FINITE ELEMENT METHODS FOR LINEAR ELASTICITY

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ABSTRACT. This lecture notes discusses finite element methods for solving linear elasticity equations, focusing on the locking phenomenon in nearly incompressible materials. The displacement formulation is introduced, but standard linear finite elements suffer from locking when the Lamé constant is large. To overcome this, a displacement-pressure formulation using stable Stokes elements and a stress-displacement formulation with $H(\mathrm{div})$-conforming spaces for symmetric matrix functions are presented.

We discuss finite element methods for solving the linear elasticity equations. For a complete background review, we refer to Introduction to Linear Elasticity and Variational Formulation of Linear Elasticity.

1. DISPLACEMENT FORMULATION

The weak form of the displacement formulation of linear elasticity reads as follows: Given a force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$2\mu(\nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + \lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{1}$$

where λ and μ are Lamé constants and the symmetric gradient $\nabla^s \boldsymbol{u} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\intercal)/2$. For nearly incompressible materials, the parameter $\lambda \gg 1$ while $\mu = \mathcal{O}(1)$.

The following identity

$$2 \operatorname{div} \nabla^s \boldsymbol{u} = \Delta \boldsymbol{u} + \operatorname{grad} \operatorname{div} \boldsymbol{u} \tag{2}$$

can be proved as follows: for $k = 1, 2, \ldots, d$,

$$(\operatorname{div} 2\nabla^s \boldsymbol{u})_k = \sum_{i=1}^{d} \partial_i(\partial_i u_k + \partial_k u_i) = \Delta \boldsymbol{u}_k + \partial_k(\operatorname{div} \boldsymbol{u}).$$

Multiplying identity (2) by $\boldsymbol{v}$ and integrating by parts, we obtain an equivalent formulation

$$a(\boldsymbol{u}, \boldsymbol{v}) := \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\lambda + \mu)(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \tag{3}$$

The well-posedness of (3) is guaranteed by

- Coercivity.
$$a(\boldsymbol{u}, \boldsymbol{u}) \geq \mu \|\nabla \boldsymbol{u}\|^2.$$
- Continuity.
$$a(\boldsymbol{u}, \boldsymbol{v}) \leq (\lambda + 2\mu)\|\nabla \boldsymbol{u}\|\|\nabla \boldsymbol{v}\|.$$

Let $V_h \subset \boldsymbol{H}_0^1(\Omega)$ be the linear finite element space, and let $u_h \in V_h$ be the finite element approximation satisfying

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h.$$

Then, by the Céa lemma, we obtain the quasi-optimal approximation

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\| \leq (2 + \lambda/\mu) \inf_{\boldsymbol{v}_h \in V_h} \|\nabla(\boldsymbol{u} - \boldsymbol{v}_h)\| \lesssim (2 + \lambda/\mu) h \|\boldsymbol{u}\|_2.$$

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For given and fixed Lamé constants (λ, μ), we do observe first-order approximation as $h \to 0$. However, for a fixed h, the approximation is of the order $\mathcal{O}(\lambda)$, indicating a loss of convergence as $\lambda \to +\infty$. This loss of convergence order for nearly incompressible materials is known as locking phenomena.

Recall that the change in volume due to deformation is linked to $\operatorname{div} \boldsymbol{u}$:

$$\delta V = \int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_V \operatorname{tr}(\boldsymbol{\varepsilon}) \, \mathrm{d}\boldsymbol{x}.$$

As indicated in the constitutive equation $\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$, a very large value of λ suggests that $\operatorname{tr}(\boldsymbol{\varepsilon}) = \operatorname{div} \boldsymbol{u}$ is very small, signifying minimal volume change under stress. Mathematically, we have

$$\lambda \| \operatorname{div} \boldsymbol{u}_h \|^2 \leq a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq \frac{1}{2} \left(\|\boldsymbol{f}\|^2 + \|\boldsymbol{u}_h\|^2 \right) < +\infty,$$

which implies $\| \operatorname{div} \boldsymbol{u}_h \| = \mathcal{O}(1/\sqrt{\lambda})$. So if $\lambda = +\infty$, then $\operatorname{div} \boldsymbol{u}_h = 0$. For linear finite elements, it can be shown that if $\operatorname{div} \boldsymbol{u}_h = 0$, then $\boldsymbol{u}_h = 0$. Therefore no approximation

