PROJECT 2: WAVE EQUATION

Due date. March 23.

Code the explicit leapforg method for solving wave equation in two dimensions.

Equation.

$$
\begin{aligned}
&(1) \qquad & u_{tt} - \Delta u &= f(x,t), \quad x \in \Omega, t \in (0,T] \\
&(2) \qquad & u(x,0) &= g(x), \quad x \in \Omega, \\
&(3) \qquad & u_t(x,0) &= h(x), \quad x \in \Omega, \\
&(4) \qquad & u &= u_D, \quad x \in \partial\Omega, t \in (0,T].
\end{aligned}
$$

Method.

$$
(5) \qquad \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{(\Delta t)^2} - (\Delta_h U^n)_j = f_j^n,
$$

where Δ_h is the discretization of Δ operator using finite difference or finite element method. (Choose the one you like.)

Initial condition. It is easy to choose U^0 by the nodal interpolation $U_j^0 = g(x_j)$. To get U^1, we introduce the ghost point U^{-1} and discretization the initial velocity (3) using central difference:

$$
(6) \qquad \frac{U_j^1 - U_j^{-1}}{2\Delta t} = h(x_j).
$$

To eliminate the ghost point, we combine the equation (5) at $n = 0$ to figure out the formula for U^1.

Test. We choose the domain as $\Omega = (0,12) \times (0,12)$ and the source term as

$$
\begin{aligned}
f(x,t) &= \exp(-7|x - x_S|) 2a(2a(t-b)^2 - 1)\exp(-a(t-b)^2), \\
a &= (\frac{\pi}{1.31})^2, \ b = 1.35 \\
x_S &= (6,6).
\end{aligned}
$$

The boundary and initial conditions

$$
g = h = 0, \quad u_D = 0.
$$

Show the evolution of the solution and verify the convergent rate.

Hint: how to compute the rate without exact solution? You can estimate the convergence rates by the formula

$$
r^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2},
$$

where e^N is the error of N unknowns. The rate for the time variable can be computed similarly.