite the correction as

$$v^1 = v^{J+1} + e_J + Q_{J-1}^t e_{J-1} + \cdots + Q_1 e_1.$$

The correction can be computed by the loop

$$e_i = e_i + I_{i-1}^i e_{i-1}, \quad i = 2 : J$$

By these discussion, SSC on a nested space decomposition will results a V-cycle multigrid method. We use notation e_i, r_i to emphasis that in each level we are solving the residual equation $A_i e_i = r_i$.

In the second loop (/) part, we add a post-smoothing step and choose R_i^t which is the transpose of the pre-smoothing operator. For example, if $R_i = (D_i + L_i)^{-1}$ is forward Gauss-Seidal method, then the post-smoothing is backward Gauss-Seidal $(D_i + U_i)^{-1}$. This choice will make the operator B symmetric and thus can be used as preconditioner.

input : a vector r, matrix A
output: a vector $e = Br$ which is an approximation of $A^{-1}r$

$r_J = r, A_J = A$;
for $i = J : -1 : 2$ **do**
  $e_i = R_i r_i$; `// pre-smoothing`
  $r_{i-1} = I_i^{i-1}(r_i - A_i e_i)$; `// form residual and restriction`
end
$e_1 = A_1^{-1} r_1$; `// exact solver in the coarsest space`
for $i = 2 : J$ **do**
  $e_i = e_i + I_{i-1}^i e_{i-1}$; `// prolongation`
  $e_i = e_i + R_i^t(r_i - A_i e_i)$; `// post-smoothing`
end

3.4. **Sharp Convergence Identities.** The analysis of parallel subspace correction methods relies on the following identity which is well known in the literature [23, 24, 12, 26].

Theorem 3.1 (Identity for PSC)**.** *If R_i is SPD on $\mathbb{V}_i$ for $i = 0, \ldots, J$, then B defined by (17) is also SPD on $\mathbb{V}$. Furthermore*

$$(B^{-1}v, v) = \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J (R_i^{-1} v_i, v_i). \tag{19}$$

On the other hand, the analysis of Successive subspace correction methods hinges on an identity of Xu and Zikatanov [26] to be described below. First we assume that each subspace smoother R_i induces a convergent iteration, i.e. the error operator $I - T_i$ is a contraction.

(T) Contraction of Subspace Error Operator: There exists $\rho < 1$ such that

$$\|I - T_i\|_{A_i} \le \rho \quad \text{for all } i = 0, 1, \cdots, J. \tag{20}$$

We associate with T_i the adjoint operator T_i^* with respect to the inner product $(\cdot,\cdot)_A$. To deal with general, possibly non-symmetric smoothers R_i, we introduce the symmetrization of T_i

$$\overline{T}_i = T_i + T_i^* - T_i^* T_i, \text{ for } i = 0, \cdots, J. \tag{21}$$

We use a simplified version of XZ identity given by Cho, Xu, and Zikatanov [8]; see also [7].

Theorem 3.2 (Identity of SSC)**.** *If assumption (***T***) is valid, then the following identity holds*

$$\Big\| \prod_{i=0}^J (I - T_i) \Big\|_A^2 = 1 - \frac{1}{K},$$

where

$$K = \sup_{\|v\|_A = 1} \inf_{\sum_{i=0}^J v_i = v} \sum_{k=0}^J \left(\overline{T}_i^{-1}(v_i + T_i^* w_i), v_i + T_i^* w_i \right)_A, \tag{22}$$

with $w_i = \sum_{j>i} v_j$.

When we choose exact local solvers, i.e., $R_i = A_i^{-1}$ and consequently $T_i = P_i$ for $i = 0, \cdots, J$, (**T**) holds with $\rho = 0$. Therefore we have a more concise formulation for such choice [26].

