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# FINITE ELEMENTS FOR DIV- AND DIVDIV-CONFORMING SYMMETRIC TENSORS IN ARBITRARY DIMENSION*

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**Abstract.** Several div-conforming and divdiv-conforming finite elements for symmetric tensors on simplexes in arbitrary dimension are constructed in this work. The shape function space is first split as the trace space and the bubble space. The later is further decomposed into the null space of the differential operator and its orthogonal complement. Instead of characterizations of these subspaces of the shape function space, characterizations of corresponding degrees of freedom in the dual spaces are provided. Vector div-conforming finite elements are first constructed as an introductory example. Then new symmetric div-conforming finite elements are constructed. The dual subspaces are then used as build blocks to construct new divdiv-conforming finite elements.

**Key words.** symmetric tensor, div-conforming finite elements, divdiv-conforming finite elements, space decomposition, dual approach

**MSC codes.** 65N30, 74S05

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## 1. Introduction.

In this paper we construct div-conforming finite elements and divdiv-conforming finite elements for symmetric tensors on simplexes in arbitrary dimension. A finite element on a geometric domain $K$ is defined as a triple $(K, V, \mathrm{DoF})$ by Ciarlet in [18], where $V$ is a finite-dimensional space consisting of the so-called shape functions and the set of degrees of freedom (DoFs) is a basis of the dual space $V'$. The shape functions are usually polynomials. The key is to identify an appropriate basis of $V'$ to enforce the continuity of the functions across the boundary of elements so that the global finite element space is a subspace of some Sobolev space $H(\mathrm{d}, \Omega)$, where $\Omega \subset \mathbb{R}^d$ is a domain and $\mathrm{d}$ is a generic differential operator.

Denote by $\mathrm{tr}^{\mathrm{d}}$ the trace operator associated to d and the bubble function space $\mathbb{B}(\mathrm{d}) := \ker(\mathrm{tr}^{\mathrm{d}}) \cap V$. We shall decompose $V = \mathbb{B}(\mathrm{d}) \oplus \mathcal{E}(\mathrm{img}(\mathrm{tr}^{\mathrm{d}}))$, where $\mathcal{E}$ is an injective extension operator $\mathcal{E} : \mathrm{img}(\mathrm{tr}^{\mathrm{d}}) \to V$, and find DoFs of each subspace by

1. characterization of $(\mathrm{img}(\mathrm{tr}^{\mathrm{d}}))'$ using the Green's formula;
2. characterization of $\mathbb{B}'(\mathrm{d})$ through the polynomial complexes.

In the characterization of $\mathbb{B}'(\mathrm{d})$, we will use the differential operator d to further split $\mathbb{B}(\mathrm{d})$ into two subspaces (see Figure 1.1)

$$E_0 := \mathbb{B}(\mathrm{d}) \cap \ker(\mathrm{d}) \quad \text{and} \quad E_0^{\perp} := \mathbb{B}(\mathrm{d})/E_0.$$

We then present a basis of $(E_0^{\perp})'$ and $E_0'$:

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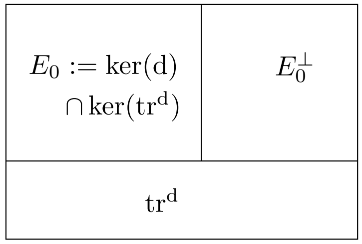


FIG. 1.1. *Decomposition of a generic finite element space.*

1. a basis of $(E_0^{\perp})'$ is given by $\{(\mathrm{d}\cdot, p),\ p \in \mathrm{d}\mathbb{B}(\mathrm{d}) = \mathrm{d}E_0^{\perp}\}$;
2. on the other part $E_0'$, there are two approaches:
   - the primary approach: $E_0$ is the image of the previous bubble space;
   - the dual approach: $E_0'$ is isomorphic to the null space of a Koszul operator.

The dual approach is simpler and more general. For example, for the elasticity complex, the previous symmetric tensor space is related to the second order differential operator inc [3]. In the dual approach, we prove that a basis of $E_0'$ is given by $\mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}))$. Here to simplify notation, we introduce operator $\mathcal{N}: U \to V'$ as $\mathcal{N}(p) := (\cdot, p)$ with $U \subseteq V$ and $(\cdot,\cdot)$ is the inner product of space $V$ which is usually the $L^2$-inner product. Generalization of inc and its bubble function space to $\mathbb{R}^d$ is unclear while $E_0' = \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}))$ holds in arbitrary dimension.

To show the main idea with easy examples, we first review the construction of the Brezzi–Douglas–Marini (BDM) element [8, 7] and the Raviart–Thomas (RT) element [27, 25] for $H(\mathrm{div})$-conforming elements. For the BDM element, the shape function space is $\mathbb{P}_k(K;\mathbb{R}^d)$, and for the RT element, it is $\mathbb{P}_{k+1}^-(K;\mathbb{R}^d) := \mathbb{P}_k(K;\mathbb{R}^d) \oplus \mathbb{H}_k(K)\boldsymbol{x}$. We determine the trace space $\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{R}^d)) = \prod_{F\in\partial K} \mathbb{P}_k(F)$. By the aid of the space decomposition $\mathbb{P}_{k-1}(K;\mathbb{R}^d) = \operatorname{grad}\mathbb{P}_k(K) \oplus \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)$ derived from the dual complex, we can show $E_0' = \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d))$. BDM and RT elements will share the same trace space and $E_0$, while

$$(E_0^{\perp})' = \begin{cases} \mathcal{N}(\operatorname{grad}\mathbb{P}_{k-1}(K)) & \text{for BDM element,} \\ \mathcal{N}(\operatorname{grad}\mathbb{P}_{k}(K)) & \text{for RT element.} \end{cases}$$

The dual space $\mathbb{B}_k'(\mathrm{div}, K) \cong (E_0^{\perp})' \oplus E_0'$ for the BDM element can be further merged as

$$\mathbb{B}_k'(\mathrm{div}, K) = \mathcal{N}\big(\mathrm{ND}_{k-2}(K)\big) := \mathcal{N}\big(\mathbb{P}_{k-2}(K;\mathbb{R}^d) \oplus \mathbb{H}_{k-2}(K;\mathbb{K})\boldsymbol{x}\big).$$

We summarize DoFs for the BDM element as

$$(\boldsymbol{v}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_k(F) \text{ for each } F \in \partial K, \tag{1.1}$$

$$(\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(K), \tag{1.2}$$

and the interior moments (1.2) for $\mathbb{B}_k'(\mathrm{div}, K)$ can be further split as

$$\begin{aligned} (E_0^{\perp})' \qquad & (\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{grad}\mathbb{P}_{k-1}(K), \\ (E_0)' \qquad & (\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d). \end{aligned} \tag{1.3}$$

Enriching (1.3) to $\mathcal{N}(\operatorname{grad}\mathbb{P}_k(K))$, we then get the RT element.

We then apply our approach to a more challenging problem: $H(\mathrm{div})$-conforming finite elements for symmetric tensors, which are used in the mixed finite element methods for the stress-displacement formulation of the elasticity system. Several $H(\mathrm{div})$-conforming finite elements for symmetric tensors were designed in [6, 1, 3, 24, 21, 23] on simplices, but our elements are new and construction is more systematical. Let $\Pi_F\boldsymbol{\tau}$ be the projection of column vectors of $\boldsymbol{\tau}$ to the plane $F$, and let $\mathcal{F}^r(K)$ be the set of subsimplexes of $K$ with co-dimension $r$ for $r = 1, \ldots, d-1$. The space of shape functions is $\mathbb{P}_k(K;\mathbb{S})$, and DoFs are

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{1.4}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1, \tag{1.5}$$

$$(\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K), \tag{1.6}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(K;\mathbb{S}).$$

The symmetry of the shape function and the trace $\boldsymbol{\tau}\boldsymbol{n}$ on $(d-1)$-dimensional faces lead to the DoFs (1.4)–(1.5), which will determine the normal-normal component $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$. The set of DoF (1.6) is for the face bubble part of the tangential-normal component $\Pi_F\boldsymbol{\tau}\boldsymbol{n}$ (cf. (1.2)), which differs from that of Hu's element in [21] for $d \geq 3$. The bubble function space $\mathbb{B}_k(\mathrm{div}, K;\mathbb{S})$ can be decomposed into two parts, $E_{0,k}(\mathbb{S}) := \mathbb{B}_k(\mathrm{div}, K;\mathbb{S}) \cap \ker(\mathrm{div})$ and $E_{0,k}^{\perp}(\mathbb{S}) := \mathbb{B}_k(\mathrm{div}, K;\mathbb{S})/E_{0,k}(\mathbb{S})$. We show that

$$E'_{0,k}(\mathbb{S}) = \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})), \quad (E_{0,k}^{\perp}(\mathbb{S}))' = \mathcal{N}(\operatorname{def}\mathbb{P}_{k-1}(K,\mathbb{R}^d)). \tag{1.7}$$

A new family of $H(\mathrm{div};\mathbb{S})$-conforming elements is devised with the shape function space $\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S}) + E_{0,k+1}^{\perp}(\mathbb{S})$, and we enrich DoF $(E_{0,k}^{\perp}(\mathbb{S}))'$ to $(E_{0,k+1}^{\perp}(\mathbb{S}))'$ so that $\operatorname{div}\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) = \mathbb{P}_k(K;\mathbb{R}^d)$.

