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Some error analysis on virtual element methods

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Abstract Some error analyses on virtual element methods (VEMs) including inverse inequalities, norm equivalence, and interpolation error estimates are developed for polygonal meshes, each element of which admits a virtual quasi-uniform triangulation. This sub-mesh regularity covers the usual ones used for theoretical analysis of VEMs, and the proofs are presented by means of standard technical tools in finite element methods.

Keywords Virtual elements · Inverse inequality · Norm equivalence · Interpolation error estimate

Mathematics Subject Classification 65N30 · 65N12

1 Introduction

Since the pioneer work in [2–4], virtual element methods (VEMs) have been widely used to approximate various partial differential equations in recent years. Compared with the standard finite element methods (cf. [10,17]), such methods have several

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significant advantages: (1) they are natively adapted to polygonal/polyhedral meshes, leading to great convenience in mesh generation for problems with complex geometries. For example, in [16] a simple and efficient interface-fitted polyhedral mesh algorithm is developed and VEM has been successfully applied to the elliptic interface problem. (2) They are suitable for attacking high-order elliptic problems. For instance, it is very difficult to construct the usual H^2-conforming finite element method for fourth-order elliptic problems, hence many nonconforming elements were devised to overcome the difficulty (see [25]). It is, however, very straightforward to construct H^2-conforming virtual element methods for this type of problems (cf. [13]). Until now, both conforming and nonconforming VEMs for elliptic problems have been developed with elaborated details (cf. [2,3,8,13,15,16,20]).

Error estimates for approximation spaces play fundamental roles in theoretical analysis of finite element methods, so do the inverse inequality and the norm equivalence between the continuous and discrete norms of a finite element function. They are equally important for the virtual element methods. Such results were stated or implied in the papers [2,3], though the detailed justifications were not presented. More recently, in the papers [7,15], the inverse inequalities ((4.9) and (4.11) in [7]) and the norm equivalence (Lemma 4.9 in [15]) were derived in detail, respectively. Let $\mathcal{T}_h$ be a polygon mesh, which consists of a finite number of simple polygons (i.e. open simply connected sets with non-self-intersecting polygonal boundaries). All the results mentioned above were obtained using the so-called generalized scaling argument (cf. [14]), based on the following assumptions on $\mathcal{T}_h$ in two-dimensional cases:

- **C1**. There exists a real number $\gamma > 0$ such that, for each element $K \in \mathcal{T}_h$, it is star-shaped with respect to a disk of radius $\rho_K \geq \gamma h_K$, where h_K is the diameter of K.
- **C2**. There exists a real number $\gamma_1 > 0$ such that, for each element $K \in \mathcal{T}_h$, the distance between any two vertices of K is $\geq \gamma_1 h_K$.

Using the similar arguments in [21], these estimates still hold if any element $K \in \mathcal{T}_h$ is the union of a finite number of polygons satisfying conditions **C1** and **C2**.

The key idea of the generalized scaling argument (still called the scaling argument in [14]) is the use of the compactness argument. To convey the basic ideas, a simple proof of the inverse inequality is presented as follows:

$$\|\nabla v\|_{0,K} \leq C h_K^{-1} \|v\|_{0,K} \quad \forall\, v \in V_K, \tag{1}$$

where V_K is a finite dimensional space of shape functions defined over a polygon $K \in \mathcal{T}_h$, and C is a generic constant independent of the mesh size h_K. With a scaling transformation, it suffices to derive the estimate (1) provided that $h_K = 1$. In this case, under the assumptions of **C1** and **C2**, the set $\mathcal{K}$ consisting of all such K can be viewed as a compact set in certain topology. Then, let

$$C(K) = \sup_{v \in V_K} \frac{\|\nabla v\|_{0,K}}{\|v\|_{0,K}}. \tag{2}$$

If $C(K)$ can be proved to be continuous with respect to $K \in \mathcal{K}$ in the sense of the aforementioned topology, then it is evident that $C(K)$ can attain its maximum C over

$\mathcal{K}$, leading to the desired estimate (1) readily; we refer the reader to the proof of Lemma 4.1 for the details of such arguments.

Hence, when applying the generalized scaling argument to derive the estimate (1) for virtual element spaces, we require to show the solution of the Poisson equation defined over K depends on the shape of K continuously, since the local space V_K is defined with the help of the Laplacian operator (for details see [2,3] or Sect. 2). In fact, such results can be obtained rigorously in a very subtle and technical way (cf. [19]).

Similarly, we remark that we should use the trace inequality or the Sobolev embedding inequality over K carefully, since the generic constant depends on the geometric nature of K implicitly.

Based on the above comments, in this paper, we aim to derive all the results mentioned above through only the mathematical tools widely-used in the community of finite element methods, to shed light on theoretical analysis of virtual element methods in an alternative way. To this end, we impose the following mesh regularity condition for a family of meshes $\{\mathcal{T}_h\}_h$ under discussion:

A1. For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is proportional to h_K. Each edge of K is a side of a certain triangle in $\mathcal{T}_K$.

It is evident that the mesh $\mathcal{T}_h$ fulfilling the conditions **C1** and **C2** naturally satisfy the above conditions. We shall derive some error analysis on VEMs including inverse inequalities, norm equivalence, and interpolation error estimates for several types of VEM spaces, under the mesh regularity conditions **A1** which cover the usual ones frequently used in the analysis of virtual element methods. The idea of using a "virtual" triangulation can be traced back to regular decomposition condition in the error analysis of mimetic finite difference methods (cf. [12]).

For triangular meshes, one can use an affine transformation to map an arbitrary triangle to a so-called reference triangle and then work on the reference triangle. Results established on the reference triangle can be pulled back to the original triangle by estimating the Jacobian of the affine map. For polygons, scaling can be still used but not the affine transformation. Therefore we cannot work on a reference polygon which does not exist for a family of polygons with general shapes. Instead we decompose a polygon K into shape regular triangles and use the scaling argument in each triangle.

Throughout this paper, we will always assume the mesh $\mathcal{T}_h$ satisfies the conditions **A1**, and the generic constant hidden in the symbol $\lesssim$ depends only on the parameters involving the shape regularity and quasi-uniformity of the auxiliary triangulation $\mathcal{T}_K$ given in **A1**. Moreover, for any two quantities a and b, "$a \eqsim b$" indicates "$a \lesssim b \lesssim a$". We will also use the standard notations and symbols for Sobolev spaces and their norms/semi-norms; the reader is referred to [1] for details.

Denote by V_K a virtual element space, whose precise definition can be found in Sect. 2. With the help of **A1**, we are going to rigorously prove: for all $v \in V_K$

1. Inverse inequality: $|v|_{1,K} \lesssim h_K^{-1} \|v\|_{0,K}$.
2. Norm equivalence: $h_K \|\boldsymbol{\chi}(v)\|_{l^2} \lesssim \|v\|_{0,K} \lesssim h_K \|\boldsymbol{\chi}(v)\|_{l^2}$, where $\boldsymbol{\chi}(v)$ is the vector formed by the degrees of freedom of v.

3. Stability estimate of the VEM formulation:

$$\|\nabla v\|_{0,K}^2 \eqsim \left\|\nabla \Pi_k^\nabla v\right\|_{0,K}^2 + \left\|\boldsymbol{\chi}\left(v - \Pi_k^\nabla v\right)\right\|_{l^2}^2,$$
$$\|\nabla v\|_{0,K}^2 \eqsim \left\|\nabla \Pi_k^\nabla v\right\|_{0,K}^2 + \left\|\boldsymbol{\chi}_{\partial K}\left(v - \Pi_k^0 v\right)\right\|_{l^2}^2,$$

   where Π_k^∇, Π_k^0 are H^1, L^2-projection to the polynomial space $\mathbb{P}_k(K)$, respectively.
4. Interpolation error estimate: if $I_K u \in V_K$ denotes the canonical interpolant defined by d.o.f. of u, then

$$\|u - I_K u\|_{0,K} + h_K |u - I_K u|_{1,K} \lesssim h_K^{k+1} \|u\|_{k+1,K} \quad \forall u \in H^{k+1}(K).$$

After completing this work, we are aware that similar studies were also developed in a recent paper [11] with respect to mesh conditions **C1** and **C2**. The analysis is based on a variety of estimates related to a mesh dependent norm

$$|\!|\!| v |\!|\!|_{k,K}^2 := h_K \sum_{e \in \partial K} \left\|\Pi_{k,e}^0 v\right\|_{0,e}^2 + \left\|\Pi_{k-2,K}^0 v\right\|_{0,K}^2,$$

which plays the role of $\|\cdot\|_{0,K}$ though not knowing their h_K-independent equivalence. For example, a different inverse inequality $|v|_{1,K} \lesssim h_K^{-1} |\!|\!| v |\!|\!|_{k,K}$ is obtained (cf. Lemma 2.19 in [11]). However, as the continuous L^2-norm is used for a VEM function, there is no discussion on the norm equivalence between this norm and the l^2-norm of its degrees of freedom. As we shall show in Sect. 4, it is by no means trivial to derive such norm equivalence. Moreover, it deserves to point out that similar results for stability and error estimates for the interpolation operators were recently presented in [6], where a different stabilization involving boundary derivatives (cf. [26]) was also analyzed.