Corollary 3.3 (Identity of SSC with exact solver)**.** *One has the following identity*

$$\Big\| \prod_{i=0}^{J} (I - P_i) \Big\|_A^2 = 1 - \frac{1}{1 + c_0},$$

where

$$c_0 = \sup_{\|v\|_A = 1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^{J} \Big\| P_i \sum_{j=i+1}^{N} v_j \Big\|_A^2. \tag{23}$$

3.5. **Convergence Analysis.** We now present a convergence analysis based on three assumptions: (**T**) on T_i and the two ones below on the space decomposition. The analysis here is adapted from Xu [24] and simplified by using the XZ identity.

(A1) Stable Decomposition: For any $v \in \mathcal{V}$, there exists a decomposition $v = \sum_{i=0}^J v_i, v_i \in \mathcal{V}_i, i = 0, \dots, J$ such that

$$\sum_{i=0}^{J} \|v_i\|_A^2 \le K_1 \|v\|_A^2. \tag{24}$$

(A2) Strengthened Cauchy Schwarz (SCS) Inequality: For any $u_i, v_i \in \mathcal{V}_i, i = 0, \cdots, J$

$$\left| \sum_{i=0}^{J} \sum_{j=i+1}^{J} (u_i, v_j)_A \right| \le K_2 \left(\sum_{i=0}^{J} \|u_i\|_A^2 \right)^{1/2} \left(\sum_{i=0}^{J} \|v_i\|_A^2 \right)^{1/2}. \tag{25}$$

Theorem 3.4 (Multilevel preconditioning)**.** *Let $\mathcal{V} = \sum_{i=0}^J \mathcal{V}_i$ be a space decomposition satisfying assumptions* (**A1**) *and* (**A2**)*, and let R_i be SPDs for $i = 0, \cdots, J$ such that*

$$K_4^{-1} \|u_i\|_A^2 \le (R_i^{-1} u_i, u_i) \le K_3 \|u_i\|_A^2. \tag{26}$$

Then B defined by (17) is SPD and

$$\kappa(BA) \le (1 + 2K_2) K_1 K_3 K_4. \tag{27}$$

Theorem 3.5 (Convergence of SSC)**.** *Let $\mathcal{V} = \sum_{i=0}^J \mathcal{V}_i$ be a space decomposition satisfying assumptions* (**A1**) *and* (**A2**)*, and let the subspace smoothers T_i satisfy* (**T**)*. We then have*

$$\Big\| \prod_{i=0}^{J} (I - T_i) \Big\|_A^2 \le 1 - \frac{1 - \rho^2}{2K_1 (1 + (1 + \rho)^2 K_2^2)}.$$

When we use exact local solvers $R_i = A_i^{-1}$, we have a simpler proof and sharper estimate.

Corollary 3.6. *Let the space decomposition satisfy* (**A1**) and (**A2**), and let $R_i = A_i^{-1}$ for all i. Then

$$\Big\| \prod_{i=0}^{J} (I - P_i) \Big\|_A^2 \le 1 - \frac{1}{1 + K_1 K_2^2}.$$

4. Multigrid Algorithms

To present the multigrid methods in the framework of subspace correction, it suffices to give stable decomposition of the finite element spaces $\mathbb{V}(\mathrm{curl},\mathcal{T})$ and $\mathbb{V}(\mathrm{div},\mathcal{T})$.

We first present the standard basis decomposition which corresponds to the smoother. Let $\mathbb{V}(\mathrm{grad},\mathcal{T}) = \mathrm{span}\{\phi_p, p\in\Lambda(p)\}, \mathbb{V}(\mathrm{curl},\mathcal{T}) = \mathrm{span}\{\phi_e, e\in\Lambda(e)\}$, and $\mathbb{V}(\mathrm{div},\mathcal{T}) = \mathrm{span}\{\phi_f, f\in\Lambda(f)\}$. We use notation $\Lambda(p)$ for the basis of $\mathbb{V}(\mathrm{grad},\mathcal{T})$ since they are mainly associated to nodes. For the same reason, we use $\Lambda(e)$ or $\Lambda(f)$ for $\mathbb{V}(\mathrm{curl},\mathcal{T})$ or $\mathbb{V}(\mathrm{div},\mathcal{T})$, respectively. Denoted by the one dimensional space $\mathbb{V}_p = \mathrm{span}\{\phi_p\}$, $\mathbb{V}_e = \mathrm{span}\{\phi_e\}$, and $\mathbb{V}_f = \mathrm{span}\{\phi_f\}$. We then have the standard basis decompositions:

$$\mathbb{V}(\mathrm{grad},\mathcal{T}) = \sum_{p\in\Lambda(p)}\mathbb{V}_p,\quad \mathbb{V}(\mathrm{curl},\mathcal{T}) = \sum_{e\in\Lambda(e)}\mathbb{V}_e,\quad \mathbb{V}(\mathrm{div},\mathcal{T}) = \sum_{f\in\Lambda(f)}\mathbb{V}_f.$$

To present a multilevel space decomposition, let us assume that we have an initial quasi-uniform triangulation $\mathcal{T}_0$ and a nested sequence of triangulations $\{\mathcal{T}_k\}_{k=0}^J$ where $\mathcal{T}_k$ is obtained by the uniform refinement of $\mathcal{T}_{k-1}$ for $k>0$. We then get a nested sequence (in the sense of trees [22]) of quasi-uniform triangulations

$$\mathcal{T}_0 \le \mathcal{T}_1 \le \cdots \le \mathcal{T}_J = \mathcal{T}_h.$$

Note that h_k, the mesh size of $\mathcal{T}_k$, satisfies $h_k \eqsim \gamma^{2k}$ for some $\gamma\in(0,1)$, and thus $J \eqsim |\log h|$.

4.1. Space decomposition of H^1 system.

Let $\mathbb{V}_k$ denote the corresponding linear finite element space of $H_0^1(\Omega)$ based on $\mathcal{T}_k$. We thus get a sequence of multilevel nested spaces

$$\mathbb{V}_0 \subset \mathbb{V}_1 ... \subset \mathbb{V}_J = \mathbb{V},$$

and a macro space decomposition

$$\mathbb{V} = \sum_{k=0}^{J}\mathbb{V}_k. \tag{28}$$

There is redundant overlapping in this multilevel decomposition, so the sum is not direct. The subspace solvers need only to take care of the "non-overlapping" components of the error (high frequencies in $\mathbb{V}_k$). For each subspace problem $A_k e_k = r_k$ posed on $\mathbb{V}_k$, we use a simple Richardson method

$$R_k = h_k^2 I_k,$$

where $I_k : \mathbb{V}_k \to \mathbb{V}_k$ is the identity and $h_k \eqsim \lambda_{\max}(A_k)$.

Let N_k be the dimension of $\mathbb{V}_k$, i.e., the number of interior vertices of $\mathcal{T}_k$. The standard nodal basis in $\mathbb{V}_k$ will be denoted by $\phi_{(k,i)}, i = 1,\cdots,N_k$. By our characterization of Richardson method, it is PSC method on the micro decomposition $\mathbb{V}_k = \sum_{i=1}^{N_k}\mathbb{V}_{(k,i)}$ with $\mathbb{V}_{(k,i)} = \mathrm{span}\{\phi_{(k,i)}\}$. In summary we choose the space decomposition:

$$\mathbb{V} = \sum_{k=0}^{J}\mathbb{V}_k = \sum_{k=0}^{J}\sum_{i=1}^{N_k}\mathbb{V}_{(k,i)}. \tag{29}$$

If we apply PSC to the decomposition (29) with $R_{(k,i)} = h_k^2 I_{(K,i)}$, we obtain

$$I_{(k,i)}R_{(k,i)}I_{(k,i)}^t u = h^{2-d}(u,\phi_{(k,i)})\phi_{(k,i)}.$$

The resulting operator B, according to (17), is the so-called BPX preconditioner [6]

$$Bu = \sum_{k=0}^{J}\sum_{i=1}^{N_k} h_k^{2-d}(u, \phi_{(k,i)})\phi_{(k,i)}. \tag{30}$$

If we apply SSC to the decomposition (29) with exact subspace solvers $R_i = A_i^{-1}$, we obtain a V-cycle multigrid method with Gauss-Seidel smoothers.