Motivated by the recent construction [22] in two and three dimensions, the previous $H(\mathrm{div})$-conforming finite elements for symmetric tensors are then revised to acquire $H(\mathrm{div}\,\mathrm{div}) \cap H(\mathrm{div})$-conforming finite elements for symmetric tensors in arbitrary dimension. Using the building blocks in the BDM element and the $H(\mathrm{div})$-conforming $\mathbb{P}_k(K;\mathbb{S})$ element, we construct the following DoFs:

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K),$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1,$$

$$(\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K),$$

$$(\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}, p)_F \quad \forall\, p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^1(K), \tag{1.8}$$

$$(\boldsymbol{\tau}, \operatorname{def}\boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-3}(K), \tag{1.9}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}). \tag{1.10}$$

The DoF (1.8) to enforce $\operatorname{div}\boldsymbol{\tau}$ is $H(\mathrm{div})$-conforming and thus $\boldsymbol{\tau} \in H(\mathrm{div}\,\mathrm{div}) \cap H(\mathrm{div})$. DoF (1.10) is $E'_{0,k}(\mathbb{S})$ shown in (1.7) and (1.8)–(1.9) are a further decomposition of $\operatorname{div} E_{0,k}^{\perp}(\mathbb{S})$ by the trace-bubble decomposition of the BDM element; cf. (1.1)–(1.2).

We then modify this element slightly to get $H(\mathrm{div}\,\mathrm{div})$-conforming symmetric finite elements generalizing the $H(\mathrm{div}\,\mathrm{div})$-conforming element in two and three dimensions [12, 11]. The DoFs are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{1.11}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\dots,r, \text{ and } r = 1,\dots,d-1, \tag{1.12}$$

$$(\Pi_F \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K), \tag{1.13}$$

$$(\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau} + \operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), p)_F \quad \forall\, p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^1(K), \tag{1.14}$$

$$(\boldsymbol{\tau}, \operatorname{def}\boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-3}(K), \qquad (\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}). \tag{1.15}$$

As we mentioned before, (1.11)–(1.13) will determine the trace $\boldsymbol{\tau}\boldsymbol{n}$, and consequently $\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n})$. The only difference is that (1.8) is replaced by (1.14), which agrees with a trace operator of the div div operator derived in [11, 12]. Such modification is from the requirement of $H(\operatorname{div}\operatorname{div})$-conformity: $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$ and $\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau} + \operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n})$ are continuous. Therefore (1.13) for $\Pi_F\boldsymbol{\tau}\boldsymbol{n}$ is considered as a local DoF to $K$, i.e., it is not single-valued across simplices.

In our recent work [11, 12], we have constructed $H(\operatorname{div}\operatorname{div})$-conforming symmetric finite elements for $d = 2, 3$. The dual space $(\operatorname{tr}^{\operatorname{div}\operatorname{div}}(\mathbb{P}_k(K;\mathbb{S})))'$ is given by DoFs (1.11)–(1.14) but without (1.13) as $\Pi_F\boldsymbol{\tau}\boldsymbol{n}$ is not part of the trace of div div operator. Let $E_0(\operatorname{div}\operatorname{div},\mathbb{S}) := \mathbb{B}_k(\operatorname{div}\operatorname{div}, K;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div})$ and $E_0^{\perp}(\operatorname{div}\operatorname{div},\mathbb{S}) := \mathbb{B}_k(\operatorname{div}\operatorname{div}, K;\mathbb{S})/E_0(\operatorname{div}\operatorname{div},\mathbb{S})$. Then the characterization

$$(E_0^{\perp}(\operatorname{div}\operatorname{div},\mathbb{S}))' = \mathcal{N}(\nabla^2\mathbb{P}_{k-2}(K))$$

is easy, but the identification of $E_0'(\operatorname{div}\operatorname{div},\mathbb{S})$ is very tricky in three dimensions. In [12], we have used the primary approach to get

$$E_0'(\operatorname{div}\operatorname{div},\mathbb{S}) = \mathcal{N}(\operatorname{sym}\operatorname{curl}\mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})),$$

which is hard to generalize to an arbitrary dimension. When using the dual approach, it turns out $\mathcal{N}(\ker(\boldsymbol{x}^{\intercal}\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{S}))$ is a strict subspace of $E_0'(\operatorname{div}\operatorname{div},\mathbb{S})$ as the dimensions cannot match. An extra DoF on one face $(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n}\times\boldsymbol{x}q)_{F_1}$, $q \in \mathbb{P}_{k-2}(F_1)$, is introduced to fill the gap. Again such a fix in three dimensions seems not easy to generalize to an arbitrary dimension. In (1.15), if we further decompose $\mathrm{ND}_{k-3}(K) = \operatorname{grad}\mathbb{P}_{k-2}(K) \oplus \mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}$, based on our new element, we can obtain a characterization

$$\begin{aligned} E_0'(\operatorname{div}\operatorname{div},\mathbb{S}) = {} & \cup_{F\in\mathcal{F}^1(K)}\mathcal{N}(\mathrm{ND}_{k-2}(F)) \\ & \oplus \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})) \oplus \mathcal{N}(\operatorname{def}\mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}). \end{aligned}$$

Furthermore, a new family of $\mathbb{P}_{k+1}^{-}(K;\mathbb{S})$ type $H(\operatorname{div}\operatorname{div}) \cap H(\operatorname{div})$-conforming and $H(\operatorname{div}\operatorname{div})$-conforming finite elements are developed. The shape function space is enriched to $\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(K)$. The range $\operatorname{div}\operatorname{div}\mathbb{P}_{k+1}^{-}(K;\mathbb{S})$ is enriched to $\mathbb{P}_{k-1}(K)$ and so is $(E_0^{\perp}(\operatorname{div}\operatorname{div},\mathbb{S}))' = \mathcal{N}(\nabla^2\mathbb{P}_{k-1}(K))$. But the trace DoFs and $E_0'(\operatorname{div}\operatorname{div},\mathbb{S})$ are unchanged. Such $\mathbb{P}_{k+1}^{-}(K;\mathbb{S})$ type div div-conforming elements for symmetric tensors are new and not easy to construct without exploring the decomposition of the dual spaces.

The rest of this paper is organized as follows. Preliminaries are given in section 2. The construction of $H(\operatorname{div})$-conforming elements is presented in section 3. In section 4, new $H(\operatorname{div})$-conforming elements for symmetric tensors are designed. And construction of $H(\operatorname{div}\operatorname{div})$-conforming elements is shown in section 5.

## 2. Preliminary.

**2.1. Notation.** Let $K \subset \mathbb{R}^d$ be a nondegenerated $d$-dimensional simplex. For $r = 1, 2, \ldots, d$, denote by $\mathcal{F}^r(K)$ the set of all $(d-r)$-dimensional faces of $K$. The superscript $r$ in $\mathcal{F}^r(K)$ represents the co-dimension of a $(d-r)$-dimensional face $f$, where $F$ is reserved for the $(d-1)$-dimensional face and $f$ for a generic lower-dimensional face. Set $\mathcal{V}(K) := \mathcal{F}^d(K)$ as the set of vertices. Similarly, for $f \in \mathcal{F}^r(K)$, define

$$\mathcal{F}^1(f) := \{e \in \mathcal{F}^{r+1}(K) : e \subset \partial f\}.$$

For any $f \in \mathcal{F}^r(K)$ with $1 \le r \le d-1$, let $\boldsymbol{n}_{f,1}, \ldots, \boldsymbol{n}_{f,r}$ be its mutually perpendicular unit normal vectors, and let $\boldsymbol{t}_{f,1}, \ldots, \boldsymbol{t}_{f,d-r}$ be its mutually perpendicular unit tangential vectors. We abbreviate $\boldsymbol{n}_{F,1}$ as $\boldsymbol{n}_F$ or $\boldsymbol{n}$ when $r = 1$. We also abbreviate $\boldsymbol{n}_{F,i}$ and $\boldsymbol{t}_{F,i}$ as $\boldsymbol{n}_i$ and $\boldsymbol{t}_i$, respectively, if not causing any confusion. For any $F \in \mathcal{F}^1(K)$ and $e \in \mathcal{F}^1(F)$, denote by $\boldsymbol{n}_{F,e}$ the unit outward normal to $\partial F$ being parallel to $F$.

Given a face $F \in \mathcal{F}^1(K)$, and a vector $\boldsymbol{v} \in \mathbb{R}^d$, define

$$\Pi_F \boldsymbol{v} = (\boldsymbol{n}_F \times \boldsymbol{v}) \times \boldsymbol{n}_F = (\boldsymbol{I} - \boldsymbol{n}_F \boldsymbol{n}_F^{\intercal})\boldsymbol{v}$$

as the projection of $\boldsymbol{v}$ onto the face $F$. For a matrix $\boldsymbol{\tau} \in \mathbb{R}^{d\times d}$, $\Pi_F \boldsymbol{\tau}$ is applied to each column vector of $\boldsymbol{\tau}$. Given a scalar function $v$, define the surface gradient as

$$\nabla_F v := \Pi_F \nabla v = \nabla v - \frac{\partial v}{\partial n_F}\boldsymbol{n}_F = \sum_{i=1}^{d-1} \frac{\partial v}{\partial t_{F,i}} \boldsymbol{t}_{F,i},$$

namely the projection of $\nabla v$ to the face $F$, which is independent of the choice of the normal vectors. Denote by $\operatorname{div}_F$ the corresponding surface divergence.