The rest of the paper is organized as follows. The virtual element method is introduced in Sect. 2. Inverse inequalities, norm equivalence, and interpolation error estimates for several types of VEM spaces are derived with technical details in Sects. 3–5, respectively.

2 Virtual element methods

A two dimensional domain Ω is decomposed into a polygonal mesh $\mathcal{T}_h$ so that each element in $\mathcal{T}_h$ is a simple polygon and a generic element is denoted by K. We work under the two dimensional setting for a clear illustration, and the generalization to higher dimensions shall be commented afterwards.

To present the main idea, we consider the simplest Poisson equation with zero Dirichlet boundary condition:

$$-\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

The weak formulation is: given an $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

2.1 Assumptions on the polygon mesh

As mentioned in the introduction, we shall carry out the analysis based on the assumption **A1**, for which some more discussions are given as follows. Recall that a triangle is shape regular if there exists a constant κ such that the ratio of the diameter of this triangle to the radius of its inscribed circle is bounded by κ. It is also equivalent to the condition that the minimum angle is bounded below by a positive constant θ. A triangulation $\mathcal{T}$ is quasi-uniform if any two triangles in the triangulation are of comparable sizes. Namely there exists a constant σ, such that $\max_{\tau\in\mathcal{T}} h_\tau \le \sigma \min_{\tau\in\mathcal{T}} h_\tau$. The term "uniform" means the constants κ, θ and σ are independent of K.

By assumption **A1**, the number of triangles of each 'virtual triangulation' $\mathcal{T}_K$ is uniformly bounded by a number L and the size of each triangle is comparable to that of the polygon, i.e. $h_K \lesssim h_\tau \le h_K, \ \forall \tau \in \mathcal{T}_K$. The constants in our inequalities depend on the shape regularity constant κ (or equivalently θ) and the quasi-uniformity constant σ (or equivalently L).

Assumption **A1** is introduced so that the estimates for finite elements on triangles can be used. If K is assumed to be star-shaped and each edge is of comparable size, e.g. assumption **C2**, then a virtual triangulation can be obtained by connecting vertices of K to the center of the star. In contrast, **A1** allows the union of star-shaped regions to form irregular polygons.

Note that such virtual triangulations can be created with additional artificial vertices in the interior of K but not on ∂K.

2.2 Spaces in virtual element methods

Let k, l be two non-negative integers. Introduce the following space on K

$$V_{k,l}(K) := \left\{ v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K),\, \Delta v \in \mathbb{P}_l(K) \right\}, \tag{4}$$

where $\mathbb{P}_l(K)$ is the space of polynomials of total degree $\le l$ on K and conventionally $\mathbb{P}_{-1}(K) := \{0\}$, and $\mathbb{B}_k(\partial K)$ is a function space on the boundary ∂K defined by

$$\mathbb{B}_k(\partial K) = \left\{ v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \text{ for all edges } e \subset \partial K \right\}.$$

That means, the restriction of $V_{k,l}(K)$ to ∂K is a standard conforming Lagrange element of degree k.The shape function in (4) is well-defined, but the point-wise value of a function $v \in V_{k,l}(K)$ requires solving a boundary value problem on K, thus being implicitly defined and not explicitly known. The novelty of VEM is that the shape function is determined implicitly, but the degrees of freedom (d.o.f.) are still enough to produce a stable discrete method directly as for the usual finite element method.

To present the d.o.f., we first introduce a scaled monomial $\mathbb{M}_r(D)$ on a d-dimensional domain D

$$\mathbb{M}_r(D) := \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_c}{h_D} \right)^{\boldsymbol{s}}, |\boldsymbol{s}| \le r \right\} \tag{5}$$

where h_D is the diameter of D, $\boldsymbol{x}_c$ the centroid of D, and r a non-negative integer. When D is a polygon, $\boldsymbol{x}_c$ is chosen as the average of coordinates of all vertices of D and thus $|\boldsymbol{x} - \boldsymbol{x}_c| \le h_D$ for all $\boldsymbol{x} \in D$.

We then introduce the dual space

$$\mathcal{X}_{k,l}(K) = \operatorname{span}\left\{ \chi_a, \chi_e^{k-2}, \chi_K^l \right\}, \tag{6}$$

where the functional vectors are

- χ_a: the values at the vertices of K;
- χ_e^{k-2}: the moments on edges up to degree $k-2$

$$\chi_e(v) = |e|^{-1}(m, v)_e \quad \forall m \in \mathbb{M}_{k-2}(e), \forall \text{ edge } e \subset \partial K;$$

- χ_K^l: the moments on element K up to degree l

$$\chi_K(v) = |K|^{-1}(m, v)_K \quad \forall m \in \mathbb{M}_l(K).$$

The verification

$$(V_{k,l}(K))' = \mathcal{X}_{k,l}(K), \tag{7}$$

is called unisovlence and has been established in [3]. See also [5] for a shorter proof.

Remark 2.1 The operator Δ used in the definition of VEM space (4) can be replaced by other operators as long as the space $V_{k,l}(K)$ contains a polynomial space with appropriate degree, which ensures the approximation property. For example, when K is triangulated to form a triangulation $\mathcal{T}_K$, we can introduce a standard k-th order Lagrange element space $S_k(\mathcal{T}_K)$ on $\mathcal{T}_K$ and impose $\Delta_h v \in \mathbb{P}_l(K)$ where Δ_h is the standard Galerkin discretization of Δ related to $S_k(\mathcal{T}_K)$. From this point of view, VEM is similar to a certain kind of subgrid upscaling.

Relabel the d.o.f. by a single index $i = 1, 2, \ldots, N_{k,l} := \dim V_{k,l}(K)$. Associated with each d.o.f., there exists a basis $\{\phi_j\}$ of $V_{k,l}(K)$ such that $\chi_i(\phi_j) = \delta_{ij}$ for $i, j = 1, \ldots, N_{k,l}$. Then every function $v \in V_{k,l}(K)$ can be expanded as

$$v(x) = \sum_{i=1}^{N_{k,l}} \chi_i(v)\phi_i(x)$$

and in numerical computation it can be identified as a vector $\boldsymbol{v} \in \mathbb{R}^{N_{k,l}}$ in the form

$$\boldsymbol{v} = (\chi_1(v), \chi_2(v), \dots, \chi_{N_{k,l}}(v))^{\intercal}.$$

The isomorphism can be denoted by

$$\boldsymbol{\chi} : V_{k,l}(K) \to \mathbb{R}^{N_{k,l}}, \quad \boldsymbol{\chi}(v) = (\chi_1(v), \chi_2(v), \dots, \chi_{N_{k,l}}(v))^{\intercal}.$$

The inverse of this isomorphism is denoted by

$$\Phi : \mathbb{R}^{N_{k,l}} \to V_{k,l}(K), \quad \Phi(\boldsymbol{v}) = \boldsymbol{\phi} \cdot \boldsymbol{v},$$

if the basis is treated as a vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_{N_{k,l}})^{\intercal}$.