We shall construct similar decomposition for $\mathbb{V}(\mathrm{curl}, \mathcal{T})$ and $\mathbb{V}(\mathrm{div}, \mathcal{T})$. We shall present two types of space decomposition. One is to write out the kernel space explicitly and another is to implicitly build into the smoother.

4.2. **Hiptmair space decomposition.**

$$\mathbb{V}(\mathrm{curl}, \mathcal{T}) = \mathbb{V}(\mathrm{curl}, \mathcal{T}_0) + \sum_{k=0}^{J}\sum_{e\in\Lambda_k(e)} \mathbb{V}_{k,e} + \sum_{k=0}^{J}\sum_{p\in\Lambda_k(p)} \mathrm{grad}\,\mathbb{V}_{k,p}, \tag{31}$$

$$\mathbb{V}(\mathrm{div}, \mathcal{T}) = \mathbb{V}(\mathrm{div}, \mathcal{T}_0) + \sum_{k=0}^{J}\sum_{F\in\Lambda_k(f)} \mathbb{V}_{k,f} + \sum_{k=0}^{J}\sum_{e\in\Lambda_k(e)} \mathrm{curl}\,\mathbb{V}_{k,e}. \tag{32}$$

4.3. **Arnold-Falk-Winther space decomposition.** We can merge some subspaces together to get Arnold-Falk-Winther type space decomposition [3]. Let us define

$$\mathbb{W}_p = \mathrm{span}\{\phi_e, \mathrm{supp}\,\phi_e \subset \omega_p\} \subset \mathbb{V}(\mathrm{curl}, \mathcal{T}),$$
$$\mathbb{W}_e = \mathrm{span}\{\phi_f, \mathrm{supp}\,\phi_e \subset \omega_e\} \subset \mathbb{V}(\mathrm{div}, \mathcal{T}).$$

The AFW decomposition is

$$\mathbb{V}(\mathrm{curl}, \mathcal{T}) = \mathbb{V}(\mathrm{curl}, \mathcal{T}_0) + \sum_{k=0}^{J}\sum_{p\in\mathcal{N}(\mathcal{T}_k)} \mathbb{W}_p, \tag{33}$$

$$\mathbb{V}(\mathrm{div}, \mathcal{T}) = \mathbb{V}(\mathrm{curl}, \mathcal{T}_0) + \sum_{k=0}^{J}\sum_{e\in\mathcal{E}(\mathcal{T}_k)} \mathbb{W}_e. \tag{34}$$

The main difference between Hiptmair and AFW decomposition is on the explicit or implicit inclusion of the kernel space of the differential operator $\mathcal{D}$. For Hiptmair decomposition, we need to form an auxiliary problem for the kernel space, for example, forming a H^1 system for a $\boldsymbol{H}^{\mathrm{curl}}$ problem. On the other hand, the local problem on $\mathbb{W}_p$ to be solved in AFW decomposition is relatively big. For example, there could be 20 edges connected to a vertex and thus $\dim \mathbb{W}_p \approx 60$.

5. HX Preconditioner

5.1. **The Auxiliary Space Method.** The method of subspace correction consists of solving a system of equations in a vector space by solving on appropriately chosen *subspaces* of the original space. Such subspaces are, however, not always available. The auxiliary space method (Xu 1996 [25]) is for designing preconditioners using auxiliary spaces which are not necessarily subspaces of the original subspace.

To solve the equation $a(u, v) = (f, v)$ in a Hilbert space $\mathcal{V}$, we consider

$$\overline{\mathcal{V}} = \mathcal{V} \times \mathcal{W}_1 \times \cdots \times \mathcal{W}_J\ , \tag{35}$$

where $\mathcal{W}_1, \ldots, \mathcal{W}_J$, $J \in \mathbb{N}$ are auxiliary (Hilbert) spaces endowed with inner products $\bar{a}_j(\cdot,\cdot)$, $j = 1, \ldots, J$.