**2.2. Polynomial spaces.** We recall some results about polynomial spaces on a bounded and topologically trivial domain $D \subset \mathbb{R}^d$. Without loss of generality, we assume $\boldsymbol{0} \in D$. Given a nonnegative integer $k$, let $\mathbb{P}_k(D)$ stand for the set of all polynomials in $D$ with the total degree no more than $k$, and let $\mathbb{P}_k(D; \mathbb{X})$ denote the tensor or vector version. Let $\mathbb{H}_k(D) := \mathbb{P}_k(D) \backslash \mathbb{P}_{k-1}(D)$ be the space of homogeneous polynomials of degree $k$. Recall that

$$\dim \mathbb{P}_k(D) = \binom{k+d}{d} = \binom{k+d}{k}, \quad \dim \mathbb{H}_k(D) = \binom{k+d-1}{d-1} = \binom{k+d-1}{k}$$

for a $d$-dimensional domain $D$.

By Euler's formula, we have

$$\boldsymbol{x} \cdot \nabla q = kq \quad \forall\, q \in \mathbb{H}_k(D), \tag{2.1}$$

$$\operatorname{div}(\boldsymbol{x} q) = (k+d)q \quad \forall\, q \in \mathbb{H}_k(D) \tag{2.2}$$

for integer $k \ge 0$.

**2.3. Dual spaces.** Consider a Hilbert space $V$ with the inner product $(\cdot, \cdot)$. Let $U \subseteq V$, then define $\mathcal{N} : U \to V'$ as follows: for any $p \in U$, $\mathcal{N}(p) \in V'$ is given by

$$\langle \mathcal{N}(p), \cdot \rangle = (\cdot, p).$$

When $V$ is a subspace of an ambient Hilbert space $W$, we use the inclusion $\mathcal{I} : V \hookrightarrow W$ to denote the embedding of $V$ into $W$. Then the dual operator $\mathcal{I}' : W' \to V'$ is onto. That is, for any $N \in W'$, $\mathcal{I}'N \in V'$ is defined as $\langle \mathcal{I}'N, v\rangle = \langle N, \mathcal{I}v\rangle$.

Consider the case of the finite-dimensional subspace $V \subseteq W$ and a subspace $P' \subseteq W'$; then to prove $V' = \mathcal{I}'P'$, it suffices to show

$$\text{for any } v \in V, \text{ if } N(v) = 0, \forall N \in P', \text{ then } v = 0. \tag{2.3}$$

Note that it means $\mathcal{I}'$ is onto but may not be bijective. That is, $\dim P'$ might be larger than $\dim V'$. It is less rigorous to write $V' \subseteq P'$ as those two dual spaces consist of functionals with different domains. The mapping $\mathcal{I}'$ is introduced as a bridge for comparison. When $\mathcal{I}' : P' \to V'$ is a bijection, we shall skip $\mathcal{I}'$ and simply write $V' = P'$. To prove $V' = P'$, besides (2.3), dimension count is applied to verify $\dim V' = \dim P'$.

The art of designing conforming finite element spaces is indeed identifying appropriate DoFs to enforce the continuity of the function across the boundary of the elements. Take $V = \mathbb{P}_k(K)$ as an example. A naive choice is $\mathcal{N}(\mathbb{P}_k(K)) = V'$ but such basis enforces no continuity on $\partial K$. To be $H^1$-conforming we need a basis for $(\mathrm{tr}(\mathbb{P}_k(K)))'$ to ensure the continuity of the trace on lower dimensional faces of an element $K$. Note that as the shape function is a polynomial inside the element, the trace is usually smoother than its Sobolev version, which is known as supersmoothness [17, 28]. Choice of dual bases is not unique. For example, for $H^1$-conforming finite elements, $V = \mathbb{P}_k(K)$, the Lagrange element, and the Hermite element will have different bases for $V'$.

When counting the dimensions, we often use the following simple fact: for a linear operator $T$ defined on a finite-dimensional linear space $V$, it holds

$$\dim V = \dim \ker(T) + \dim \mathrm{img}(T).$$

**2.4. Simplex and barycentric coordinates.** For $i = 1, \dots, d$, denote by $\boldsymbol{e}_i \in \mathbb{R}^d$ the $d$-dimensional vector whose $j$th component is $\delta_{ij}$ for $j = 1, \dots, d$. Let $K \subset \mathbb{R}^d$ be a nondegenerated simplex with vertices $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_d$. Let $F_i \in \mathcal{F}^1(K)$ be the $(d-1)$-dimensional face opposite to vertex $\boldsymbol{x}_i$ and $\lambda_i$ be the barycentric coordinate of $\boldsymbol{x}$ corresponding to vertex $\boldsymbol{x}_i$ for $i = 0, 1, \dots, d$. Then $\lambda_i(\boldsymbol{x})$ is a linear polynomial and $\lambda_i|_{F_i} = 0$. For any subsimplex $S$ not containing $\boldsymbol{x}_i$ (and thus $S \subseteq F_i$), $\lambda_i|_S = 0$. On the other hand, for a polynomial $p \in \mathbb{P}_k(K)$, if $p|_{F_i} = 0$, then $p = \lambda_i q$ for some $q \in \mathbb{P}_{k-1}(K)$. As $F_i$ is contained in the zero level set of $\lambda_i$, $\nabla\lambda_i$ is orthogonal to $F_i$ and a simple scaling calculation shows the relation $\nabla\lambda_i = -|\nabla\lambda_i|\boldsymbol{n}_i$, where $\boldsymbol{n}_i$ is the unit outward normal to the face $F_i$ of the simplex $K$ for $i = 0, 1, \dots, d$. Clearly $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \dots, \boldsymbol{n}_d\}$ spans $\mathbb{R}^d$. We will identify its dual basis $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \dots, \boldsymbol{l}_d\}$, i.e., $(\boldsymbol{l}_i, \boldsymbol{n}_j) = \delta_{ij}$ for $i, j = 1, 2, \dots, d$. Here the index 0 is singled out for ease of notation. We can set an arbitrary vertex as the origin.

Set $\boldsymbol{t}_{i,j} := \boldsymbol{x}_j - \boldsymbol{x}_i$ for $0 \le i \ne j \le d$. By computing the constant directional derivative $\boldsymbol{t}_{i,j} \cdot \nabla\lambda_\ell$ by values on the two vertices, we have

$$\boldsymbol{t}_{i,j} \cdot \nabla\lambda_\ell = \delta_{j\ell} - \delta_{i\ell} = \begin{cases} 1 & \text{if } \ell = j, \\ -1 & \text{if } \ell = i, \\ 0 & \text{if } \ell \ne i, j. \end{cases} \tag{2.4}$$

Then it is straightforward to verify $\{\boldsymbol{l}_i := |\nabla\lambda_i|\boldsymbol{t}_{i,0}\}$ is dual to $\{\boldsymbol{n}_i\}$. Note that in general neither $\{\boldsymbol{n}_i\}$ nor $\{\boldsymbol{l}_i\}$ is an orthonormal basis unless $K$ is a scaling of the reference simplex $\hat{K}$ with vertices $\boldsymbol{0}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d$. By using the basis $\{\boldsymbol{n}_i, i = 1, 2, \dots, d\}$, we avoid the pull back from the reference simplex.

Following notation in [5], denote by $\mathbb{N}^d$ the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ with integer $\alpha_i \geq 0$ and by $\mathbb{N}_0^d$ the set of all multi-indices $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ with integer $\alpha_i \geq 0$. For $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $\alpha \in \mathbb{N}^d$, define $\boldsymbol{x}^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $|\alpha| := \sum_{i=1}^d \alpha_i$. Similarly, for $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_d)$ and $\alpha \in \mathbb{N}_0^d$, define $\boldsymbol{\lambda}^\alpha := \lambda_0^{\alpha_0}\lambda_1^{\alpha_1} \cdots \lambda_d^{\alpha_d}$ and $|\alpha| := \sum_{i=0}^d \alpha_i$. The Bernstein basis for the space $\mathbb{P}_k(K)$ consists of all monomials of degree $k$ in the variables $\lambda_i$, i.e., the basis functions are given by

$$\{\boldsymbol{\lambda}^\alpha := \lambda_0^{\alpha_0}\lambda_1^{\alpha_1} \cdots \lambda_d^{\alpha_d} : \alpha \in \mathbb{N}_0^d,\ |\alpha| = k\}.$$

Then $\mathbb{P}_k(K) = \{\sum_{\alpha \in \mathbb{N}_0^d, |\alpha|=k} c_\alpha \boldsymbol{\lambda}^\alpha : c_\alpha \in \mathbb{R}\}$.