Among different choices of the index (k, l) in $V_{k,l}(K)$, the first VEM space in [3] is

$$V_k(K) := V_{k,k-2}(K). \tag{8}$$

Later on, in order to compute the L^2-projection of VEM functions, a larger space is introduced in [2] with the form

$$\widetilde{V}_k(K) := V_{k,k}(K), \tag{9}$$

from which a new VEM space is given by

$$W_k(K) := \left\{ w \in \widetilde{V}_k(K) : \left(w - \Pi_k^{\nabla} w, q^* \right)_K = 0 \quad \forall q^* \in \mathbb{M}_k(K) \backslash \mathbb{M}_{k-2}(K) \right\}, \tag{10}$$

where Π_k^{∇} stands for the H^1-projection to $\mathbb{P}_k(K)$, defined in the next section. The spaces $V_k(K)$ and $W_k(K)$ are different but share the same d.o.f. For the same vector $\boldsymbol{v} \in \mathbb{R}^{N_{k,k-2}}$, we can then have different functions $\Phi_V(\boldsymbol{v}) \in V_k(K)$ and $\Phi_W(\boldsymbol{v}) \in W_k(K)$ and in general $\Phi_V(\boldsymbol{v}) \neq \Phi_W(\boldsymbol{v})$.

Function spaces in each element are used to design a H^1-conforming virtual element space on the whole domain Ω in the standard way. Concretely speaking, given a polygon mesh $\mathcal{T}_h$ of Ω and a given integer $k \geq 1$, we define

$$\begin{aligned}
V_h^{k,l} &= \left\{ v \in H^1(\Omega) : v|_K \in V_{k,l}(K) \quad \forall K \subset \mathcal{T}_h \right\}, \\
V_h &= \left\{ v \in H^1(\Omega) : v|_K \in V_k(K) \quad \forall K \subset \mathcal{T}_h \right\}, \\
\widetilde{V}_h &= \left\{ v \in H^1(\Omega) : v|_K \in \widetilde{V}_k(K) \quad \forall K \subset \mathcal{T}_h \right\}, \\
W_h &= \left\{ v \in H^1(\Omega) : v|_K \in W_k(K) \quad \forall K \subset \mathcal{T}_h \right\}.
\end{aligned}$$

The d.o.f. can be defined for the global space in the natural way.

For the pure diffusion problem, the choice of V_h is enough to produce numerical solutions with optimal accuracy. However, when dealing with second order elliptic equations with lower-order terms (e.g., reaction-diffusion problems), the use of the function spaces W_h and $\widetilde{V}_h$ are more efficient (see [2]).

2.3 Approximate stiffness matrix

A conforming virtual finite element space $V_h^0 := V_h \cap H_0^1(\Omega)$ is chosen to discretize (3). We cannot, however, compute the Galerkin projection of u to V_h^0 since the traditional way of computing $a(u_h, v_h)$ using numerical quadrature requires point-wise information of functions and their gradient inside each element. In virtual element methods, only d.o.f is enough to assemble an approximated stiffness matrix.

Define a local H^1 projection $\Pi_k^\nabla : H^1(K) \to \mathbb{P}_k(K)$ as follows: given $v \in H^1(K)$, let $\Pi_k^\nabla v \in \mathbb{P}_k(K)$ satisfy

$$\left(\nabla \Pi_k^\nabla v, \nabla p\right)_K = (\nabla v, \nabla p)_K \quad \text{for all } p \in \mathbb{P}_k(K).$$

The right hand side can be written as

$$(\nabla v, \nabla p)_K = -(v, \Delta p)_K + \langle v, n \cdot \nabla p \rangle_{\partial K}.$$

When v is in a VEM space with $l \geq k-2$ (including $V_k(K)$, $\widetilde{V}_k(K)$ or $W_k(K)$), the above quantity can be computed using d.o.f. of v since, for $p \in \mathbb{P}_k(K)$, $\Delta p \in \mathbb{P}_{k-2}(K)$ and $\nabla p \cdot n \in \mathbb{P}_{k-1}(e)$, $e \in \partial K$. The operator Π_k^∇ can be naturally extended to the global space $V_h^{k,l}$ in an element-wise way.

As $(\nabla \cdot, \nabla \cdot)$ is only semi-positive definite, a constraint should be imposed to eliminate the constant kernel. When Π_k^∇ is applied to a VEM function, we shall choose the constraint

$$\int_K v \,\mathrm{d}x = \int_K \Pi_k^\nabla v \,\mathrm{d}x \quad \text{if } l \geq 0$$

or in the lowest order case

$$\int_{\partial K} v \,\mathrm{d}s = \int_{\partial K} \Pi_k^\nabla v \,\mathrm{d}s \quad \text{if } l = -1.$$

Both constraints can be expressed in terms of the d.o.f. of a VEM function.

For later uses, let us next recall the following Poincaré–Friedrichs inequality for $v \in H_0^1(K)$

$$\|v\|_{0,K} \leq h_K \|\nabla v\|_{0,K}, \tag{11}$$

and the following result established in [9].

Lemma 2.2 (Poincaré–Friedrichs inequality [9]) *The following Poincaré–Friedrichs inequality holds*

$$\left\| v - \Pi_k^\nabla v \right\|_{0,K} \lesssim h_K \left\| \nabla \left(v - \Pi_k^\nabla v \right) \right\|_{0,K} \quad \forall v \in H^1(K). \tag{12}$$

The scaling factor h_K is not presented in the form in [9] but can be easily obtained by the following scaling argument. The transformation $\hat{\boldsymbol{x}} = (\boldsymbol{x} - \boldsymbol{x}_c)/h_K$ is applied on $\boldsymbol{x} \in K$, so that $\hat{K}$, the image of K, is contained in the unit disk. The transformed triangulation $\mathcal{T}_{\hat{K}}$ is still shape regular so that we can apply results in [9]. Then the constant h_K can be obtained by scaling back to K. As pointed out in [9], the generic constant depends only on the shape regularity not the quasi-uniformity of the triangulation $\mathcal{T}_K$.

With the help of the projection operator Π_k^{∇}, the first part of the approximated stiffness matrix of the virtual element method can be formed from the following bilinear form

$$a\left(\Pi_k^{\nabla} u, \Pi_k^{\nabla} v\right).$$

2.4 Stabilization

The approximate bilinear form $a(\Pi_k^{\nabla} u, \Pi_k^{\nabla} v)$ alone does not lead to a stable method, since it is not coercive in general, and hence a stabilization term should be added correspondingly. To ensure the stability while maintaining the accuracy, the following assumptions on the element-wise stabilization term $S_K(\cdot, \cdot)$ are imposed in VEM (cf. [3]).

- k-consistency: for $p_k \in \mathbb{P}_k(K)$

$$S_K(p_k, v) = 0 \quad \forall v \in V_h.$$

- stability:

$$S_K(\tilde{u}, \tilde{u}) \eqsim (\nabla \tilde{u}, \nabla \tilde{u})_K \quad \forall \tilde{u} \in \left(I - \Pi_k^{\nabla}\right) V_h.$$

We then define

$$a_h(u, v) := a\left(\Pi_k^{\nabla} u, \Pi_k^{\nabla} v\right) + \sum_{K \in \mathcal{T}_h} S_K(u, v).$$

Now, we are ready to propose a VEM discretization of (3) as follows.

Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, \Pi_h v_h) \quad \forall v_h \in V_h, \tag{13}$$

where $\Pi_h v_h = \Pi_1^{\nabla} v_h$ for $k = 1$ and $\Pi_h v_h = \Pi_{k-2}^0 v_h$ for $k \geq 2$.

It is mentioned that VEMs are a family of numerical methods different in the choice of stabilization terms. The k-consistency implies the above method passes the usual *Patch Test*, *i.e.*, if $u \in \mathbb{P}_k(\Omega)$, then

$$a(u, v_h) = a_h(u, v_h) \quad \text{for all } v_h \in V_h.$$

The stability implies

$$a(v, v) \eqsim a_h(v, v) \quad \text{for all } v \in V_h.$$

An abstract error estimate of VEM with stabilization satisfying k-consistency and stability is given in [3]. See also [6,11] for recent progress along this line.

In the continuous level, a stabilization term can be a scaled L^2-inner product

$$h_K^{-2}\left(u-\Pi_k^{\nabla}u, v-\Pi_k^{\nabla}v\right)_K. \tag{14}$$

The k-consistency is obvious as Π_k^{∇} preserves polynomials of degree $\leq k$. The stability can be proved using an inverse inequality and the Poincaré–Friedrichs type inequality, and it will be proved rigorously later on.

In the implementation, the stabilization (14) is realized as

$$S_{\chi}(u,v) := \chi\left(\left(I-\Pi_k^{\nabla}\right)u\right)\cdot\chi\left(\left(I-\Pi_k^{\nabla}\right)v\right), \tag{15}$$

that is, the l^2-inner product of the d.o.f. vectors is used to approximate the L^2-inner product of the functions involved. The scaling factor h_K^{-2} is absorbed into the definition of d.o.f. through the scaling of the monomials [cf. (5)]. The norm equivalence of l^2 and L^2 norm is well-known for standard finite element spaces. Rigorous justification for functions in VEM spaces will be established in Sect. 4 (see also Lemma 4.9 in [15]).