A distinctive feature of the auxiliary space method is the presence of $\mathcal{V}$ in (35), but as a component of $\overline{\mathcal{V}}$. The space $\mathcal{V}$ is equipped with an inner product $d(\cdot,\cdot)$ different from $a(\cdot,\cdot)$. The operator $D:\mathcal{V}\mapsto\mathcal{V}$ induced by $d(\cdot,\cdot)$ on $\mathcal{V}$ leads to the smoother $S=D^{-1}$. For each $\mathcal{W}_j$ we need $\Pi_j:\mathcal{W}_j\mapsto\mathcal{V}$ which gives

$$\Pi := Id\times\Pi_1\times\cdots\times\Pi_J:\overline{\mathcal{V}}\mapsto\mathcal{V}, \tag{36}$$

with properties

$$\|\Pi_j w_j\|_A \le c_j\bar{a}(w_j,w_j)^{1/2}\ , \qquad \text{for all } w_j\in\mathcal{W}_j, j=1,\cdots,J\ , \tag{37}$$

$$\|v\|_A \le c_s d(v,v)^{1/2}\ , \qquad \text{for all } v\in\mathcal{V}\ , \tag{38}$$

and for every $v\in\mathcal{V}$, there exist $v_0\in\mathcal{V}$ and $w_j\in\mathcal{W}_j$ such that $v=v_0+\sum_{j=1}^J\Pi_j w_j$ and

$$d(v_0,v_0)^{1/2}+\sum_{j=1}^J\bar{a}_j(w_j,w_j)^{1/2}\le c_0\|v\|_A\ . \tag{39}$$

Let $\bar{A}_i$, for $i=1,\cdots,J$, be operators induced by $(\cdot,\cdot)_{A_i}$. Then the auxiliary space preconditioner is given by

$$B=S+\sum_{j=1}^J\Pi_j\bar{a}_j^{-1}\Pi_j^*. \tag{40}$$

The estimate of the condition number $\kappa(BA)$ is given below.

Theorem 5.1. *Let $\Pi=Id\times\Pi_1\times\cdots\times\Pi_J:\bar{\mathcal{V}}=\mathcal{V}\times\mathcal{W}_1\times\cdots\times\mathcal{W}_J\mapsto\mathcal{V}$ satisfy properties (37), (38), and (39). Then the auxiliary space preconditioner B given in (40) admits the following estimate:*

$$\kappa(BA)\le c_0^2(c_s^2+c_1^2+\cdots+c_J^2)\ . \tag{41}$$

5.2. **Discrete regular decomposition.** In this section we shall introduce discrete regular decomposition for finite element spaces $\mathbb{V}(\mathrm{curl},\mathcal{T})$ and $\mathbb{V}(\mathrm{div},\mathcal{T})$ which is firstly introduced by Hiptmair and Xu in [19].

Theorem 5.2 (Discrete regular decomposition of $\mathbb{V}(\mathrm{curl},\mathcal{T})$)**.** *Let $\mathbb{V}(\mathrm{grad},\mathcal{T}),\mathbb{V}(\mathrm{curl},\mathcal{T})$ be a pair in the four exact sequences and $\mathbb{V}^3\subset\mathbb{V}(\mathrm{curl},\mathcal{T})$. For any $\boldsymbol{v}\in\mathbb{V}(\mathrm{curl},\mathcal{T})$, there exist $\tilde{\boldsymbol{v}}\in\mathbb{V}(\mathrm{curl},\mathcal{T}),\boldsymbol{\phi}\in\mathbb{V}^3$ and $u\in\mathbb{V}(\mathrm{grad},\mathcal{T})$ such that*

$$\boldsymbol{v}=\tilde{\boldsymbol{v}}+\boldsymbol{\phi}+\mathrm{grad}\,u,\ \ \textit{and} \tag{42}$$

$$\|h^{-1}\tilde{\boldsymbol{v}}\|+\|\boldsymbol{\phi}\|_1+\|u\|_1\lesssim\|\boldsymbol{v}\|_{A^c}. \tag{43}$$