**2.5. Tensors.** Denote by $\mathbb{S}$ and $\mathbb{K}$ the subspace of symmetric matrices and skew-symmetric matrices of $\mathbb{R}^{d\times d}$, respectively. The set of symmetric tensors $\{\boldsymbol{T}_{i,j} := \boldsymbol{t}_{i,j}\boldsymbol{t}_{i,j}^\intercal\}_{0\leq i<j\leq d}$ is dual to $\{\boldsymbol{N}_{i,j}\}_{0\leq i<j\leq d}$, where

$$\boldsymbol{N}_{i,j} := \frac{1}{2(\boldsymbol{n}_i^\intercal \boldsymbol{t}_{i,j})(\boldsymbol{n}_j^\intercal \boldsymbol{t}_{i,j})}(\boldsymbol{n}_i\boldsymbol{n}_j^\intercal + \boldsymbol{n}_j\boldsymbol{n}_i^\intercal).$$

That is, by direct calculation [9, (3.2)],

$$\boldsymbol{T}_{i,j} : \boldsymbol{N}_{k,\ell} = \delta_{ik}\delta_{j\ell}, \quad 0 \leq i < j \leq d,\ 0 \leq k < \ell \leq d,$$

where : is the Frobenius inner product of matrices. Assuming $\sum_{0\leq i<j\leq d} c_{ij}\boldsymbol{T}_{i,j} = \boldsymbol{0}$, then apply the Frobenius inner product with $\boldsymbol{N}_{k,\ell}$ to conclude $c_{k\ell} = 0$ for all $0 \leq k < \ell \leq d$. Therefore both $\{\boldsymbol{T}_{i,j}\}_{0\leq i<j\leq d}$ and $\{\boldsymbol{N}_{i,j}\}_{0\leq i<j\leq d}$ are bases of $\mathbb{S}$. The basis $\{\boldsymbol{T}_{i,j}\}_{0\leq i<j\leq d}$ is introduced in [15, 21] and $\{\boldsymbol{N}_{i,j}\}_{0\leq i<j\leq d}$ is in [15, 9].

**2.6. Characterization of DoFs for bubble spaces.** We give a characterization of DoFs for bubble spaces and a decomposition of the bubble spaces through the bubble complex.

LEMMA 2.1. *Assume finite-dimensional Hilbert spaces $\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n$ with the inner product $(\cdot,\cdot)$ form an exact Hilbert complex*

$$0 \xrightarrow{\subset} \mathbb{B}_1 \xrightarrow{\mathrm{d}_1} \ldots \mathbb{B}_i \xrightarrow{\mathrm{d}_i} \ldots \mathbb{B}_n \to 0,$$

*where $\mathbb{B}_i \subseteq \ker(\operatorname{tr}^{\mathrm{d}_i})$ for $i = 1, 2, \ldots, n-1$. Then the bubble space $\mathbb{B}_i$, for $i = 1, \ldots, n-1$, is uniquely determined by the DoFs*

$$(v, \mathrm{d}_i^* q) \quad \forall\ q \in \mathrm{d}_i\mathbb{B}_i, \tag{2.5}$$

$$(v, q) \quad \forall\ q \in \mathbb{Q} \cong (\mathrm{d}_{i-1}\mathbb{B}_{i-1})', \tag{2.6}$$

*where $\mathrm{d}_i^*$ is the adjoint of $\mathrm{d}_i : \mathbb{B}_i \to \mathbb{B}_{i+1}$ with respect to the inner product $(\cdot,\cdot)$ and the isomorphism $\mathbb{Q} \to (\mathrm{d}_{i-1}\mathbb{B}_{i-1})'$ is given by $p \to (p,\cdot)$ for $p \in \mathbb{Q}$.*

*Proof.* By the splitting lemma in [20] (see also Theorem 2.2 in [10]),

$$\mathbb{B}_i = \mathrm{d}_i^*\, \mathrm{d}_i\mathbb{B}_i \oplus \mathrm{d}_{i-1}\mathbb{B}_{i-1}. \tag{2.7}$$

Since $\mathrm{d}_i^*$ restricted to $\mathrm{d}_i\mathbb{B}_i$ is injective, the number of DoFs (2.5)–(2.6) is the same as $\dim \mathbb{B}_i$. Assume $v \in \mathbb{B}_i$ and all the DoFs (2.5)–(2.6) vanish. By the decomposition (2.7), there exist $v_1 \in \mathbb{B}_i$ and $v_2 \in \mathbb{B}_{i-1}$ such that $v = \mathrm{d}_i^*\, \mathrm{d}_i v_1 + \mathrm{d}_{i-1} v_2$. The vanishing (2.5) yields $\mathrm{d}_i v = 0$, that is, $\mathrm{d}_i\, \mathrm{d}_i^*(\mathrm{d}_i v_1) = 0$. Noting that $\mathrm{d}_i\, \mathrm{d}_i^* : \mathrm{d}_i\mathbb{B}_i \to \mathrm{d}_i\mathbb{B}_i$ is isomorphic, we get $\mathrm{d}_i v_1 = 0$ and thus $v = \mathrm{d}_{i-1} v_2$. Now apply the vanishing (2.6) to get $v = 0$. ∎

When the bubble function space $\mathbb{B}$ can be characterized precisely, we can simply use $\mathcal{N}(\mathbb{B})$, i.e., $(v,q), q \in \mathbb{B}$ as DoFs. When $\mathbb{Q} = \mathrm{d}_{i-1}\mathbb{B}_{i-1}$, Lemma 2.1 is the same as Proposition 5.44 for the finite element systems in [16]. However, Lemma 2.1 tells us it suffices to identify the dual space without knowing the explicit form of the bubble functions. In the following, we present a way to identify $\mathbb{B}'$ by a decomposition of the dual space.

LEMMA 2.2. *Consider linear map* $\mathrm{d}: V \to P$ *between two finite-dimensional Hilbert spaces sharing the same inner product* $(\cdot,\cdot)$. *Let* $\mathbb{B} = \ker(\mathrm{tr}^{\mathrm{d}}) \cap V$, $E_0 = \ker(\mathrm{d}) \cap \mathbb{B}$, *and* $E_0^{\perp} = \mathbb{B}/E_0$. *Assume*

(B1) $\mathbb{B}' = \mathcal{I}'\mathcal{N}(U)$ *for some subspace* $U \subseteq V$;

(B2) *there exists an operator* $\kappa : U \to \kappa U$ *leading to the inclusion*

$$U \subseteq \mathrm{d}^*(H(\mathrm{d}^*)) \oplus (\ker(\kappa) \cap U), \tag{2.8}$$

*where* $\mathrm{d}^*$ *is the adjoint of* $\mathrm{d}: \mathbb{B} \to \mathrm{d}\mathbb{B}$ *with respect to the inner product* $(\cdot,\cdot)$ *and can be continuously extended to the space* $H(\mathrm{d}^*)$.

*Then*

$$(E_0^{\perp})' = \mathcal{N}(\mathrm{d}^*(\mathrm{d}\mathbb{B})), \tag{2.9}$$

$$E_0' = \mathcal{I}'\mathcal{N}(\ker(\kappa) \cap U). \tag{2.10}$$

*Proof.* The characterization (2.9) is straightforward as $\mathrm{d}: E_0^{\perp} \to \mathrm{d}\mathbb{B}$ is a bijection. To prove (2.10), it suffices to show that for any $u \in E_0$, if $(u,p) = 0$ for all $p \in \ker(\kappa) \cap U$, then $u = 0$. First of all, as $u \in E_0$, $u \perp \mathrm{d}^*(H(\mathrm{d}^*))$, i.e.,

$$(u, \mathrm{d}^* p) = (\mathrm{d}u, p) = 0 \quad \forall\, p \in H(\mathrm{d}^*).$$

Combined with the assumption (B2), (2.8), we have $(u,p) = 0$ for all $p \in U$ and conclude $u = 0$ by assumption (B1) $\mathbb{B}' = \mathcal{I}'\mathcal{N}(U)$. ∎

As we mentioned before, in (2.10), $\mathcal{I}'$ could be onto. For example, one can choose $U = V$. We want to choose the smallest subspace $U$ to get $E_0' = \mathcal{N}(\ker(\kappa) \cap U)$. One guideline is the dimension count. On one hand, we have the following identity:

$$\dim E_0 = \dim \mathbb{B} - \dim E_0^{\perp} = \dim V - \dim(\mathrm{img}(\mathrm{tr}^d)) - \dim E_0^{\perp}.$$

On the other hand, we have

$$\dim(\ker(\kappa) \cap U) = \dim U - \dim(\kappa U).$$

For specific examples, we only need to figure out the dimension, not exact identification of subspaces.

Next we enrich space $V$ to derive another finite element.