3 Inverse inequalities

In this section we shall establish the inverse inequality

$$\|\nabla v\|_{0,K} \leq C h_K^{-1}\|v\|_{0,K} \quad \text{for all } v\in V_{k,l}(K).$$

As previously mentioned in the introduction, one approach is to use the fact that all norms are equivalent on a finite dimensional space like $V_{k,l}(K)$. However, this argument cannot show the dependence of the generic constant C on the geometric nature of K. To overcome this difficulty, we shall derive the inequality with the help of a shape regular and quasi-uniform 'virtual triangulation' $\mathcal{T}_K$ and using the fact that $\Delta v\in\mathbb{P}_l$.

Note that if the definition of virtual element spaces is modified by using the discrete Laplacian operator (cf. Remark 2.1), then the inverse inequality is trivially true as now the function in VEM space is a finite element function on the virtual triangulation.

We first establish an inverse inequality for polynomial spaces on polygons.

Lemma 3.1 (Inverse inequality of polynomial spaces on a polygon) *There holds*

$$\|g\|_{0,K} \lesssim h_K^{-i}\|g\|_{-i,K} \quad \textit{for all } g\in\mathbb{P}_k,\ i=1,2.$$

Proof Restricted to one triangle $\tau\in\mathcal{T}_K$, g is a polynomial. Using the scaling argument, one has $\|g\|_{0,\tau}\lesssim h_\tau^{-i}\|g\|_{-i,\tau}$, for $i=1,2$. According to the definition of a dual norm, we easily know $\|g\|_{-i,\tau} \leq \|g\|_{-i,K}$. Therefore

$$\|g\|_{0,K}^2 = \sum_{\tau \in \mathcal{T}_K} \|g\|_{0,\tau}^2 \lesssim \sum_{\tau \in \mathcal{T}_K} h_\tau^{-2i} \|g\|_{-i,\tau}^2 \lesssim h_K^{-2i} \|g\|_{-i,K}^2,$$

as required. □

Let $S_k(\mathcal{T}_K)$ be the standard continuous k-th order Lagrange finite element space on $\mathcal{T}_K$ and $S_k^0(\mathcal{T}_K) := S_k(\mathcal{T}_K) \cap H_0^1(K)$. Define $Q_K : V_{k,l}(K) \to S_k(\mathcal{T}_K)$ as follows:

1. $Q_K v|_{\partial K} = v|_{\partial K}$;
2. $(Q_K v, \phi)_K = (v, \phi)_K$ for all $\phi \in S_k^0(\mathcal{T}_K)$.

That means, the projection function preserves the boundary value of the original VEM function, and the interior nodal values are further determined by the orthogonality conditions imposed. Now, let us prove the following stability result of Q_K.

Lemma 3.2 (Weighted stability of Q_K) *For any $\epsilon > 0$, there holds*

$$h_K^{1/2} \|Q_K v\|_{0,\partial K} + \|Q_K v\|_{0,K} \lesssim (1 + \epsilon^{-1}) \|v\|_{0,K} + \epsilon h_K \|\nabla v\|_{0,K}, \quad v \in V_{k,l}(K),$$

where the generic constant is independent of the parameter ϵ.

Proof First of all, write $Q_K v = v_{\partial,h} + v_{0,h}$, where $v_{\partial,h}$ is a function in $S_k(\mathcal{T}_K)$ which vanishes on the interior nodes of $S_k(\mathcal{T}_K)$ and is equal to v on ∂K. It is evident that $v_{0,h} = Q_K v - v_{\partial,h} \in S_k^0(\mathcal{T}_K)$. Therefore

$$(Q_K v, Q_K v)_K = (Q_K v, v_{\partial,h})_K + (Q_K v, v_{0,h})_K =: \mathrm{I}_1 + \mathrm{I}_2.$$

The first term can be bounded by

$$\mathrm{I}_1 \leq \|Q_K v\|_{0,K} \|v_{\partial,h}\|_{0,K}.$$

By the definition of Q_K, the second term can be bounded as

$$\mathrm{I}_2 = (v, v_{0,h})_K \leq \|v\|_{0,K} \|v_{0,h}\|_{0,K} \leq \|v\|_{0,K} \left(\|v_{\partial,h}\|_{0,K} + \|Q_K v\| \right).$$

Hence, we have by Young's inequality that

$$\|Q_K v\|_{0,K} \lesssim \|v\|_{0,K} + \|v_{\partial,h}\|_{0,K}. \tag{16}$$

So the key is to estimate the boundary term $\|v_{\partial,h}\|_{0,K}$. For a boundary edge e, denote by τ_e the triangle in $\mathcal{T}_K$ with e as an edge. By the definition of $v_{\partial,h}$, we have

$$\|v_{\partial,h}\|_{0,K}^2 \eqsim \sum_{e \subset \partial K} \|v_{\partial,h}\|_{0,\tau_e}^2 \lesssim \sum_{e \subset \partial K} \|v_{\partial,h}\|_{0,e}^2 h_e = \sum_{e \subset \partial K} \|v\|_{0,e}^2 h_e,$$

where, in the derivation of the last step, we also use the fact $v_{\partial,h}|_{\partial K} = Q_K v|_{\partial K} = v|_{\partial K}$.

On the other hand, for a bounded domain ω with Lipschitz boundary, the estimate $\|v\|_{0,\partial\omega}^2 \lesssim \|v\|_{0,\omega}\|\nabla v\|_{0,\omega}$ holds for any $v \in H^1(\omega)$ (see [10]). Hence, it follows from the scaling argument and Young's inequality that on each triangle τ_e, there holds the following weighted trace estimate

$$h_e\|v\|_{0,e}^2 \lesssim \epsilon^{-2}\|v\|_{0,\tau_e}^2 + \epsilon^2 h_e^2\|\nabla v\|_{0,\tau_e}^2.$$

Summing over $e \subset \partial K$ and taking square root yield

$$\|v_{\partial,h}\|_{0,K} \lesssim h_K^{1/2}\|Q_K v\|_{0,\partial K} \lesssim \epsilon^{-1}\|v\|_{0,K} + \epsilon h_K\|\nabla v\|_{0,K}, \tag{17}$$

from which and (16) the desired inequality for $\|Q_K v\|_{0,K}$ follows readily. The inequality (17) also implies the desired estimate for $h_K^{1/2}\|Q_K v\|_{0,\partial K}$ directly. The proof is complete. □

To develop various estimates for a function in VEM spaces, we shall separate it into two functions, related to the moment and the trace of the function, respectively.

Lemma 3.3 (An H^1-orthogonal decomposition) *Every function $v \in H^1(K)$ admits the decomposition*

$$v = v_1 + v_2,$$

where

1. $v_1 \in H^1(K)$, $v_1|_{\partial K} = v|_{\partial K}$, $\Delta v_1 = 0$ *in* K,
2. $v_2 \in H_0^1(K)$, $\Delta v_2 = \Delta v$ *in* K.

Furthermore the decomposition is H^1-orthogonal in the sense that

$$\|\nabla v\|_{0,K}^2 = \|\nabla v_1\|_{0,K}^2 + \|\nabla v_2\|_{0,K}^2.$$

Proof One can simply choose v_2 as the H^1-projection of v to $H_0^1(K)$, i.e., $v_2 \in H_0^1(K)$ satisfies the variational equation

$$(\nabla v_2, \nabla\phi)_K = (\nabla v, \nabla\phi)_K \quad \text{for all } \phi \in H_0^1(K),$$

and then set $v_1 = v - v_2$. Equivalently, one can set v_1 to be a harmonic function in K which has the same boundary value of v, and then let $v_2 = v - v_1$. □

For the harmonic part, we have the following inequality.

Lemma 3.4 (A weighted inequality of the harmonic part of a VEM function) *For any function $v \in V_{k,l}(K)$, let $v_1 \in H^1(K)$, $v_1|_{\partial K} = v|_{\partial K}$, $\Delta v_1 = 0$ in K. Then for any $\epsilon > 0$, there holds*

$$\|\nabla v_1\|_{0,K} \lesssim h_K^{-1}(1+\epsilon^{-1})\|v\|_{0,K} + \epsilon\|\nabla v\|_{0,K},$$

where the generic constant is independent of ϵ.