Theorem 5.3 (Discrete regular decomposition of $\mathbb{V}(\mathrm{div},\mathcal{T})$)**.** *Let $\mathbb{V}(\mathrm{curl},\mathcal{T}),\mathbb{V}(\mathrm{div},\mathcal{T})$ be a pair in the four exact sequences and $\mathbb{V}^3\subset\mathbb{V}(\mathrm{div},\mathcal{T})$. For any $\boldsymbol{v}\in\mathbb{V}(\mathrm{div},\mathcal{T})$, there exist $\tilde{\boldsymbol{v}}\in\mathbb{V}(\mathrm{div},\mathcal{T}),\boldsymbol{\phi}\in\mathbb{V}^3$ and $\boldsymbol{u}\in\mathbb{V}(\mathrm{curl},\mathcal{T})$ such that*

(1) $\boldsymbol{v}=\tilde{\boldsymbol{v}}+\boldsymbol{\phi}+\mathrm{curl}\,\boldsymbol{u}$,
(2) $\|h^{-1}\tilde{\boldsymbol{v}}\|+\|\boldsymbol{\phi}\|_1+\|\boldsymbol{u}\|_{A^c}\lesssim\|\boldsymbol{v}\|_{A^d}$.

For the lowest order space $\mathbb{V}(\mathrm{curl},\mathcal{P}_1^-,\mathcal{T})$, since $\mathbb{V}^3\not\subseteq\mathbb{V}(\mathrm{curl},\mathcal{P}_1^-,\mathcal{T})$, the decomposition is slightly different. The interpolation operator $\Pi^{\mathcal{D}}$ stays in the decomposition. The proof is similar and thus leave to the readers.

Corollary 5.4. *For any $\boldsymbol{v} \in \mathbb{V}(\operatorname{curl}, \mathcal{P}_1^-, \mathcal{T})$, there exists $\tilde{\boldsymbol{v}} \in \mathbb{V}(\operatorname{curl}, \mathcal{P}_1^-, \mathcal{T}), \boldsymbol{\phi} \in \mathbb{V}^3$ and $u \in \mathbb{V}$ such that*

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + \Pi^{\text{curl}}\boldsymbol{\phi} + \operatorname{grad} u, \quad \textit{and} \tag{44}$$

$$\|h^{-1}\tilde{\boldsymbol{v}}\| + \|\boldsymbol{\phi}\|_1 + \|u\|_1 \lesssim \|\boldsymbol{v}\|_{A^c}. \tag{45}$$

Corollary 5.5. *For any $\boldsymbol{v} \in \mathbb{V}(\operatorname{div}, \mathcal{P}_1^-, \mathcal{T})$, there exists $\tilde{\boldsymbol{v}} \in \mathbb{V}(\operatorname{div}, \mathcal{P}_1^-, \mathcal{T}), \boldsymbol{\phi} \in \mathbb{V}^3$ and $\boldsymbol{u} \in \mathbb{V}(\operatorname{curl}, \mathcal{P}_2, \mathcal{T})$ such that*

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} + \Pi^{\text{div}}\boldsymbol{\phi} + \operatorname{curl} \boldsymbol{u}, \quad \textit{and} \tag{46}$$

$$\|h^{-1}\tilde{\boldsymbol{v}}\| + \|\boldsymbol{\phi}\|_1 + \|\boldsymbol{u}\|_1 \lesssim \|\boldsymbol{v}\|_{A^c}. \tag{47}$$

5.3. **HX preconditioner.** We present an *auxiliary space preconditioner* for $H(\operatorname{curl})$ and $H(\operatorname{div})$ systems developed in Hiptmair and Xu [19] (see also R. Beck [5] for a special case). The basic idea is to apply an auxiliary space preconditioner framework in [25], to the discrete regular decompositions of $\mathbb{V}(\operatorname{curl}, \mathcal{T})$ or $\mathbb{V}(\operatorname{div}, \mathcal{T})$. The resulting preconditioner for the $H(\operatorname{curl})$ systems is

$$B^{\text{curl}} = S^{\text{curl}} + \Pi^{\text{curl}}\boldsymbol{B}^{\text{grad}}(\Pi^{\text{curl}})^t + \operatorname{grad}\ B^{\text{grad}}(\operatorname{grad})^t. \tag{48}$$