LEMMA 2.3. *Consider linear map* $\mathrm{d}: V \to P$ *between two finite-dimensional Hilbert spaces sharing the same inner product* $(\cdot,\cdot)$. *Let* $\mathbb{B} = \ker(\mathrm{tr}^{\mathrm{d}}) \cap V$, $E_0 = \ker(\mathrm{d}) \cap \mathbb{B}$, *and* $E_0^{\perp} = \mathbb{B}/E_0$. *With a finite-dimensional Hilbert space* $\mathbb{H}$, *we enrich the space* $V$ *to* $V + \mathbb{H}$ *and let* $\mathbb{B}^+ = \ker(\mathrm{tr}^{\mathrm{d}}) \cap (V + \mathbb{H})$. *Assume*

(H1) $V \cap \mathbb{H} = \{0\}$ *and* $\mathrm{d}V \cap \mathrm{d}\mathbb{H} = \{0\}$;

(H2) $\mathrm{tr}^{\mathrm{d}}(\mathbb{H}) \subseteq \mathrm{tr}^{\mathrm{d}}(V)$;

(H3) $\mathrm{d}: \mathbb{H} \to \mathrm{d}\mathbb{H}$ *is bijective;*

(H4) $E_0 = \mathcal{N}(\mathbb{Q}), (E_0^{\perp})' = \mathcal{N}(\,\mathrm{d}^*\mathbb{P})$;
(H5) $\big(\,\mathrm{d}\mathbb{B}^+\big)' = \mathcal{I}'\mathcal{N}(\mathbb{P} \oplus \,\mathrm{d}\mathbb{H})$,

*where* $\mathbb{P}$ *and* $\mathbb{Q}$ *are finite-dimensional Hilbert spaces. Then*

$$(\mathbb{B}^+)' = \mathcal{N}(\mathbb{Q}) \oplus \mathcal{N}\big(\,\mathrm{d}^*(\mathbb{P} \oplus \,\mathrm{d}\mathbb{H})\big), \tag{2.11}$$

*i.e., a function* $v \in \mathbb{B}^+$ *is uniquely determined by DoFs*

$$(v,\,\mathrm{d}^*q) \quad \forall\, q \in \mathbb{P}, \tag{2.12}$$

$$(v,\,\mathrm{d}^*q) \quad \forall\, q \in \,\mathrm{d}\mathbb{H}, \tag{2.13}$$

$$(v, q) \quad \forall\, q \in \mathbb{Q}. \tag{2.14}$$

*Proof.* As $\mathrm{tr}^{\,\mathrm{d}}(\mathbb{H}) \subseteq \mathrm{tr}^{\,\mathrm{d}}(V)$, $\dim \mathbb{B}^+ - \dim \mathbb{B} = \dim \mathbb{H}$. On the other hand, since $\mathrm{d}^*\,\mathrm{d} : \mathbb{H} \to \,\mathrm{d}^*\,\mathrm{d}\mathbb{H}$ is bijective, the number of DoFs increased is also $\dim \mathbb{H}$. Thus the dimensions in (2.11) are equal. Take a $v \in \mathbb{B}^+$ and assume all the DoFs (2.12)–(2.14) vanish. Thanks to the vanishing DoFs (2.12) and (2.13), we get from (H5) that $\mathrm{d}v = 0$, which together with $\,\mathrm{d}V \cap \,\mathrm{d}\mathbb{H} = \{0\}$ implies $v \in V$. Finally $v = 0$ follows from the vanishing DoFs (2.12) and (2.14). ∎

Assumptions (H1)–(H3) are built into the construction of $\mathbb{H}$. Assumption (H4) can be verified from the characterization of $\mathbb{B}'$ in Lemma 2.2. Only (H5) requires some work. One can show the kernel of $\,\mathrm{d}$ in the bubble space remains unchanged as $E_0$ but its image is enriched. The dual space is enriched from $(\,\mathrm{d}\mathbb{B})' = \mathcal{N}(\mathbb{P})$ to $(\,\mathrm{d}\mathbb{B}^+)' = \mathcal{N}(\mathbb{P} \oplus \,\mathrm{d}\mathbb{H})$. Note that the precise characterization of $\mathbb{B}^+$ is not easy and $\mathbb{H}$ may not be in $\mathbb{B}^+$.

## 3. $\boldsymbol{H}$(div)-conforming finite elements.

In this section we shall construct the well-known $H(\mathrm{div})$-conforming finite elements: BDM [8, 7, 26] and RT elements [27, 25]. We start with this simple example to illustrate our approach and build some elementary blocks.

### 3.1. Div operator.

We begin with the following result on the div operator.

LEMMA 3.1. *Let integer* $k \geq 0$. *The mapping* $\mathrm{div} : \boldsymbol{x}\mathbb{H}_k(D) \to \mathbb{H}_k(D)$ *is bijective. Consequently* $\mathrm{div} : \mathbb{P}_{k+1}(D; \mathbb{R}^d) \to \mathbb{P}_k(D)$ *is surjective.*

*Proof.* It is a simple consequence of the Euler's formulae (2.1) and (2.2). ∎

### 3.2. Trace space.

The trace operator for $H(\mathrm{div}, K)$ space

$$\mathrm{tr}^{\mathrm{div}} : H(\mathrm{div}, K) \to H^{-1/2}(\partial K)$$

is a continuous extension of $\mathrm{tr}^{\mathrm{div}}\boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial K}$ defined on smooth functions. We then focus on the restriction of the trace operator to the polynomial space. Denote by $\mathbb{P}_k(\mathcal{F}^1(K)) := \{q \in L^2(\partial K) : q|_F \in \mathbb{P}_k(F) \text{ for each } F \in \mathcal{F}^1(K)\}$, which is a Hilbert space with inner product $\sum_{F \in \mathcal{F}^1(K)} (\cdot, \cdot)_F$. Obviously $\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K; \mathbb{R}^d)) \subseteq \mathbb{P}_k(\mathcal{F}^1(K))$. We prove it is indeed surjective.

LEMMA 3.2. *For integer* $k \geq 1$, *the mapping* $\mathrm{tr}^{\mathrm{div}} : \mathbb{P}_k(K; \mathbb{R}^d) \to \mathbb{P}_k(\mathcal{F}^1(K))$ *is onto. Consequently*

$$\dim \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K; \mathbb{R}^d)) = \dim \mathbb{P}_k(\mathcal{F}^1(K)) = (d+1)\binom{k+d-1}{k}.$$

*Proof.* By the linearity of the trace operator, it suffices to prove that for any $F_i \in \mathcal{F}^1(K)$ and any $p \in \mathbb{P}_k(F_i)$, we can find a $\boldsymbol{v} \in \mathbb{P}_k(K;\mathbb{R}^d)$ s.t. $\boldsymbol{v}\cdot\boldsymbol{n}|_{F_i} = p$ and $\boldsymbol{v}\cdot\boldsymbol{n}|_{F_j} = 0$ for other $F_j \in \mathcal{F}^1(K)$ with $j \neq i$. Without loss of generality, we can assume $i = 0$.

For any $p \in \mathbb{P}_k(F_0)$, it can be expanded in Bernstein basis $p = \sum_{\alpha\in\mathbb{N}^d,|\alpha|=k} c_\alpha \boldsymbol{\lambda}^\alpha$, which can be naturally extended to the whole simplex by the definition of barycentric coordinates. Again by the linearity, we only need to consider one generic term still denoted by $p = c_\alpha\boldsymbol{\lambda}^\alpha$ for a multi-index $\alpha \in \mathbb{N}^d, |\alpha| = k$. As $\sum_{i=1}^d \alpha_i = k > 0$, there exists an index $1 \le i \le d$ s.t. $\alpha_i \neq 0$. Then we can write $p = \lambda_i q$ with $q \in \mathbb{P}_{k-1}(K)$.

Now we let $\boldsymbol{v} = \lambda_i q\,\boldsymbol{l}_i/(\boldsymbol{l}_i,\boldsymbol{n}_0)$. By construction,

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{n}_0 &= \lambda_i q = p,\\ (\boldsymbol{v}\cdot\boldsymbol{n}_j)|_{F_j} &= \lambda_i q|_{F_j}\,(\boldsymbol{l}_i,\boldsymbol{n}_j)/(\boldsymbol{l}_i,\boldsymbol{n}_0) = 0, \quad j = 1,2,\ldots,d.\end{aligned}$$

That is, we find $\boldsymbol{v} \in \mathbb{P}_k(K;\mathbb{R}^d)$ s.t. $(\mathrm{tr}^{\mathrm{div}}\boldsymbol{v})|_{F_0} = p$ and $(\mathrm{tr}^{\mathrm{div}}\boldsymbol{v})|_{F_j} = 0$ for $j = 1,\ldots,d$. ∎

With this identification of the trace space, we clearly have $\mathcal{N}(\mathbb{P}_k(\mathcal{F}^1(K))) = (\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{R}^d)))'$, and through $(\mathrm{tr}^{\mathrm{div}})'$, we embed $\mathcal{N}(\mathbb{P}_k(\mathcal{F}^1(K))$ into $\mathbb{P}_k'(K;\mathbb{R}^d)$.

LEMMA 3.3. *Let integer $k \ge 1$. For any $\boldsymbol{v} \in \mathbb{P}_k(K;\mathbb{R}^d)$, if the DoFs*

$$(\boldsymbol{v}\cdot\boldsymbol{n}, p)_F = 0 \quad \forall\ p \in \mathbb{P}_k(F), F \in \mathcal{F}^1(K),$$

*vanish, then* $\mathrm{tr}^{\mathrm{div}}\boldsymbol{v} = 0$.

*Proof.* Due to Lemma 3.2, the dual operator $(\mathrm{tr}^{\mathrm{div}})' : \mathbb{P}_k'(\mathcal{F}^1(K)) \to \mathbb{P}_k'(K;\mathbb{R}^d)$ is injective. Taking $N = (p,\cdot)_F \in \mathbb{P}_k'(\mathcal{F}^1(K))$ for any $F \in \mathcal{F}^1(K)$ and $p \in \mathbb{P}_k(F)$, we have

$$((\mathrm{tr}^{\mathrm{div}})'N)(\boldsymbol{v}) = N(\mathrm{tr}^{\mathrm{div}}\boldsymbol{v}) = (p,\mathrm{tr}^{\mathrm{div}}\boldsymbol{v})_F = (p,\boldsymbol{n}\cdot\boldsymbol{v})_F.$$

By the assumption, we have $\boldsymbol{v} \perp \mathrm{img}((\mathrm{tr}^{\mathrm{div}})')$, which indicates $\boldsymbol{v} \in \ker(\mathrm{tr}^{\mathrm{div}})$. ∎

Another basis of $(\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{R}^d)))'$ can be obtained by a geometric decomposition of vector Lagrange elements; see [13] for details.