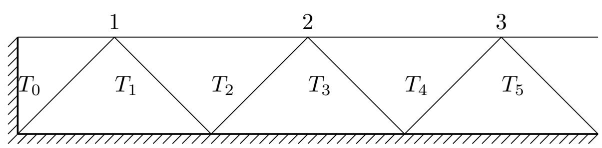

FIGURE 1. There are two DoFs $\boldsymbol{u}(x_1)$ at vertex x_1. Due to the zero boundary condition and two constraints $\operatorname{div} \boldsymbol{u} = 0$ in T_0 and T_1, we conclude $\boldsymbol{u}(x_1) = 0$. By repeating this argument, we can conclude $\boldsymbol{u}(x_2) = \boldsymbol{u}(x_3) = 0$ and so on.

can be provided for $\boldsymbol{u}$. In general, locking is unavoidable if the subspace $\ker(\operatorname{div}) \cap V_h$ is not big enough.

2. DISPLACEMENT AND PRESSURE FORMULATION

We introduce an artificial pressure $p = \lambda \operatorname{div} \boldsymbol{u}$ and rewrite the equation (1) into perturbed Stokes equations: Find $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} 2\mu(\nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v}) + (p, \operatorname{div} \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ (\operatorname{div} \boldsymbol{u}, q) - \frac{1}{\lambda}(p, q) &= 0, \qquad\quad \text{for all } q \in L_0^2(\Omega). \end{aligned} \tag{4}$$

The well-posedness of the saddle point problem (4) is guaranteed by the inf-sup condition, which holds for the limiting case $1/\lambda = 0$; see Inf-sup Conditions for Operator Equations. The large continuity constant now turns to the denominator:

$$\frac{1}{\lambda}(p, q) \leq \frac{1}{\lambda} \|p\| \|q\|.$$

An equivalent formulation based on (3) is:

$$\begin{aligned} \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (p, \operatorname{div} \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ (\operatorname{div} \boldsymbol{u}, q) - \frac{1}{\lambda + \mu}(p, q) &= 0, \qquad\quad \text{for all } q \in L_2(\Omega). \end{aligned} \tag{5}$$

Then we can use stable finite element methods developed for Stokes equations. Choose spaces $V_h \times P_h \subset \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ satisfying the discrete inf-sup condition

$$\inf_{q_h \in P_h} \sup_{v_h \in V_h} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_1 \|q_h\|} = \beta > 0.$$

Find $\boldsymbol{u}_h \in V_h, p_h \in P_h$ s.t.

$$\begin{aligned} \mu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + (p_h, \operatorname{div} \boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h), && \text{for all } \boldsymbol{v}_h \in V_h, \\ (\operatorname{div} \boldsymbol{u}_h, q_h) - \frac{1}{\lambda + \mu}(p_h, q_h) &= 0, && \text{for all } q_h \in P_h. \end{aligned} \tag{6}$$

In general, $\operatorname{div} V_h \not\subseteq P_h$. In $(\operatorname{div} \boldsymbol{u}_h, q_h)$, the operator is $Q_h \operatorname{div}$, where Q_h is the L^2 projection to P_h.