The implementation makes use of the input data: the $\boldsymbol{H}(\operatorname{curl})$ stiffness matrix A, the coordinates of the grid points, along with the discrete gradient grad (for the lowest order Nédélec element case, it is simply the "vertex"-to-"edge" mapping with entries 1 or -1). Based on the coordinates, one can easily construct the interpolation operator Π_h^{curl}. Then the "Auxiliary space Maxwell solver" consists of the following three components:

(1) The smoother S^{curl} of A (it could be the standard Jacobi or symmetric Gauss-Seidel methods).
(2) An algebraic multigrid (AMG) solver B^{grad} for $\operatorname{grad}^t A \operatorname{grad}$
(3) An (vector) AMG solver $\boldsymbol{B}^{\text{grad}}$ for $\left(\Pi^{\text{curl}}\right)^T A\Pi^{\text{curl}}$.

Similarly

$$\begin{aligned} B^{\text{div}} &= S^{\text{div}} + \Pi^{\text{div}}\boldsymbol{B}^{\text{grad}}(\Pi_h^{\text{div}})^t + \operatorname{curl}\ B^{\text{curl}}(\operatorname{curl})^t \\ &= S^{\text{div}} + \Pi^{\text{div}}\boldsymbol{B}^{\text{grad}}(\Pi^{\text{div}})^t + \operatorname{curl}\ S^{\text{curl}}(\operatorname{curl})^t + \operatorname{curl}\Pi^{\text{curl}}\boldsymbol{B}^{\text{grad}}(\Pi^{\text{curl}})^t(\operatorname{curl})^t. \end{aligned}$$

This preconditioner consists of 4 Poisson solvers B^{grad} for $\boldsymbol{H}(\operatorname{curl})$ (and 6 for $\boldsymbol{H}(\operatorname{div})$) as well as 1 simple relaxation method (S^{curl}) such as point Jacobi for $\boldsymbol{H}(\operatorname{curl})$ (and 2 relaxation methods for $\boldsymbol{H}(\operatorname{div})$).

The point here is that we can use well-developed AMG for H^1 systems for the Poisson solver B^{grad} to obtain robust AMG methods for $\boldsymbol{H}(\operatorname{curl})$ and $\boldsymbol{H}(\operatorname{div})$ systems. These classes of preconditioners are in some way a "grey-box" AMG as it makes use of information on geometric grids (and associated interpolation operators). But the overhead is minimal and it requires very little programming effort. It has been proved in [19] that it is optimal and efficient for problems on unstructured grids.

To interpret B^{curl} as an auxiliary space preconditioner, we choose $\mathbb{V} = \mathbb{V}(\operatorname{curl}, \mathcal{T})$ and $\mathcal{W}_1 = \mathcal{W}_2 = \mathbb{V}(\operatorname{grad}, \mathcal{T})$. The inner product for the smoother is induced by the diagonal matrix of A^{curl} and the inner product $\bar{A}_1, \bar{A}_2$ is induced by $(\boldsymbol{B}^{\text{grad}})^{-1}$. The operator $\Pi_1 : \mathcal{W}_1 \to \mathbb{V}$ is the interpolation Π^{curl} and $\Pi_2 = \operatorname{grad} : \mathcal{W}_1 \to \mathbb{V}$.

Theorem 5.6. *Suppose B^{grad} is an SPD matrix such that $((B^{\text{grad}})^{-1}u, u) \eqsim (u, u)_1$. Then the preconditioner B^{curl} defined by (48) admits the estimate*

$$\kappa(B^{\text{curl}}A^{\text{curl}}) \lesssim 1.$$

We state a similar result for $B^{\rm div}$ below and leave the proof to readers.