Proof Using the fact $\Delta v_1 = 0$ in K, one has

$$\|\nabla v_1\|_{0,K} = \inf_{w \in H^1(K), w|_{\partial K} = v_1|_{\partial K}} \|\nabla w\|_{0,K}. \tag{18}$$

Observe that $Q_K v|_{\partial K} = v_1|_{\partial K}$ and $Q_K v \in H^1(K)$. Therefore, from the principle of energy minimization (18), the inverse inequality for functions in $S_k(\mathcal{T}_K)$, and the weighted stability of Q_K, it follows that

$$\|\nabla v_1\|_{0,K} \leq \|\nabla (Q_K v)\|_{0,K} \lesssim h_K^{-1} \|Q_K v\|_{0,K} \leq h_K^{-1}(1+\epsilon^{-1})\|v\|_{0,K} + \epsilon \|\nabla v\|_{0,K},$$

as required. □

We now estimate the second part in the decomposition.

Lemma 3.5 (Inverse inequality of non-zero moments part) *For any function $v \in V_{k,l}(K)$, let $v_2 \in H_0^1(K)$ satisfies $\Delta v_2 = \Delta v$ in K. Then*

$$\|\nabla v_2\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K}.$$

Proof As $v_2 \in H_0^1(K)$ and $\Delta v_2 = \Delta v$ in K, applying the integration by parts yields

$$\|\nabla v_2\|_{0,K}^2 = -(\Delta v_2, v_2)_K = -(\Delta v, v_2)_K \leq \|\Delta v\|_{0,K} \|v_2\|_{0,K},$$

which along with the Poincaré–Friedrichs inequality (11) for $v_2 \in H_0^1(\mathrm{K})$ implies

$$\|\nabla v_2\|_{0,K} \leq h_K \|\Delta v\|_{0,K}. \tag{19}$$

For $v \in V_{k,l}(K)$, one can apply the inverse inequality to $\Delta v \in \mathbb{P}_l$:

$$\|\Delta v\|_K \lesssim h_K^{-2} \|\Delta v\|_{-2,K} \leq h_K^{-2} \|v\|_{0,K}. \tag{20}$$

Hence, the combination of (19) and (20) immediately leads to the desired estimate. □

Now, we summarize our main result in this section as follows.

Theorem 3.6 (Inverse inequality of a VEM function) *The following inverse inequality holds:*

$$\|\nabla v\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K} \quad \textit{for all } v \in V_{k,l}(K).$$

Proof By Lemmas 3.4 and 3.5, one has

$$\|\nabla v\|_{0,K} \leq \|\nabla v_1\|_{0,K} + \|\nabla v_2\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K} + \epsilon \|\nabla v\|_{0,K}.$$

Choose ϵ small enough and absorb the term $\epsilon \|\nabla v\|_{0,K}$ to the left hand side to get the desired inverse inequality. □

As an application of the inverse inequality, we prove the L^2-stability of the projection operators Q_K and Π_k^{∇} restricted to VEM spaces.

Corollary 3.7 (L^2-stability of Q_K) *The operator $Q_K : V_{k,l}(K) \to S_k(\mathcal{T}_K)$ is L^2-stable, i.e.,*

$$\|Q_K v\|_{0,K} \lesssim \|v\|_{0,K} \quad \textit{for all } v \in V_{k,l}(K).$$

Proof Simply apply the inverse inequality to bound $h_K \|\nabla v\|_{0,K} \lesssim \|v\|_{0,K}$ in Lemma 3.2 to get the desired result. □

Corollary 3.8 (L^2-stability of Π_k^{∇}) *Let k, l be two positive integers and $l \geq k-2$. The operator $\Pi_k^{\nabla} : V_{k,l}(K) \to \mathbb{P}_k(K)$ is L^2-stable, i.e.,*

$$\left\|\Pi_k^{\nabla} v\right\|_{0,K} \lesssim \|v\|_{0,K}, \quad \textit{for all } v \in V_{k,l}(K).$$

Proof By the triangle inequality and the Poincaré–Friedrichs inequality, we have

$$\left\|\Pi_k^{\nabla} v\right\|_{0,K} \leq \|v\|_{0,K} + \left\|v - \Pi_k^{\nabla} v\right\|_{0,K} \lesssim \|v\|_{0,K} + h_K \left\|\nabla\left(v - \Pi_k^{\nabla} v\right)\right\|_{0,K}.$$

Then by the H^1-stability of Π_k^{∇} and the inverse inequality

$$h_K \left\|\nabla\left(v - \Pi_k^{\nabla} v\right)\right\|_{0,K} \lesssim h_K \|\nabla v\|_{0,K} \lesssim \|v\|_{0,K}.$$

The proof is thus completed. □

4 Norm equivalence

We shall prove the norm equivalence between L^2-norm of a VEM function and l^2-norm of the corresponding vector representation using d.o.f. In light of this result, we are able to derive two stabilization methods used in VEM formulation.

4.1 Norm equivalence of polynomial spaces on a polygon

We begin with a norm equivalence of polynomial spaces on polygons.

Lemma 4.1 (Norm equivalence of polynomial spaces on a polygon) *Let $g = \sum_\alpha g_\alpha m_\alpha$ be a polynomial on K. Denote by $\boldsymbol{g} = (g_\alpha)$ the coefficient vector. Then the following norm equivalence holds*

$$h_K \|\boldsymbol{g}\|_{l^2} \lesssim \|g\|_{0,K} \lesssim h_K \|\boldsymbol{g}\|_{l^2}.$$

Proof The inequality $\|g\|_{0,K} \lesssim h_K \|\boldsymbol{g}\|_{l^2}$ is straightforward. As $\boldsymbol{x}_c$ is the average of coordinates of all vertices of the polygon, we have $\|m_\alpha\|_{\infty,K} \leq 1$ and thus $\|m_\alpha\|_{0,K} \lesssim h_K$. Then by the triangle inequality and the Cauchy–Schwarz inequality,

$$\|g\|_{0,K} \leq \sum_\alpha |g_\alpha| \|m_\alpha\|_{0,K} \lesssim h_K \|\boldsymbol{g}\|_{l^2}.$$

The proof of the lower bound $h_K \|\boldsymbol{g}\|_{l^2} \lesssim \|g\|_{0,K}$ is technical. Again the standard scaling argument cannot be applied since there is no reference polygon. Instead we choose a circle S_τ inside a triangle $\tau \in \mathcal{T}_K$ such that the radius satisfies $r_\tau = \delta h_K$, where the constant $\delta \in (0, 1)$ depending only on the shape regularity and quasi-uniformity of the triangulation $\mathcal{T}_K$. After applying an affine map $\hat{\boldsymbol{x}} = (\boldsymbol{x} - \boldsymbol{x}_c)/h_K$, the transformed circle $\hat{S}_\tau$ with radius δ is contained in the unit disk centered at the origin. As $S_\tau \subset K$, we have

$$\|g\|_{0,K} \geq \|g\|_{0,S_\tau} = \|\hat{g}\|_{0,\hat{S}_\tau} h_K, \tag{21}$$

where $\hat{g}(\hat{\boldsymbol{x}}) := g(\boldsymbol{x})$. Let $\hat{M}_{ij} = \int_{\hat{S}_\tau} \hat{m}_i \hat{m}_j \, \mathrm{d}\hat{x}$ and $\hat{M} = (\hat{M}_{ij})$. Then

$$\|\hat{g}\|^2_{0,\hat{S}_\tau} = \boldsymbol{g}^\intercal \hat{M} \boldsymbol{g} \geq \lambda_{\min}(\hat{M}) \|\boldsymbol{g}\|^2_{l^2}, \tag{22}$$

where $\lambda_{\min}(\hat{M})$ denotes the smallest eigenvalue of the mass matrix $\hat{M}$. It is evident to check that the entry $\hat{M}_{ij}$ of the mass matrix is a continuous function of the center $\boldsymbol{c}$ of the circle $\hat{S}_\tau$. Hence, we simply write $\lambda_{\min}(\hat{M})$ as $\lambda_{\min}(\boldsymbol{c})$, which is also continuous with respect to $\boldsymbol{c}$. On the other hand, by the construction, $\boldsymbol{c}$ is contained in the unit disk. We then let $\lambda^* = \min_{\boldsymbol{c},|\boldsymbol{c}|\leq 1} \lambda_{\min}(\boldsymbol{c})$ and obtain a uniform bound $\|\hat{g}\|^2_{0,\hat{S}_\tau} \geq \lambda^* \|\boldsymbol{g}\|^2_{l^2}$. Notice that after the scaling, the proof is done on a reference circle and thus the constant λ^* depends only on the radius δ of $\hat{S}_\tau$.