Solving $p_h = (\lambda + \mu) Q_h \operatorname{div} \boldsymbol{u}_h$ and substituting back to the first equation in (6), we obtain a modified discretization of the displacement formulation: Find $u_h \in V_h$ s.t.

$$\mu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + (\lambda + \mu)(Q_h \operatorname{div} \boldsymbol{u}_h, Q_h \operatorname{div} \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall v_h \in V_h.$$

The modified L^2-inner product $(Q_h \operatorname{div} \boldsymbol{u}_h, Q_h \operatorname{div} \boldsymbol{v}_h)$ can be thought of as reduced integration. A specific example is $P_2 - P_0$ Stokes element. If we compute $(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{v}_h)$ exactly (e.g., using a three-edge midpoint quadrature rule), essentially it is using the unstable Stokes pair $P_2 - P_1^{-1}$. Instead, if we use a one-point quadrature rule (the center of the element), it computes $(Q_h \operatorname{div} \boldsymbol{u}_h, Q_h \operatorname{div} \boldsymbol{v}_h)$ corresponding to the stable pair $P_2 - P_0$. It is surprising and counterintuitive that inexact numerical quadrature yields better results.

When the domain of interest is smooth or convex, we have the regularity result:

$$\|\boldsymbol{u}\|_2 + \lambda \|\operatorname{div} \boldsymbol{u}\|_1 \leq C \|\boldsymbol{f}\|.$$

Based on this regularity result, we can prove the error analysis for the displacement-pressure formulation which is robust to λ. Indeed as $\boldsymbol{u} \in H^2, p = \lambda \operatorname{div} \boldsymbol{u} \in H^1$, by the standard error analysis, for $P_2 - P_0$ solution of (6), we have

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\| \lesssim h(\|\boldsymbol{u}\|_2 + \|p\|_1) \lesssim h \|\boldsymbol{f}\|,$$

where the constant is independent of λ.

One might opt for nonconforming finite element methods, as the $P_1^{\mathrm{CR}} - P_0$ pair constitutes a stable Stokes combination, ensuring locking-free behavior when applied to (3). It is worth noting that since P_1^{CR} does not belong to $\boldsymbol{H}_0^1$, (1) and (3) are no longer equivalent at the discrete level.

When dealing with the pure traction boundary condition $\boldsymbol{\sigma n} = \boldsymbol{t}_N$ on $\partial\Omega$, using the bilinear form $\mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$ with strong coercivity loses its equivalence as integration by parts from (2) will have non-vanishing boundary terms. Instead, $2\mu(\nabla^s \boldsymbol{u}_h, \nabla^s \boldsymbol{v}_h)$ is employed, and Korn's second inequality in the discrete setting may not hold for the linear nonconforming elements; see [2].

3. Stress and Displacement Formulation

We utilize the mixed formulation involving stress and displacement. Considering this, the natural finite element spaces would be $P_k(T; \mathbb{S}) - P_{k-1}^{-1}(T; \mathbb{R}^3)$ for $k \geq 1$. The primary challenge lies in constructing $H(\text{div})$-conforming finite element space for symmetric matrix functions. Arnold and Winther designed the first polynomial symmetric stress element in 2D [1]. Here, we follow Hu and Zhang [3] to present a simple construction based on the modification of Lagrange elements.

3.1. **Lagrange Elements.** A simplicial lattice, known as the principal lattice [4], of degree k and dimension d is a multi-index set of $d+1$ components and with a fixed sum k, i.e.,

$$\mathbb{T}^d_k = \left\{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{0:d} : |\alpha| = \alpha_0 + \alpha_1 + \ldots + \alpha_d = k\right\}.$$

We can embed the simplicial lattice into a geometric simplex by using α/k as the barycentric coordinate of node α. Let T be a simplex with vertices $\{\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_d\}$ and λ_i be the corresponding barycentric coordinate. The geometric embedding is

$$x : \mathbb{T}^d_k \to T, \quad x(\alpha) = \sum_{i=0}^{d} \lambda_i(\alpha)\mathrm{v}_i, \ \lambda_i(\alpha) = \alpha_i/k, i = 0, \ldots, d.$$

Define $\mathcal{X}_T = \{x(\alpha), \alpha \in \mathbb{T}^d_k\}$ and call it the set of interpolation nodes. See Fig. 2 for an illustration in 2D and 3D. The Bernstein basis of polynomial of degree k on a simplex T is

$$P_k(T) = \mathrm{span}\{\lambda^\alpha := \lambda_0^{\alpha_0}\lambda_1^{\alpha_1}\ldots\lambda_d^{\alpha_d}, \alpha \in \mathbb{T}^d_k\}. \tag{7}$$