Theorem 5.7. *Suppose $B^{\rm grad}$ is an SPD matrix such that $((B^{\rm grad})^{-1}u,u) \eqsim (u,u)_1$. Then the preconditioner*

$$B^{\rm div} = S^{\rm div} + \Pi^{\rm div}\boldsymbol{B}^{\rm grad}(\Pi^{\rm div})^t + \operatorname{curl}\ S^{\rm curl}(\operatorname{curl})^t + \operatorname{curl}\Pi^{\rm curl}\boldsymbol{B}^{\rm grad}(\Pi^{\rm curl})^t(\operatorname{curl})^t$$

admits the estimate

$$\kappa(B^{\rm div}A^{\rm div}) \lesssim 1.$$

For $\boldsymbol{H}(\rm curl)$ systems, the preconditioners have been included and tested in LLNL's *hypre* package [9, 10, 11] based on its parallel algebraic multigrid solver "BoomerAMG" [13]. It is a parallel implementation, almost a 'black-box" as it requires only discrete gradient matrix plus vertex coordinates, it can handle complicated geometries and coefficient jumps, scales with the problem size and on large parallel machines, supports simplified magnetostatics mode, and can utilize Poisson matrices, when available. Extensive numerical experiments demonstrate that this preconditioner is also efficient and robust for more general equations (see Hiptmair and Xu [19], and Kolev and Vassilevski [20, 21]) such as

$$\operatorname{curl}(\mu(x)\operatorname{curl} u) + \sigma(x)u = f \tag{49}$$

where μ and σ may be discontinuous, degenerate, and exhibit large variations.

For this type of general equations, we may not expect that the simple Poisson solvers are sufficient to handle possible variations of μ and σ. Let us argue roughly what the right equations are to replace the Poisson equations. Let us assume our problems has sufficient regularity (e.g., Ω is convex). We then have

$$\|\operatorname{grad}\boldsymbol{u}\|^2 \eqsim \|\operatorname{curl}\boldsymbol{u}\|^2 + \|\operatorname{div}\boldsymbol{u}\|^2.$$

If $\boldsymbol{u}(=\operatorname{curl} w) \in N(\operatorname{curl})^\perp$, then $\|\operatorname{grad}\boldsymbol{u}\| = \|\operatorname{curl}\boldsymbol{u}\|$. Roughly, we get the following equivalence:

$$(\mu\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{u}) + (\sigma\boldsymbol{u},\boldsymbol{u}) \eqsim (\mu\operatorname{grad}\boldsymbol{u}, \operatorname{grad}\boldsymbol{u}) + (\sigma\boldsymbol{u},\boldsymbol{u}),$$

which corresponds to the following operator:

$$\boldsymbol{L}_1\boldsymbol{u} \equiv -\operatorname{div}(\mu(x)\operatorname{grad}\boldsymbol{u}) + \sigma(x)\boldsymbol{u}. \tag{50}$$

On the other hand, if $\boldsymbol{u},\boldsymbol{v} \in N(\operatorname{curl})$, $\boldsymbol{u} = \operatorname{grad} p$ and $\boldsymbol{v} = \operatorname{grad} q$,

$$(\mu\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{v}) + (\sigma\boldsymbol{u},\boldsymbol{v}) = (\sigma\operatorname{grad} p, \operatorname{grad} q)$$

which corresponds to the following operator:

$$L_2\boldsymbol{u} \equiv -\operatorname{div}(\sigma(x)\operatorname{grad} p). \tag{51}$$

We obtain the following preconditioner for the general equation (49):

$$B^{\rm curl} = S^{\rm curl} + \Pi^{\rm curl}\boldsymbol{B}_1^{\rm grad}(\Pi^{\rm curl})^t + \operatorname{grad}\ B_2^{\rm grad}(\operatorname{grad})^t$$

where is $\boldsymbol{B}_1^{\rm grad}$ is a preconditioner for the operator in the equation (50) and $B_2^{\rm grad}$ is a preconditioner for the operator in the equation (51).

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