Combining (21) and (22), the following desired inequality is obtained with a constant depending only on the shape regularity and quasi-uniform constants of the triangulation $\mathcal{T}_K$:

$$h_K \|\boldsymbol{g}\|_{l^2} \lesssim \|g\|_{0,K},$$

as required. □

4.2 Norm equivalence for VEM spaces

In this subsection, we are going to prove the norm equivalence of the L^2-norm of VEM functions to the l^2-norm of their corresponding d.o.f. vectors.

Lemma 4.2 (Lower bound) *For any $v \in V_{k,l}(K)$, the following estimate holds:*

$$h_K \|\boldsymbol{\chi}(v)\|_{l^2} \lesssim \|v\|_{0,K}.$$

Proof The d.o.f.s are grouped into two categories: $\boldsymbol{\chi}_{\partial K}(\cdot)$ are d.o.f.s associated with the boundary of K, and $\boldsymbol{\chi}_K(\cdot)$ are moments in K.

Restricted to the boundary, $v|_{\partial K} \in \mathbb{B}_k(K)$ consists of standard Lagrange elements. A standard scaling argument yields

$$h_K \|\boldsymbol{\chi}_{\partial K}(v)\|_{l^2} \eqsim h_K^{1/2} \|v\|_{0,\partial K}.$$

Apply the weighted trace theorem in Lemma 3.2, and the inverse inequality of functions in VEM spaces to obtain

$$h_K^{1/2} \|v\|_{0,\partial K} \lesssim \|v\|_{0,K} + h_K \|\nabla v\|_{0,K} \lesssim \|v\|_{0,K}.$$

For the d.o.f.s of interior moments, applying the Cauchy–Schwarz inequality gives

$$|K|^{-1} \int_K vm \,\mathrm{d}x \le |K|^{-1} \|v\|_{0,K} \|m\|_{0,K} \lesssim h_K^{-1} \|v\|_{0,K} \quad \text{for all } m \in \mathbb{M}_l(K).$$

Combining the estimate of $\boldsymbol{\chi}_{\partial K}(\cdot)$ and $\boldsymbol{\chi}_K(\cdot)$ finishes the proof. $\square$

The proof of the estimate of the upper bound turns out to be technical. Again we shall use the H^1 decomposition presented in Lemma 3.3.

Lemma 4.3 (Upper bound for the harmonic part) *For any $v \in V_{k,l}(K)$, let $v_1 \in H^1(K)$ satisfy $v_1|_{\partial K} = v|_{\partial K}$ and $\Delta v_1 = 0$ in K. Then*

$$\|v_1\|_{0,K} \lesssim h_K \|\boldsymbol{\chi}_{\partial K}(v)\|_{l^2}.$$

Proof By the construction v_1 can be written as

$$v_1 = \sum_{i=1}^{N_{\partial K}} \chi_i(v_1)\phi_i(x),$$

where $\{\phi_i|_{\partial K}\} \subset \mathbb{B}_k(\partial K)$ is a dual basis of $\boldsymbol{\chi}_{\partial K}$ on the boundary and $\Delta\phi_i = 0$ inside K. By the Cauchy–Schwarz inequality, it suffices to prove $\|\phi_i\|_{0,K} \lesssim h_K$.

Restricting ϕ_i to the boundary, one can use the scaling argument for each edge and conclude $\|\phi_i\|_{\infty,\partial K} \lesssim 1$. As ϕ_i is harmonic, by the maximum principle, $\|\phi_i\|_{\infty,K} \le \|\phi_i\|_{\infty,\partial K} \lesssim 1$. Then $\|\phi_i\|_{0,K} \lesssim h_K$ follows. $\square$

Lemma 4.4 (Upper bound for the moment part) *For any $v \in V_{k,l}(K)$, let $v_2 \in H_0^1(K)$ satisfy $\Delta v_2 = \Delta v$ in K. Then*

$$\|v_2\|_{0,K} \lesssim h_K \|\boldsymbol{\chi}(v)\|_{l^2}.$$

Proof Let $g = -\Delta v = -\Delta v_2$. Then by integration by parts

$$\|\nabla v_2\|_{0,K}^2 = -(\Delta v_2, v_2)_K = (g, v_2)_K = (g, v)_K - (g, v_1)_K. \tag{23}$$

Expand g in the basis m_α, *i.e.* $g = \sum_\alpha g_\alpha m_\alpha$ and denote by $\boldsymbol{g} = (g_\alpha)$. Then by the Cauchy–Schwarz inequality and the norm equivalence for g in Lemma 4.1, one has

$$(g, v)_K = |K| \sum_\alpha g_\alpha \chi_\alpha(v) \lesssim h_K^2 \|\boldsymbol{g}\|_{l^2} \|\boldsymbol{\chi}_K(v)\|_{l^2} \lesssim h_K \|g\|_{0,K} \|\boldsymbol{\chi}_K(v)\|_{l^2}.$$

An upper bound of $\|\nabla v_2\|_{0,K}$ is then obtained by substituting the above estimate into (23):

$$\|\nabla v_2\|_{0,K}^2 \lesssim h_K \|\Delta v_2\|_{0,K} \left(\|\boldsymbol{\chi}_K(v)\|_{l^2} + \|v_1\|_{0,K}\right) \lesssim \|\nabla v_2\|_{0,K} \|\boldsymbol{\chi}(v)\|_{l^2}, \quad (24)$$

i.e.,

$$\|\nabla v_2\|_{0,K} \lesssim \|\boldsymbol{\chi}(v)\|_{l^2}.$$

In the derivation of (24), we have also used the inverse inequality and the upper bound for v_1 established in Lemma 4.3 and the inverse inequality for $\Delta v_2 \in \mathbb{P}_{k-2}$ (cf. Lemma 3.1).

Finally the proof is completed by using the Poincaré–Friedrichs inequality $\|v_2\|_{0,K} \lesssim h_K \|\nabla v_2\|_{0,K}$ for $v_2 \in H_0^1(K)$. □

In summary, the following theorem holds.

Theorem 4.5 (Norm equivalence between L^2 and l^2-norms) *For any $v \in V_{k,l}(K)$, the following norm equivalence holds*

$$h_K \|\boldsymbol{\chi}(v)\|_{l^2} \lesssim \|v\|_{0,K} \lesssim h_K \|\boldsymbol{\chi}(v)\|_{l^2}.$$

For functions in space $V_k(K)$, Theorem 4.5 can be applied directly. For space $W_k(K) \subset V_{k,k}(K)$, if Theorem 4.5 is applied to functions in $V_{k,k}(K)$, additional moments in $\boldsymbol{\chi}_K^k \backslash \boldsymbol{\chi}_K^{k-2}$ are involved. Henceforth we shall show that no additional moments are required for $W_k(K)$.

Corollary 4.6 (Norm equivalence between L^2 and l^2-norms for $W_k(K)$) *For any $v \in W_k(K)$, the following norm equivalence holds:*

$$h_K \|\boldsymbol{\chi}(v)\|_{l^2} \lesssim \|v\|_{0,K} \lesssim h_K \|\boldsymbol{\chi}(v)\|_{l^2}.$$

Proof The lower bound $h_K \|\boldsymbol{\chi}(v)\|_{l^2} \lesssim \|v\|_{0,K}$ is trivial, since $W_k(K)$ is a subspace of $V_{k,k}(K)$, and the d.o.f.s in $V_{k,k}(K)$, comparing with that of $W_k(K)$, contain additional moments with weights $\boldsymbol{\chi}_K^k \backslash \boldsymbol{\chi}_K^{k-2}$. To prove the upper bound, it suffices to bound these additional moments by the other degrees of freedom.