**3.3. Bubble space.** After we characterize the range of the trace operator, we focus on its null space. Define the polynomial bubble space

$$\mathbb{B}_k(\mathrm{div},K) = \ker(\mathrm{tr}^{\mathrm{div}}) \cap \mathbb{P}_k(K;\mathbb{R}^d).$$

As $\{\boldsymbol{n}_i, i = 1,2,\ldots,d\}$ is a basis of $\mathbb{R}^d$, it is obvious that for $k = 0$, $\mathbb{B}_0(\mathrm{div},K) = \{\boldsymbol{0}\}$. As a direct consequence of dimension count (see Lemma 3.4 below), $\mathbb{B}_1(\mathrm{div},K)$ is also the zero space.

LEMMA 3.4. *Let integer $k \ge 1$. It holds that*

$$\dim\mathbb{B}_k(\mathrm{div},K) = d\binom{k+d}{k} - (d+1)\binom{k+d-1}{k} = (k-1)\binom{k+d-1}{k}.$$

*Proof.* By the characterization of the trace space, we can count the dimension

$$\dim\mathbb{B}_k(\mathrm{div},K) = \dim\mathbb{P}_k(K;\mathbb{R}^d) - \dim\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{R}^d)),$$

as required. ∎

Next we find different bases of $\mathbb{B}'_k(\operatorname{div}, K)$. The primary approach is to find a basis for $\mathbb{B}_k(\operatorname{div}, K)$, which induces a basis of $\mathcal{N}(\mathbb{B}_k(\operatorname{div}, K))$. For example, one can show

$$\mathbb{B}_k(\operatorname{div}, K) = \sum_{0\leq i<j\leq d} \lambda_i\lambda_j\mathbb{P}_{k-2}(K)\boldsymbol{t}_{i,j} \quad \text{for } k \geq 2.$$

Verification $\lambda_i\lambda_j\mathbb{P}_{k-2}(K)\boldsymbol{t}_{i,j} \subseteq \mathbb{B}_k(\operatorname{div}, K)$ is from the fact

$$\lambda_i\lambda_j\boldsymbol{t}_{i,j}\cdot\boldsymbol{n}_\ell|_{F_\ell} = 0, \quad \ell = 0, 1, \dots, d.$$

Indeed if $\ell = i$ or $\ell = j$, then $\lambda_i\lambda_j|_{F_\ell} = 0$. Otherwise $\boldsymbol{t}_{i,j}\cdot\boldsymbol{n}_\ell = 0$ by (2.4). To show every function in $\mathbb{B}_k(\operatorname{div}, K)$ can be written as a linear combination of $\lambda_i\lambda_j\boldsymbol{t}_{i,j}$ is tedious and will be skipped. Obviously $\dim\mathbb{B}_k(\operatorname{div}, K) \neq d(d+1)/2\dim\mathbb{P}_{k-2}(K)$ as $\{\lambda_i\lambda_j\mathbb{P}_{k-2}(K)\boldsymbol{t}_{i,j}, 0 \leq i < j \leq d\}$ is linearly dependent. One can further expand the polynomials in $\mathbb{P}_{k-2}(K)$ in the Bernstein basis and $\boldsymbol{t}_{i,j}$ in terms of $\mathrm{d}\lambda$ and add a constraint on the multi-index to find a basis from this generating set; see [5]. Another systematical way to identify $\mathbb{B}_k(\operatorname{div}, K)$ is through a geometric decomposition of vector Lagrange elements and a $t - n$ basis decomposition at each subsimplex; see [13] for details.

Fortunately we are interested in the dual space, which can let us find a basis of $\mathbb{B}'_k(\operatorname{div}, K)$ without knowing one for $\mathbb{B}_k(\operatorname{div}, K)$. Following Lemma 2.2, we first find a larger space containing $\mathbb{B}'_k(\operatorname{div}, K)$.

LEMMA 3.5. *Let integer $k \geq 1$. We have*

$$\mathbb{B}'_k(\operatorname{div}, K) = \mathcal{I}'\mathcal{N}(\mathbb{P}_{k-1}(K;\mathbb{R}^d)),$$

*where $\mathcal{I} : \mathbb{B}_k(\operatorname{div}, K) \to \mathbb{P}_k(K;\mathbb{R}^d)$ is the inclusion map.*

*Proof.* It suffices to show that for any $\boldsymbol{v} \in \mathbb{B}_k(\operatorname{div}, K)$ satisfying

$$(\boldsymbol{v}, \boldsymbol{q})_K = 0 \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(K;\mathbb{R}^d), \tag{3.1}$$

then $\boldsymbol{v} = \boldsymbol{0}$.

Expand $\boldsymbol{v}$ in terms of $\{\boldsymbol{l}_i\}$ as $\boldsymbol{v} = \sum_{i=1}^d v_i\boldsymbol{l}_i$. Then $\boldsymbol{v}\cdot\boldsymbol{n}_i|_{F_i} = 0$ implies $v_i|_{F_i} = 0$ for $i = 1, \dots, d$, i.e., $v_i = \lambda_i p_i$ for some $p_i \in \mathbb{P}_{k-1}(K)$. Choose $\boldsymbol{q} = \sum_{i=1}^d p_i\boldsymbol{n}_i \in \mathbb{P}_{k-1}(K;\mathbb{R}^d)$ in (3.1) to get $\int_K \boldsymbol{v}\cdot(\sum_{i=1}^d p_i\boldsymbol{n}_i)\,\mathrm{d}x = \int_K \sum_{i=1}^d \lambda_i p_i^2\,\mathrm{d}x = 0$, which implies $p_i = 0$, i.e., $v_i = 0$ for all $i = 1, 2, \dots, d$. Thus $\boldsymbol{v} = \boldsymbol{0}$. ∎

Again the dimension count $\dim\mathbb{B}_k(\operatorname{div}, K) \neq \dim\mathbb{P}_{k-1}(K;\mathbb{R}^d)$ implies $\mathcal{I}'$ is not injective. We need to further refine our characterization. We use the div operator to decompose $\mathbb{B}_k(\operatorname{div}, K)$ into

$$E_0 := \mathbb{B}_k(\operatorname{div}, K) \cap \ker(\operatorname{div}), \quad E_0^\perp := \mathbb{B}_k(\operatorname{div}, K)/E_0.$$

We can characterize the dual space of $E_0^\perp$ through $\operatorname{div}^*$, which is $-\operatorname{grad}$ restricted to the bubble function space and can be continuously extended to $H^1(K)$.

LEMMA 3.6. *Let integer $k \geq 1$. The mapping*

$$\operatorname{div} : E_0^\perp \to \mathbb{P}_{k-1}(K)/\mathbb{R}$$

*is a bijection and consequently*

$$\dim E_0^\perp = \dim\mathbb{P}_{k-1}(K) - 1 = \binom{k-1+d}{d} - 1.$$

*Proof.* The inclusion $\operatorname{div}(\mathbb{B}_k(\operatorname{div}, K)) \subseteq \mathbb{P}_{k-1}(K)/\mathbb{R}$ is proved through integration by parts

$$(\operatorname{div} \boldsymbol{v}, p)_K = -(\boldsymbol{v}, \operatorname{grad} p)_K = 0 \quad \forall\, p \in \mathbb{R} = \ker(\operatorname{grad}).$$

If $\operatorname{div}(\mathbb{B}_k(\operatorname{div}, K)) \neq \mathbb{P}_{k-1}(K)/\mathbb{R}$, then there exists a $p \in \mathbb{P}_{k-1}(K)/\mathbb{R}$ and $p \perp \operatorname{div}(\mathbb{B}_k(\operatorname{div}, K))$, which is equivalent to $\nabla p \perp \mathbb{B}_k(\operatorname{div}, K)$. Expand the vector $\nabla p$ in the basis $\{\boldsymbol{n}_i, i = 1, \ldots, d\}$ as $\nabla p = \sum_{i=1}^d q_i \boldsymbol{n}_i$ with $q_i \in \mathbb{P}_{k-2}(K)$. Then set $\boldsymbol{v}_p = \sum_{i=1}^d q_i \lambda_0 \lambda_i \boldsymbol{l}_i = \sum_{i=1}^d |\nabla \lambda_i| q_i \lambda_0 \lambda_i \boldsymbol{t}_{i,0} \in \mathbb{B}_k(\operatorname{div}, K)$. We have

$$(\operatorname{grad} p, \boldsymbol{v}_p)_K = \sum_{i=1}^d \int_K q_i^2 \lambda_0 \lambda_i \, \mathrm{d}x = 0,$$

which implies $q_i = 0$ for all $i = 1, 2, \ldots, d$, i.e., $\operatorname{grad} p = 0$ and $p = 0$ as $p \in \mathbb{P}_{k-1}(K)/\mathbb{R}$.