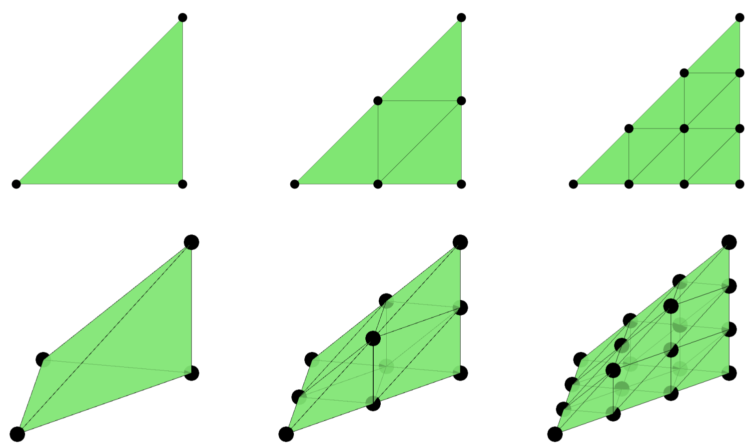

FIGURE 2. Simplicial Lattices in 2D and 3D.

Lemma 3.1 (Lagrange Interpolation Basis Functions [4])**.** *A basis function of the k-th order Lagrange finite element space on T is given by:*

$$\phi_\alpha(x) = \frac{1}{\alpha!}\prod_{i=0}^{d}\prod_{j=0}^{\alpha_i-1}(k\lambda_i(x) - j), \quad \alpha \in \mathbb{T}^d_k,$$

where the degrees of freedom (DoFs) are defined as the function values at the interpolation points:

$$N_\alpha(u) = u(x_\alpha), \quad x_\alpha \in \mathcal{X}_T. \tag{8}$$

Proof. The duality of the basis and DoFs can be verified as follows:

$$N_\beta(\phi_\alpha) = \phi_\alpha(x_\beta) = \delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$|\mathbb{T}^d_k| = \binom{d+k}{k} = \dim P_k(T),$$

the set $\{\phi_\alpha, \alpha \in \mathbb{T}_k^d\}$ forms a basis of $P_k(T)$, and $\{N_\alpha, \alpha \in \mathbb{T}_k^d\}$ forms a basis of the dual space $P_k^*(T)$. ☐

Based on a conforming triangulation $\mathcal{T}_h$ of Ω, the k-th order Lagrange finite element space $V_k^{\mathrm{L}}(\mathcal{T}_h)$ is defined as

$$V_k^{\mathrm{L}}(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_T \in P_k(T), T \in \mathcal{T}_h, \text{ and DoFs (8) are single valued}\}.$$

The interpolation points $\mathcal{X}_T$ is element-wise defined and for $x_\alpha \in \mathcal{X}_{T_1} \cap \mathcal{X}_{T_2}$, the DoF $u(x_\alpha)$ is single valued means it is independent of the element containing x_α. This ensures the continuity of piecewisely defined polynomials.

Exercise 3.2. Prove $V_k^{\mathrm{L}}(\mathcal{T}_h)$ is continuous and thus $V_k^{\mathrm{L}}(\mathcal{T}_h) \subset H^1(\Omega)$.

3.2. **Hu-Zhang element.** We can simply take the tensor product $V_h^L \otimes \mathbb{S}$ of the Lagrange element. That is, at each interpolation point, the scalar function $u(x_\alpha)$ is changed to a symmetric matrix function $(\sigma_{11}(x_\alpha), \sigma_{22}(x_\alpha), \sigma_{12}(x_\alpha))$. Here the subscript 1 or 2 refers to the Cartesian coordinate used to describe a point, i.e., $x = (x_1, x_2)^\intercal$. A symmetric matrix function $\boldsymbol{\sigma} = (\sigma_{ij})$ is stretched to a vector $(\sigma_{11}, \sigma_{22}, \sigma_{12})$.