By the definition of $W_k(K)$,

$$(v, m)_K = \left(\Pi_k^\nabla v, m\right)_K \quad \text{for all } m \in \mathbb{M}_k(K) \backslash \mathbb{M}_{k-2}(K).$$

Thus, by the Cauchy–Schwarz inequality and the bound $\|m\|_{0,K} \lesssim h_K$, it suffices to bound $\|\Pi_k^\nabla v\|_{0,K}$. Using the d.o.f.s of $v \in W_k(K)$, we can find another function $\tilde{v} \in V_k(K)$ such that $\boldsymbol{\chi}(\tilde{v}) = \boldsymbol{\chi}(v)$. Notice that the projection Π_k^∇ is uniquely determined by the d.o.f.s, so

$$\Pi_k^\nabla v = \Pi_k^\nabla \tilde{v}.$$

Then by the L^2-stability of Π_k^∇ in Corollary 3.8 and the norm equivalence for $\tilde{v} \in V_k(K)$, we obtain

$$\left\|\Pi_k^\nabla v\right\|_{0,K} = \left\|\Pi_k^\nabla \tilde{v}\right\|_{0,K} \lesssim \|\tilde{v}\|_{0,K} \lesssim h_K\|\boldsymbol{\chi}(\tilde{v})\|_{l^2} = h_K\|\boldsymbol{\chi}(v)\|_{l^2}.$$

With the above estimate in mind, we have, for $\chi \in \boldsymbol{\chi}_K^k \backslash \boldsymbol{\chi}_K^{k-2}$,

$$|\chi(v)| = |K|^{-1}|(v, m)_K| \lesssim h_K^{-1}\left\|\Pi_k^\nabla v\right\|_{0,K} \lesssim \|\boldsymbol{\chi}(v)\|_{l^2}.$$

The proof is complete. □

4.3 Norm equivalence of VEM formulation

With Theorem 4.5, we can obtain the following stability result.

Theorem 4.7 (Norm equivalence for stabilization using Π_k^∇) *For $v \in V_k(K)$ or $W_k(K)$, the following norm equivalence holds*

$$\|\nabla v\|_{0,K}^2 \eqsim \left\|\nabla \Pi_k^\nabla v\right\|_{0,K}^2 + \left\|\boldsymbol{\chi}\left(v - \Pi_k^\nabla v\right)\right\|_{l^2}^2.$$

Proof By the definition of Π_k^∇, the orthogonality holds:

$$\|\nabla v\|_{0,K}^2 = \left\|\nabla \Pi_k^\nabla v\right\|_{0,K}^2 + \left\|\nabla\left(v - \Pi_k^\nabla v\right)\right\|_{0,K}^2. \tag{25}$$

Using the inverse inequality and norm equivalence for L^2-norm, one can obtain

$$\left\|\nabla\left(v - \Pi_k^\nabla v\right)\right\|_{0,K} \lesssim h_K^{-1}\left\|v - \Pi_k^\nabla v\right\|_{0,K} \lesssim \left\|\boldsymbol{\chi}\left(v - \Pi_k^\nabla v\right)\right\|_{l^2}.$$

It is noted that for $v \in W_k(K) \subset V_{k,k}(K)$, additional moments in $\boldsymbol{\chi}_K^k \backslash \boldsymbol{\chi}_K^{k-2}$ are involved when the norm equivalence is applied for functions $V_{k,k}(K)$. However, these moments vanish for $v - \Pi_k^\nabla v$, according to the definition of $W_k(K)$.

To prove the lower bound, we shall apply the Poincaré–Friedrichs inequality in Lemma 2.2 and the lower bound in the norm equivalence to get

$$\left\|\boldsymbol{\chi}\left(v - \Pi_k^\nabla v\right)\right\|_{l^2} \lesssim h_K^{-1}\left\|v - \Pi_k^\nabla v\right\|_{0,K} \lesssim \left\|\nabla\left(v - \Pi_k^\nabla v\right)\right\|_{0,K},$$

as required. □

Following [2], we introduce the L^2-projection $\Pi_k^0 : W_k(K) \to \mathbb{P}_k(K)$ and verify the stability of another stabilization using Π_k^0. For moments up to $k-2$, the d.o.f.s of VEM function $v \in W_k(K)$ can be used, and $\Pi_k^\nabla v$ is used for higher moments. That is: given $v \in W_k(K)$, define $\Pi_k^0 v \in \mathbb{P}_k(K)$ such that

$$\begin{cases} \left(\Pi_k^0 v, m\right)_K = (v, m)_K & \text{for all } m \in \mathbb{P}_{k-2}(K), \\ \left(\Pi_k^0 v, m\right)_K = \left(\Pi_k^\nabla v, m\right)_K & \text{for all } m \in \mathbb{P}_k(K) \backslash \mathbb{P}_{k-2}(K). \end{cases}$$

Using the slice operator $I - \Pi_k^0$, the stabilization can be reduced to the d.o.f.s on the boundary only.

Corollary 4.8 (Norm equivalence for stabilization using Π_k^0) *For $v \in W_k(K)$, the following norm equivalence holds*

$$\|\nabla v\|_{0,K}^2 \eqsim \left\|\nabla \Pi_k^\nabla v\right\|_{0,K}^2 + \left\|\boldsymbol{\chi}_{\partial K}\left(v - \Pi_k^0 v\right)\right\|_{l^2}^2.$$

Proof As both Π_k^∇ and Π_k^0 preserve polynomial of degree k, $(I - \Pi_k^0)v = (I - \Pi_k^0)(I - \Pi_k^\nabla)v$ and $(I - \Pi_k^\nabla)v = (I - \Pi_k^\nabla)(I - \Pi_k^0)v$.

Using the stability of Π_k^∇ in H^1-seminorm and the inverse inequality for VEM functions, we get

$$\begin{aligned} \left\|\nabla\left(I - \Pi_k^\nabla\right)v\right\|_{0,K} &= \left\|\nabla\left(I - \Pi_k^\nabla\right)\left(I - \Pi_k^0\right)v\right\|_{0,K} \le \left\|\nabla\left(I - \Pi_k^0\right)v\right\|_{0,K} \\ &\lesssim h_k^{-1}\left\|\left(I - \Pi_k^0\right)v\right\|_{0,K}. \end{aligned}$$

Going backwards, using the approximation property of the L^2-projection yields

$$\left\|\left(I - \Pi_k^0\right)v\right\|_{0,K} = \left\|\left(I - \Pi_k^0\right)\left(I - \Pi_k^\nabla\right)v\right\|_{0,K} \lesssim h_K\left\|\nabla\left(I - \Pi_k^\nabla\right)v\right\|_{0,K}.$$

In summary, the following norm equivalence is obtained

$$h_K^{-1}\left\|\left(I - \Pi_k^0\right)v\right\|_{0,K} \eqsim \left\|\nabla\left(I - \Pi_k^\nabla\right)v\right\|_{0,K}.$$

Furthermore, observing that the moment d.o.f.s $\boldsymbol{\chi}_K$ for $v - \Pi_k^0 v$ vanish, we have from Theorem 4.5 that $h_K^{-1}\|(I - \Pi_k^0)v\|_{0,K}$ is equivalent to $\|\boldsymbol{\chi}_{\partial K}(v - \Pi_k^0 v)\|_{l^2}$. This combined with (25) implies the desired result readily. $\square$

Remark 4.9 An L^2-projection Π_k^0 to $\mathbb{P}_k(K)$ can be defined using moments d.o.f.s of a VEM function in $V_{k,k}(K)$. Given a function $v \in V_{k,k}(K)$, denote by $v^0 = \Pi_k^0 v$ and $v^b = v|_{\partial K}$, then (v^b, v^0) is a variant of the so-called weak function introduced in the weak Galerkin methods (cf. [24]). The stabilization term can be formulated as

$$\left(\boldsymbol{\chi}_{\partial K}(u^b - u^0), \boldsymbol{\chi}_{\partial K}(v^b - v^0)\right).$$

The approximated gradient $\nabla\Pi_k^\nabla v$ is indeed a variant of a weak gradient of the weak function (v^b, v^0). It is also equivalent to a special version of HDG: the embedded discontinuous Galerkin method (cf. [18,22]).