We have proved $\operatorname{div}(\mathbb{B}_k(\operatorname{div}, K)) = \mathbb{P}_{k-1}(K)/\mathbb{R}$, and thus $\operatorname{div} : E_0^{\perp} \to \mathbb{P}_{k-1}(K)/\mathbb{R}$ is a bijection as $E_0^{\perp} = \mathbb{B}_k(\operatorname{div}, K)/\ker(\operatorname{div})$. ∎

As an example of (2.9), we have the following characterization of $(E_0^{\perp})'$.

Corollary 3.7. *Let integer $k \geq 1$. We have*

$$(E_0^{\perp})' = \mathcal{N}(\operatorname{grad} \mathbb{P}_{k-1}(K)).$$

*That is, a function $\boldsymbol{v} \in E_0^{\perp}$ is uniquely determined by DoFs*

$$(\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{grad} \mathbb{P}_{k-1}(K).$$

After we know the dimensions of $\mathbb{B}_k(\operatorname{div}, K)$ and $E_0^{\perp}$, we can calculate the dimension of $E_0$.

Corollary 3.8. *Let integer $k \geq 1$. It holds that*

$$\dim E_0 = \dim \mathbb{B}_k(\operatorname{div}, K) - \dim E_0^{\perp} = d\binom{k+d-1}{d} - \binom{k+d}{d} + 1. \tag{3.2}$$

The most difficult part is to characterize $E_0$. Using the de Rham complex, we can identify the null space of $\operatorname{div} : \mathbb{B}_k(\operatorname{div}, K) \to \mathbb{P}_{k-1}(K)/\mathbb{R}$ as the image of another polynomial bubble space. For example,

$$E_0 = \begin{cases} \operatorname{curl}\left(\mathbb{P}_{k+1}(K) \cap H_0^1(K)\right) & \text{for } d = 2, \\ \operatorname{curl}\left(\mathbb{P}_{k+1}(K; \mathbb{R}^3) \cap H_0(\operatorname{curl}, K)\right) & \text{for } d = 3. \end{cases}$$

Generalization of the curl operator can be done for the de Rham complex, but it will be hard for the elasticity complex and the divdiv complex.

Instead, we take the dual approach. To identify the dual space of $E_0$, we resort to a polynomial decomposition of $\mathbb{P}_{k-1}(K; \mathbb{R}^d)$.

Lemma 3.9. *Let integer $k \geq 1$. We have the polynomial space decomposition*

$$\mathbb{P}_{k-1}(K; \mathbb{R}^d) = \operatorname{grad} \mathbb{P}_k(K) \oplus (\ker(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}(K; \mathbb{R}^d)). \tag{3.3}$$

*Proof.* Clearly it holds that

$$\operatorname{grad} \mathbb{P}_k(K) \oplus (\ker(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}(K; \mathbb{R}^d)) \subseteq \mathbb{P}_{k-1}(K; \mathbb{R}^d).$$

And the sum is direct as by Euler's formula (2.1).

The mapping $\cdot\boldsymbol{x} : \mathbb{P}_{k-1}(K;\mathbb{R}^d) \to \mathbb{P}_k(K)\backslash\mathbb{R}$ is surjective, thus

$$\dim \mathbb{P}_{k-1}(K;\mathbb{R}^d) = \dim(\mathbb{P}_k(K)\backslash\mathbb{R}) + \dim(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)).$$

As $\dim(\mathbb{P}_k(K)\backslash\mathbb{R}) = \dim \operatorname{grad} \mathbb{P}_k(K)$, we obtain the decomposition (3.3) by dimension count. ∎

COROLLARY 3.10. *Let integer $k \geq 1$. We have*

$$E_0' = \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)).$$

*That is, a function $\boldsymbol{v} \in E_0$ is uniquely determined by*

$$(\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d).$$

*Proof.* We apply Lemma 2.2 with $V = \mathbb{P}_k(K;\mathbb{R}^d), U = \mathbb{P}_{k-1}(K;\mathbb{R}^d)$, and $\kappa = \cdot\boldsymbol{x}$. Lemmas 3.5 and 3.9 verify the assumptions (B1)–(B2), and we only need to count the dimension

$$\begin{aligned}\dim(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)) &= \dim \mathbb{P}_{k-1}(K;\mathbb{R}^d) - \dim \mathbb{P}_k(K) + 1 \\ &= d\binom{k+d-1}{d} - \binom{k+d}{d} + 1 \\ &= \dim E_0,\end{aligned}$$

where the last step is based on (3.2). The desired result then follows. ∎

An explicit characterization of $\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)$ is shown in [25, Proposition 1], that is, $\boldsymbol{v} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{H}_{k-1}(K;\mathbb{R}^d)$ is equivalent to $\boldsymbol{v} \in \mathbb{H}_{k-1}(K;\mathbb{R}^d)$ such that the symmetric part of $\nabla^{k-1}\boldsymbol{v}$ vanishes. We shall give another characterization of $\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)$.

LEMMA 3.11. *Let integer $k \geq 1$. It holds that*

$$\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d) = \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}, \tag{3.4}$$

*where recall $\mathbb{K}$ is the subspace of skew-symmetric matrices of $\mathbb{R}^{d\times d}$, and*

$$\mathbb{P}_{k-1}(K;\mathbb{R}^d) = \operatorname{grad} \mathbb{P}_k(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}. \tag{3.5}$$

*Proof.* Clearly we have $\mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x} \subseteq \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{R}^d)$. By (3.3), it suffices to prove (3.5). Take $\boldsymbol{q} \in \mathbb{P}_{k-1}(K;\mathbb{R}^d)$. Without loss of generality, by linearity, it is sufficient to assume $\boldsymbol{q} = \boldsymbol{x}^\alpha \boldsymbol{e}_\ell$ with $|\alpha| = k-1$ and $1 \leq \ell \leq d$. Let

$$p = \frac{1}{k}\boldsymbol{x}\cdot\boldsymbol{q} = \frac{1}{k}\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell} \in \mathbb{P}_k(K), \;\; \boldsymbol{\tau} = \frac{1}{k}\sum_{i=1}^d \alpha_i \boldsymbol{x}^{\alpha-\boldsymbol{e}_i}(\boldsymbol{e}_\ell\boldsymbol{e}_i^\intercal - \boldsymbol{e}_i\boldsymbol{e}_\ell^\intercal) \in \mathbb{P}_{k-2}(K;\mathbb{K}).$$

Then

$$\begin{aligned}\operatorname{grad} p + \boldsymbol{\tau}\boldsymbol{x} &= \frac{1}{k}\sum_{i=1}^d(\alpha_i + \delta_{i\ell})\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i + \frac{1}{k}\sum_{i=1}^d \alpha_i\boldsymbol{x}^{\alpha-\boldsymbol{e}_i}(\boldsymbol{e}_\ell x_i - \boldsymbol{e}_i x_\ell) \\ &= \frac{1}{k}\sum_{i=1}^d(\alpha_i + \delta_{i\ell})\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i + \frac{1}{k}\sum_{i=1}^d \alpha_i\boldsymbol{x}^{\alpha}\boldsymbol{e}_\ell - \frac{1}{k}\sum_{i=1}^d \alpha_i\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i \\ &= \frac{1}{k}\boldsymbol{x}^\alpha\boldsymbol{e}_\ell + \frac{|\alpha|}{k}\boldsymbol{x}^\alpha\boldsymbol{e}_\ell = \boldsymbol{x}^\alpha\boldsymbol{e}_\ell.\end{aligned}$$

Therefore it follows $\boldsymbol{q} = \operatorname{grad} p + \boldsymbol{\tau}\boldsymbol{x}$, i.e.,

$$\mathbb{P}_{k-1}(K;\mathbb{R}^d) \subseteq \operatorname{grad}\mathbb{P}_k(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}.$$

This combined with the fact $\operatorname{grad}\mathbb{P}_k(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x} \subseteq \mathbb{P}_{k-1}(K;\mathbb{R}^d)$ gives (3.5). ∎

The decomposition (3.3) and characterization (3.4) can be summarized as the following double-directional complex:

$$\mathbb{R} \underset{\pi_0}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_k(K) \underset{\boldsymbol{v}\cdot\boldsymbol{x}}{\overset{\nabla}{\rightleftarrows}} \mathbb{P}_{k-1}(K;\mathbb{R}^d) \underset{\boldsymbol{\tau}\boldsymbol{x}}{\overset{\operatorname{skw}\nabla}{\rightleftarrows}} \mathbb{P}_{k-2}(K;\mathbb{K}),$$

where the skew-symmetric operator $\operatorname{skw} : \mathbb{R}^{d\times d} \to \mathbb{K}$ is defined by $\operatorname{skw}\boldsymbol{\tau} := \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^{\intercal})$.

Define, for an integer $k \geq 0$,

$$\mathrm{ND}_k(K) := \mathbb{P}_k(K;\mathbb{R}^d) \oplus \mathbb{H}_k(K;\mathbb{K})\boldsymbol{x},$$

which is the shape function space of the first kind of Nedéléc edge element in arbitrary dimension [25, 4].