The Lagrange element is too continuous to satisfy the inf-sup condition. As we know from the Stokes equation, the pair $P_k - P_{k-1}^-$ is not div-stable in general. In 2D, $k \geq 4$ and a non-singular vertex condition on the triangulation can guarantee the discrete inf-sup condition. The condition in 3D remains unclear and non-realistic.

For a symmetric matrix function $\boldsymbol{\sigma}$ and vector function $\boldsymbol{v}$, we have the integration by parts formulae

$$-\int_T \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_T \boldsymbol{\sigma} : \nabla^s \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_{\partial T} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{v} \,\mathrm{d}S. \tag{9}$$

From that, we conclude $\sigma \in H(\operatorname{div}, \Omega; \mathbb{S})$ if and only if $\boldsymbol{\sigma n}$ is continuous across edges (2D) and faces (3D) of the triangulation. So we only need to impose normal continuity.

The key idea is to choose different coordinate systems on different sub-simplices. In 2D, at vertices and inside the triangle, we use the default Cartesian coordinate. On each edge, we use $t - n$ coordinate: the orthonormal coordinate formed by a unit tangential and a unit normal vector. In 3D, at vertices and inside the element, we use the default Cartesian coordinate $\{\boldsymbol{e}_i\}_{i=1}^3$. On each face, we choose two orthonormal tangential vectors and a unit normal vector which forms an orthonormal basis. On each edge, we choose two normal vectors $\{\boldsymbol{n}_i^e\}_{i=1}^2$ depending only on e together with an unit tangential vector t^e. The coordinate depends on the sub-simplex but not the element containing it. See Fig. 3.

In the sequel, we will focus on two dimensions. But most results and proofs can be easily generalize to three and higher dimensions with notation change. We use $P_k \otimes \mathbb{S}$ as the shape function space. A Lagrange element is defined by the degrees of freedom (DoFs):

$$\sigma_{11}(x_\alpha), \sigma_{22}(x_\alpha), \sigma_{12}(x_\alpha), \quad \alpha \in \mathbb{T}_k^2.$$

To ensure continuity of $\boldsymbol{\sigma n}$, we use $t - n$ coordinates at edge interpolation points and adjust DoFs on the interior of edges to:

$$\sigma_{tt}(x_\alpha), \sigma_{nn}(x_\alpha), \sigma_{tn}(x_\alpha), \quad \alpha \in \mathbb{T}_k^2, x_\alpha \in \mathring{E}.$$

The unisolvence remains unchanged as $(\sigma_{tt}, \sigma_{nn}, \sigma_{tn})$ forms a basis of $\mathbb{S}$. Locally it is still a Lagrange element.

We enforce desirable normal continuity by ensuring $\sigma_{nn}(x_\alpha)$ and $\sigma_{tn}(x_\alpha)$ are single-valued, while $\sigma_{tt}(x_\alpha)$ could be double-valued. In other words, the DoF $\sigma_{tt}(x_\alpha)$ might differ across different triangles. We denote the resulting space as Σ_k.

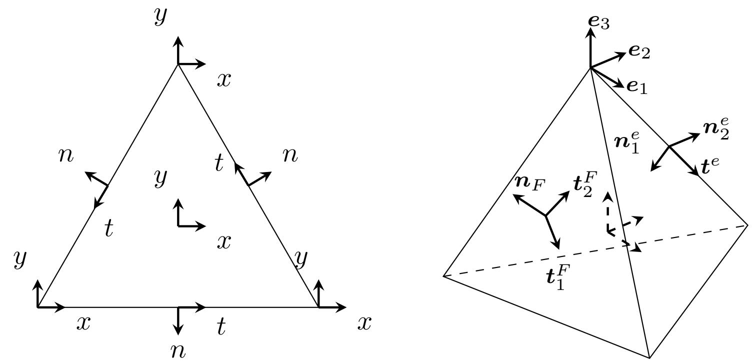

FIGURE 3. Different coordinates at different sub-simplex.