5 Interpolation error estimates

In this section, we shall provide interpolation error estimates for several interpolations to VEM spaces. The following projection and interpolants of a function $v \in H^1(K) \cap C^0(\bar{K})$ are used in this section:

- $v_\pi \in \mathbb{P}_k(K)$: the L^2 projection of v to the polynomial space;
- $v_c \in S_k(\mathcal{T}_K)$: the standard nodal interpolant to finite element space $S_k(\mathcal{T}_K)$ based on the auxiliary triangulation $\mathcal{T}_K$ of K;
- $v_I \in V_k(K)$ defined as the solution of the local problem

$$\Delta v_I = \Delta v_\pi \text{ in } K, \quad v_I = v_c \text{ on } \partial K.$$

- $I_K v \in V_k(K)$ defined by d.o.f., i.e.,

$$I_K v = v_c \text{ on } \partial K, \quad (I_K v, p)_K = (v, p)_K, \ \forall p \in \mathbb{P}_{k-2}(K).$$

- $I_K^W v \in W_k(K)$ defined by d.o.f., i.e.,

$$I_K^W v = v_c \text{ on } \partial K, \quad (I_K^W v, p)_K = (v, p)_K, \ \forall p \in \mathbb{P}_{k-2}(K).$$

Error estimates of v_π and v_c are well known (see e.g. [10]): for $w_K = v_c$ or v_π

$$\|v - w_K\|_{0,K} + h_K|v - w_K|_{1,K} \lesssim h_K^{k+1}\|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K). \tag{26}$$

Remark 5.1 Error estimate for v_π is usually presented for a star-shaped domain but can be generalized to a domain which is a union of star shaped sub-domains (see [21]). Under Assumption **A1**, the polygon K satisfies the previous condition, so the estimate (26) holds for $w_K = v_\pi$.

The following error estimate can be found in [23, Proposition 4.2]. For completeness, we present a shorter proof by comparing v_I with v_c.

Lemma 5.2 (Interpolation error estimate of u_I) *The following optimal order error estimate holds:*

$$\|v - v_I\|_{0,K} + h_K|v - v_I|_{1,K} \lesssim h_K^{k+1}\|v\|_{k+1,K} \quad \forall v \in H^{k+1}(K). \tag{27}$$

Proof By the triangle inequality, it suffices to estimate the difference $v_I - v_c \in H_0^1(K)$. By the Poincaré–Friedrichs inequality $\|v\|_{0,K} \le h_K\|\nabla v\|_{0,K}$ for $v \in H_0^1(K)$, it suffices to bound the H^1-seminorm of $v_I - v_c$.

Recalling the definition of v_I and noting $v_I - v_c \in H_0^1(K)$, we have

$$(\nabla v_I, \nabla(v_I - v_c))_K = (\nabla v_\pi, \nabla(v_I - v_c))_K.$$

Therefore

$$\|\nabla(v_I - v_c)\|_{0,K}^2 = (\nabla(v_\pi - v_c), \nabla(v_I - v_c))_K.$$

By the Cauchy–Schwarz inequality and the triangle inequality, there holds

$$\|\nabla(v_I - v_c)\|_{0,K} \le \|\nabla(v_\pi - v_c)\|_{0,K} \le \|\nabla(v - v_c)\|_{0,K} + \|\nabla(v - v_\pi)\|_{0,K}.$$

The desired result then follows readily from error estimates for v_c and v_π together. □

Now we estimate $v - I_K v$ by comparing $I_K v$ with v_I.

Theorem 5.3 (Interpolation error estimate of $I_K v$) *For $v \in H^{k+1}(K)$, the following optimal order error estimate holds in both L^2 and H^1-norm*

$$\|v - I_K v\|_{0,K} + h_K |v - I_K v|_{1,K} \lesssim h_K^{k+1} \|v\|_{k+1,K}.$$

Proof By the triangle inequality and error estimate on v_I in (27), it suffices to estimate $v_I - I_K v \in H_0^1(K)$ as follows:

$$\begin{aligned}(\nabla(v_I - I_K v), \nabla(v_I - I_K v))_K &= -(\Delta(v_I - I_K v), v_I - I_K v)_K \\ &= (\Delta(v_I - I_K v), v - v_I)_K \\ &\le \|\Delta(v_I - I_K v)\|_{0,K} \|v - v_I\|_{0,K} \\ &\lesssim h_K^{-1} \|\nabla(v_I - I_K v)\|_{0,K} h_K^{k+1} \|v\|_{k+1,K}.\end{aligned}$$

The first step involves integration by parts and the fact $v_I - I_K v \in H_0^1(K)$. The term $(\Delta(v_I - I_K v), v - I_K v) = 0$ is due to $\Delta(v_I - I_K v) \in \mathbb{P}_{k-2}(K)$ and the moment preservation of the canonical interpolation. The last step uses the inverse inequality for $\Delta(v_I - I_K v) \in \mathbb{P}_{k-2}(K)$ in (3.1) and error estimate of $v - v_I$ in (27). The desired error estimate then follows from canceling one $\|\nabla(v_I - I_K v)\|_{0,K}$. □

Next, we present the interpolation error estimate of $v - I_K^W v$ by comparing $I_K^W v$ with $I_K v$.

Theorem 5.4 (Interpolation error estimate of $I_K^W v$) *For $v \in H^{k+1}(K)$, the optimal order error estimate holds in both L^2 and H^1-norm*

$$\left\| v - I_K^W v \right\|_{0,K} + h_K \left\| \nabla \left(v - I_K^W v \right) \right\|_{0,K} \lesssim h_K^{k+1} \|v\|_{k+1,K}.$$

Proof Again by the triangle inequality and the obtained error estimate for $v - I_K v$, it suffices to estimate $I_K^W v - I_K v \in H_0^1(K)$. A crucial observation is that both interpolants, although in different VEM spaces, share the same d.o.f., i.e., $\boldsymbol{\chi}(I_K^W v) = \boldsymbol{\chi}(I_K v)$. Therefore $\Pi_k^\nabla I_K^W v = \Pi_k^\nabla I_K v = \Pi_k^\nabla v$.

Using the norm equivalence in Theorem 4.7, we have:

$$
\begin{aligned}
\left\| \nabla \left(I_K^W v - I_K v \right) \right\|_{0,K} &\le \left\| \nabla \left(I - \Pi_k^\nabla \right) I_K^W v \right\|_{0,K} + \left\| \nabla \left(I - \Pi_k^\nabla \right) I_K v \right\|_{0,K} \\
&\lesssim \left\| \boldsymbol{\chi} \left(I - \Pi_k^\nabla \right) I_K^W v \right\|_{l^2} + \left\| \nabla \left(I - \Pi_k^\nabla \right) I_K v \right\|_{0,K} \\
&= \left\| \boldsymbol{\chi} \left(I - \Pi_k^\nabla \right) I_K v \right\|_{l^2} + \left\| \nabla \left(I - \Pi_k^\nabla \right) I_K v \right\|_{0,K} \\
&\lesssim \left\| \nabla \left(I - \Pi_k^\nabla \right) I_K v \right\|_{0,K} \\
&\lesssim \left\| \nabla \left(v - \Pi_k^\nabla v \right) \right\|_{0,K} + \| \nabla (v - I_K v) \|_{0,K} \\
&\lesssim \| \nabla (v - I_K v) \|_{0,K} ,
\end{aligned}
$$

as required. □

Remark 5.5 Notice that the norm equivalence to $I_K^W v - I_K v$ cannot be applied directly since they are in different spaces. Here we use the relations $\Pi_k^\nabla I_K^W v = \Pi_k^\nabla I_K v = \Pi_k^\nabla v$ and $\boldsymbol{\chi}(I_K^W v) = \boldsymbol{\chi}(I_K v)$ as a bridge to switch the estimate for $I_K^W v$ to that of $I_K v$.

6 Conclusion and future work

In this paper we have established the inverse inequality, norm equivalence between the norm of a virtual element function and its degrees of freedom, and interpolation error estimates for several VEM spaces on a polygon which admits a virtual quasi-uniform triangulation, i.e., Assumption **A1**.

We note that **A1** rules out polygons with high aspect ratio. Equivalently the constant is not robust to the aspect ratio of K. For example, a rectangle K with two sides $h_{\max}$ and $h_{\min}$. It can be decomposed into union of shape regular rectangles but the number depends on the aspect ratio $h_{\max}/h_{\min}$. In numerical simulation, however, VEM is also robust to the aspect ratio of the elements. In a forthcoming paper, we will examine anisotropic error analysis of VEM based on certain maximum angle conditions.

We present our proofs in two dimensions but it is possible to extend the techniques to three dimensions. The outline is given as follows. Given a polyhedral region K, we need to assume **A1** holds for each face $F \subset \partial K$ and are able to prove results restricted to each face. Then we assume **A1** holds for K and prove results as for the 2-D case. It is our ongoing study to develop the details in this case.

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