TAKE-HOME EXAMS FOR MATH290A

LONG CHEN

You must complete at least three problems. You may receive partial credit for a problem only if you made substantial progress towards a solution. Please justify all statements you make.

You are encouraged to work in group. But one group consists of at most two students. The take-home exam is due on Dec 10. Please make an appointment to present your solutions.

1. Consider the system given in polar coordinates by

$$r' = r - r^3,$$
$$\theta' = \sin^2(\theta) + a.$$

For $-1 < a < 0, a = 0$, and $a > 0$, find all equilibrium points and limit cycles, determine their stability, and draw the phase portraits, respectively.

2. Consider a nonlinear pendulum system with a constant torque:

$$\theta' = v,$$
$$v' = -bv - \sin(\theta) + k,$$

where we assume that $k \geq 0$. Since θ is measured mod 2π, we may think of this system as being defined on the cylinder $S^1 \times \mathbb{R}$, where S^1 denotes the unit circle.

(1) Find all equilibrium points for this system and determine their stability.
(2) Determine the regions in the bk-parameter plane for which there are different numbers of equilibrium points.
(3) Describe the qualitative features of a Poincaré map defined on the line $\theta = 0$ for this system.
(4) Prove that when $k > 1$ there is a unique periodic solution for this system. *Hint:* Consider the energy function

$$E(\theta, \gamma) = \frac{1}{2}\gamma^2 - \cos\theta + 1$$

and use the fact that the total change of E along any periodic solution must be 0.
(5) Prove that there are parameter values for which a stable equilibrium and a periodic solution coexist.
(6) Describe the bifurcation that must occur when the periodic solution ceases to exist.

3. Apply Poincaré-Bendixson theorem to prove the following facts.

(1) Let Γ be a closed orbit and let U be the open region in the interior of Γ. Then U contains either an equilibrium point or a limit cycle.
(2) Let Γ be a closed orbit that forms the boundary of an open set U. Then U contains an equilibrium point.

Date: November 29, 2010.

4. Consider the SIR model of the spread of infectious diseases:

$$S' = -\beta S I,$$
$$I' = \beta S I - \nu I,$$

where S stands for the population of susceptible individuals and I for the infected population, and β and ν are positive parameters.

Prove that given any initial population (S_0, I_0) with $S_0 > \nu/\beta$ and $I_0 > 0$, the susceptible population decreases monotonically, while the infected population at first rises, but eventually reaches a maximum at $S = \nu/\beta$ and then decline to 0. Draw the phase portrait for the SIR system.