Exercise 3.3. Write out the DoFs in 3D and prove that the resulting element is $H(\mathrm{div})$-conforming and symmetric.

3.3. **Inf-sup conditions.** The displacement space consists of discontinuous P_{k-1} polynomials. Next, we verify the inf-sup condition $\mathrm{div} : \Sigma_k \to P_{k-1}^{-1}$.

For a continuous function $\boldsymbol{\sigma}$, $\mathrm{tr}_T^{\mathrm{div}}\boldsymbol{\sigma} = \boldsymbol{\sigma}\boldsymbol{n}\mid_{\partial T}$ where $\boldsymbol{n}$ is the outwards normal vector. Define the bubble function

$$B_k(\mathrm{div}, T; \mathbb{S}) = P_k(T; \mathbb{S}) \cap \ker(\mathrm{tr}_T^{\mathrm{div}}).$$

For an edge e with vertices x_i and x_j, denote by $b_e = \lambda_i\lambda_j$ the edge bubble function. For a polynomial defined on edge e, we can expand using barycentric coordinates, cf. (7), which naturally defines an extension to the triangle. Let $\mathcal{E}_h$ be the set of edges. We denote by $\boldsymbol{n}_i$ the normal vector of the edge opposite to vertex i.

Lemma 3.4. $\{\mathrm{sym}(\boldsymbol{n}_i \otimes \boldsymbol{n}_j), e_{ij} \in \mathcal{E}_h(T)\}$ *and* $\{\boldsymbol{t}_e \otimes \boldsymbol{t}_e, e \in \mathcal{E}_h(T)\}$ *are bases of* $\mathbb{S}$.

Proof. It is straight forward to verify that, in the Frobenius inner product $(\cdot,\cdot)_F$, the set $\{\mathrm{sym}(\boldsymbol{n}_i \otimes \boldsymbol{n}_j), e_{ij} \in \mathcal{E}_h(T)\}$ is dual to $\{\boldsymbol{t}_e \otimes \boldsymbol{t}_e, e \in \mathcal{E}_h(T)\}$. So both of them are bases of $\mathbb{S}$. ☐

Lemma 3.5.

(10) $$\mathrm{span}\,\{p(e)b_e\boldsymbol{t}_e \otimes \boldsymbol{t}_e, p \in P_{k-2}(e), e \in \mathcal{E}_h(T)\} \subseteq B_k(\mathrm{div}, T; \mathbb{S}) \quad k \geq 2.$$

Proof. Let $\phi_e = b_e\boldsymbol{t}_e \otimes \boldsymbol{t}_e$. For edge $\tilde{e} \neq e$, $b_e \mid_{\tilde{e}} = 0$. On the edge e, $(\boldsymbol{t}_e \otimes \boldsymbol{t}_e)\boldsymbol{n}_e = 0$. So $\mathrm{tr}_T^{\mathrm{div}}\phi_e = 0$. ☐

Denote by $RM = \ker(\nabla^s) = \{\boldsymbol{a} + b(x_2, -x_1)^{\intercal}\}$.

Lemma 3.6.

(11) $$\mathrm{div}\, B_k(\mathrm{div}, T; \mathbb{S}) = P_{k-1}(T; \mathbb{R}^2)/RM \quad k \geq 2.$$

Proof. For $\boldsymbol{\sigma} \in B_k(\mathrm{div}, T; \mathbb{S})$, using integration by parts, we have

$$(\mathrm{div}\,\boldsymbol{\sigma}, \boldsymbol{v})_T = -(\boldsymbol{\sigma}, \nabla^s\boldsymbol{v})_T = 0, \quad \forall \boldsymbol{v} \in RM = \ker(\nabla^s).$$

So we have proved $\mathrm{div}\, B_k(\mathrm{div}, T; \mathbb{S}) \subseteq P_{k-1}(T; \mathbb{R}^2)/RM$.

Assume $\operatorname{div} B_k(\operatorname{div}, T; \mathbb{S}) \neq P_{k-1}(T; \mathbb{R}^2)/RM$. Then there exists $\boldsymbol{u} \in P_{k-1}(T; \mathbb{R}^2)/RM$, $\boldsymbol{u} \neq 0$, and $(\operatorname{div} \boldsymbol{\sigma}, \boldsymbol{u})_T = 0$ for all $\boldsymbol{\sigma} \in B_k(\operatorname{div}, T; \mathbb{S})$. Expand $\nabla^s u$ in the basis $\operatorname{sym}(\boldsymbol{n}_i \otimes \boldsymbol{n}_j)$ as

$$\nabla^s \boldsymbol{u} = \sum_{e = e_{ij} \in \mathcal{E}_h(T)} p_e \operatorname{sym}(\boldsymbol{n}_i \otimes \boldsymbol{n}_j), \quad p_e \in P_{k-2}.$$

Set $\boldsymbol{\sigma} = \sum_e p_e b_e \boldsymbol{t}_e \otimes \boldsymbol{t}_e / (\boldsymbol{t}_e \cdot \boldsymbol{n}_i \ \boldsymbol{t}_e \cdot \boldsymbol{n}_j) \in B_k(\operatorname{div}, T; \mathbb{S})$. Then

$$0 = -(\operatorname{div} \boldsymbol{\sigma}, \boldsymbol{u})_T = \sum_e \int_T p_e^2 b_e \, \mathrm{d}x.$$

As $b_e \geq 0$ and $b_e > 0$ for $x \in \mathring{T}$, we conclude $p_e = 0$ and thus $\boldsymbol{u} = 0$. Contradicts with $\boldsymbol{u} \neq 0$. Therefore $\operatorname{div} B_k(\operatorname{div}, T; \mathbb{S}) = P_{k-1}(T; \mathbb{R}^2)/RM$. □

Theorem 3.7. *In two dimensions, we have the inf-sup condition*

$$\inf_{\boldsymbol{v}_h \in P_{k-1}^{-1}} \sup_{\boldsymbol{\sigma}_h \in \Sigma_k} \frac{(\operatorname{div} \boldsymbol{\sigma}_h, \boldsymbol{v}_h)}{\|\boldsymbol{\sigma}_h\|_{\operatorname{div}} \|\boldsymbol{v}_h\|} = \beta > 0, \quad k \geq d+1. \tag{12}$$

Proof. Given $\boldsymbol{v}_h \in P_{k-1}^{-1} \in \boldsymbol{L}^2(\Omega)$, by the inf-sup condition of Stokes equations, there exists $\boldsymbol{\sigma} \in \boldsymbol{H}^1(\Omega; \mathbb{S})$ s.t. $\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{v}_h$ and $\|\boldsymbol{\sigma}\|_1 \lesssim \|\boldsymbol{v}_h\|$.

We define a quasi-interpolation from $\mathcal{I}_h : \boldsymbol{H}^1(\Omega; \mathbb{S}) \to V_3^L \otimes \mathbb{S}$ such that

$$\int_e \mathcal{I}_h \boldsymbol{\sigma} q = \int_e \boldsymbol{\sigma} q \quad \forall q \in P_1(e), \quad e \in \partial T, \tag{13}$$

which is possible as V_3^L contains 2 DoFs on each edge (on each edge, expand in the basis $P_1(e) b_e$ and solve a Gram matrix equation in the inner product $\int_e (\cdot) b_e \, \mathrm{d}s$ to satisfy (13)), and

$$\|\boldsymbol{\sigma} - \mathcal{I}_h \boldsymbol{\sigma}\| + h \|\boldsymbol{\sigma} - \mathcal{I}_h \boldsymbol{\sigma}\|_1 \lesssim h \|\boldsymbol{\sigma}\|_1 \lesssim h \|\boldsymbol{v}_h\|.$$

Now using integration by parts, we conclude

$$(\operatorname{div}(\boldsymbol{\sigma} - \mathcal{I}_h \boldsymbol{\sigma}), q)_T = 0 \quad \forall q \in RM$$

and thus $\operatorname{div}(\boldsymbol{\sigma} - \mathcal{I}_h \boldsymbol{\sigma}) \in P_{k-1}(T; \mathbb{R}^2)/RM$. By Lemma 3.6, we can find $\boldsymbol{\sigma}_b \in B_k(\operatorname{div}, T; \mathbb{S})$ s.t. $\operatorname{div} \boldsymbol{\sigma}_b = \operatorname{div}(\boldsymbol{\sigma} - \mathcal{I}_h \boldsymbol{\sigma})$. Set $\boldsymbol{\sigma}_h = \mathcal{I}_h \boldsymbol{\sigma} + \boldsymbol{\sigma}_b$. Then

$$\operatorname{div} \boldsymbol{\sigma}_h = \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{v}_h.$$

And

$$\|\boldsymbol{\sigma}_h\| \leq \|\mathcal{I}_h \boldsymbol{\sigma}\| + \|\boldsymbol{\sigma}_b\| \lesssim \|\boldsymbol{v}_h\|.$$

□

Remark 3.8. When generalize to $\mathbb{R}^d$, the key is to ensure there are d interior interpolation points on each $d-1$ face of the element; see (13). So the condition $k \geq d+1$ is required.

3.4. **Mixed finite element.** The mixed formulation of (1) seeks $\boldsymbol{\sigma} \in \widehat{\boldsymbol{H}}(\operatorname{div}, \Omega, \mathbb{S})$ and $u \in L^2(\Omega)$ such that:

$$\begin{aligned} (\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau}) + (\boldsymbol{u}, \operatorname{div} \boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \widehat{\boldsymbol{H}}(\operatorname{div}, \Omega, \mathbb{S}), \\ (\operatorname{div} \boldsymbol{\sigma}, \boldsymbol{v}) &= -(\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in L^2(\Omega), \end{aligned} \tag{14}$$

where

$$\widehat{\boldsymbol{H}}(\operatorname{div}, \Omega, \mathbb{S}) = \{\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}, \Omega, \mathbb{S}) : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{x} = 0\}.$$

The boundary condition $\boldsymbol{u}|_{\partial \Omega} = 0$ is imposed weakly in the first equation of (14).

Define $\widehat{\Sigma}_k = \Sigma_k \cap \widehat{\boldsymbol{H}}(\mathrm{div}, \Omega, \mathbb{S})$. The mixed finite element methods is: Find $\boldsymbol{\sigma}_h \in \widehat{\Sigma}_k$ and $\boldsymbol{u}_h \in P_{k-1}^{-1}(\Omega)$ such that:

$$(\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\boldsymbol{u}_h, \mathrm{div}\, \boldsymbol{\tau}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in \widehat{\Sigma}_k,$$
$$(\mathrm{div}\, \boldsymbol{\sigma}_h, \boldsymbol{v}_h) = -(\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in P_{k-1}^{-1}(\Omega).$$

As $\mathrm{div}\, c\boldsymbol{I} = 0$ for any constant c, we can subtract the average of the trace so that (12) also holds for $\widehat{\Sigma}_k$. The robust coercivity

$$a(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) \geq \alpha \|\boldsymbol{\sigma}_h\|^2 \quad \text{for all } \boldsymbol{\sigma}_h \in \widehat{\Sigma}_k \cap \ker(\mathrm{div}), \tag{15}$$

with α independent of λ can be proved as in the continuous level; see Section 2.4 in Variational Formulation of Linear Elasticity.

Based on the inf-sup conditions (15) and (12), we have the quasi-optimal approximation and optimal order error convergence

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div}} + \|\boldsymbol{u} - \boldsymbol{u}_h\| \lesssim h^k (\|\boldsymbol{\sigma}\|_{k+1} + \|\boldsymbol{u}\|_{k+1}).$$

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