COROLLARY 3.12. *Let integer $k \geq 2$. We have*

$$\mathbb{B}_k'(\operatorname{div}, K) = \mathcal{N}(\operatorname{grad}\mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}) = \mathcal{N}(\mathrm{ND}_{k-2}(K)). \tag{3.6}$$

*Proof.* The first identity is a direct consequence of Corollaries 3.7 and 3.10 and Lemma 3.11. We then write $\mathbb{P}_{k-2}(K;\mathbb{K}) = \mathbb{P}_{k-3}(K;\mathbb{K}) \oplus \mathbb{H}_{k-2}(K;\mathbb{K})$ and use the decomposition (3.3) to conclude the second identity. ∎

*Remark* 3.13. The space $\mathrm{ND}_{k-2}(K)$ can be abbreviated as $\mathbb{P}^-_{k-1}\Lambda^1$ in the terminology of FEEC [4, 2]. The characterization of $\mathbb{B}_k'(\operatorname{div}, K)$ (3.6) can be written as

$$(\overset{\circ}{\mathbb{P}}_k \Lambda^{n-1}(K))^* = \mathbb{P}^-_{k-1}\Lambda^1(K),$$

which is well documented for the de Rham complex [5] but not easy for general complexes. Therefore we still stick to the vector/matrix calculus notation. ∎

We acquire the uni-solvence of BDM element from Lemma 3.3 and Corollary 3.12.

THEOREM 3.14 (BDM element). *Let integer $k \geq 1$. Choose the shape function space $V = \mathbb{P}_k(K;\mathbb{R}^d)$. We have the following set of DoFs for $V$:*

$$\begin{aligned} &(\boldsymbol{v}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_k(F), F \in \mathcal{F}^1(K), \\ &(\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(K) = \operatorname{grad}\mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}. \end{aligned} \tag{3.7}$$

Define the global BDM element space

$$\begin{aligned} \boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^d) : \boldsymbol{v}|_K \in \mathbb{P}_k(K;\mathbb{R}^d) \text{ for each } K \in \mathcal{T}_h, \\ \text{the DoF (3.7) is single-valued}\}. \end{aligned}$$

Since $\boldsymbol{v}\cdot\boldsymbol{n}|_F \in \mathbb{P}_k(F)$, the single-valued DoF (3.7) implies $\boldsymbol{v}\cdot\boldsymbol{n}$ is continuous across the boundary of elements, hence $\boldsymbol{V}_h \subset \boldsymbol{H}(\operatorname{div},\Omega)$.

**3.4. Raviart–Thomas element.** In this subsection, we assume integer $k \geq 0$. The space of shape functions for the RT element is enriched to

$$V^{\mathrm{RT}} := \mathbb{P}_k(K;\mathbb{R}^d) \oplus \mathbb{H}_k(K)\boldsymbol{x}.$$

The DoFs are

$$(\boldsymbol{v}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_k(F), F \in \mathcal{F}^1(K), \tag{3.8}$$

$$(\boldsymbol{v}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(K, \mathbb{R}^d) = \operatorname{grad}\mathbb{P}_k(K) \oplus \mathbb{P}_{k-2}(K;\mathbb{K})\boldsymbol{x}. \tag{3.9}$$

Note that the DoFs to determine the trace remain the same and only the interior moments are increased from $\mathrm{ND}_{k-2}(K)$ to $\mathbb{P}_{k-1}(K;\mathbb{R}^d)$. The range space is also increased, i.e., $\operatorname{div} V^{\mathrm{RT}} = \mathbb{P}_k(K)$, and therefore the approximation of $\operatorname{div}\boldsymbol{u}$ will be one order higher.

We follow our construction procedure to identify the dual spaces of each block.

Lemma 3.15. *Let integer $k \geq 0$. It holds that*

$$\operatorname{tr}^{\operatorname{div}}(V^{\mathrm{RT}}) = \mathbb{P}_k(\mathcal{F}^1(K)). \tag{3.10}$$

*Proof.* When $k \geq 1$, by Lemma 3.2 and the fact that $\boldsymbol{n}\cdot\boldsymbol{x}|_{F_i}$ is a constant, it follows that

$$\operatorname{tr}^{\operatorname{div}}(V^{\mathrm{RT}}) = \operatorname{tr}^{\operatorname{div}}(\mathbb{P}_k(K;\mathbb{R}^d)) = \mathbb{P}_k(\mathcal{F}^1(K)).$$

Consider the case $k = 0$. It is clear that $\operatorname{tr}^{\operatorname{div}}(V^{\mathrm{RT}}) \subseteq \mathbb{P}_0(\mathcal{F}^1(K))$. To prove the other side, by the linearity, assume $q \in \mathbb{P}_0(\mathcal{F}^1(K))$ such that $q|_{F_0} = c \in \mathbb{R}$ and $q|_{F_i} = 0$ for $i = 1, \ldots, d$. Set $\boldsymbol{v} = \frac{c}{(\boldsymbol{x}_1 - \boldsymbol{x}_0)\cdot\boldsymbol{n}_0}(\boldsymbol{x} - \boldsymbol{x}_0) \in V^{\mathrm{RT}}$, then $\operatorname{tr}^{\operatorname{div}}\boldsymbol{v} = q$. ∎

Define the bubble space

$$\mathbb{B}_{k+1}^{-}(\operatorname{div}, K) := \ker(\operatorname{tr}^{\operatorname{div}}) \cap V^{\mathrm{RT}}.$$

By (3.10), $\dim \mathbb{B}_{k+1}^{-}(\operatorname{div}, K) = \dim V^{\mathrm{RT}} - \dim \operatorname{tr}^{\operatorname{div}}(V^{\mathrm{RT}}) = d\binom{k+d-1}{d}$ for $k \geq 1$ and $\dim \mathbb{B}_1^{-}(\operatorname{div}, K) = 0$. We show the intersection of the null space of div operator and $\mathbb{B}_{k+1}^{-}(\operatorname{div}, K)$ remains unchanged.

Lemma 3.16. *Let integer $k \geq 0$. It holds that*

$$\mathbb{B}_{k+1}^{-}(\operatorname{div}, K) \cap \ker(\operatorname{div}) = E_0, \tag{3.11}$$

*where $E_0 := \{\mathbf{0}\}$ for $k = 0$.*

*Proof.* For $\boldsymbol{v} \in V^{\mathrm{RT}}$, if $\operatorname{div}\boldsymbol{v} = 0$, then $\boldsymbol{v} \in \mathbb{P}_k(K;\mathbb{R}^d)$ as $\operatorname{div} : \mathbb{H}_k(K)\boldsymbol{x} \to \mathbb{H}_k(K)$ is bijective. Then the desired result follows. ∎

Define $E_0^{\perp,-} := \mathbb{B}_{k+1}^{-}(\operatorname{div}, K)/E_0$. We give a characterization of $(E_0^{\perp,-})'$.

Lemma 3.17. *Let integer $k \geq 0$. It holds that*

$$(E_0^{\perp,-})' = \mathcal{N}(\operatorname{grad}\mathbb{P}_k(K)). \tag{3.12}$$

*Proof.* We first prove, given a $\boldsymbol{v} \in E_0^{\perp,-}$, i.e., $\operatorname{tr}^{\operatorname{div}}\boldsymbol{v} = 0$ and $\boldsymbol{v} \perp E_0$, if

$$(\boldsymbol{v}, \operatorname{grad} p) = 0 \quad \forall\, p \in \mathbb{P}_k(K), \tag{3.13}$$

then $\boldsymbol{v} = \mathbf{0}$. Indeed integration by parts of (3.13) and the fact $\operatorname{div}\boldsymbol{v} \in \mathbb{P}_k(K)$ imply $\operatorname{div}\boldsymbol{v} = 0$, i.e., $\boldsymbol{v} \in E_0$. Then the only possibility to have $\boldsymbol{v} \perp E_0$ is $\boldsymbol{v} = \mathbf{0}$.

Then the dimension count gives

$$\dim E_0^{\perp,-} = \dim \mathbb{B}_{k+1}^-(\operatorname{div}, K) - \dim E_0 = \binom{k+d}{d} - 1 = \dim \operatorname{grad} \mathbb{P}_k(K),$$

which indicates (3.12). ∎

Hence we acquire the uni-solvence of RT element from (3.10), (3.11), (3.12), and Corollary 3.10. The global version of finite element space can be defined similarly.

THEOREM 3.18 (uni-solvence of RT element). *Let integer $k \geq 0$. The DoFs* (3.8)–(3.9) *are uni-solvent for $V^{\mathrm{RT}}$.*

When $k \geq 1$, the RT element can be enriched from the BDM element by applying Lemma 2.3 with $\mathrm{d} = \operatorname{div}$, $V = \mathbb{P}_k(K; \mathbb{R}^d)$, $\mathbb{H} = \mathbb{H}_k(K)\boldsymbol{x}$, $\mathbb{P} = \mathbb{P}_{k-1}(K)/\mathbb{R}$, and $\mathbb{Q} = \mathbb{P}_{k-2}(K; \mathbb{K})\boldsymbol{x}$.

**4. Symmetric $\boldsymbol{H}(\operatorname{div})$-conforming finite elements.** In this section we shall construct $H(\operatorname{div})$-conforming finite elements for symmetric matrices. For space $V = \mathbb{P}_k(K, \mathbb{S})$, our element is slightly different from Hu's element constructed in [21]. A new family of $\mathbb{P}_{k+1}^-(K, \mathbb{S})$ type finite elements is also constructed. The trace space for symmetric $H(\operatorname{div})$-conforming element seems hard to characterize; instead we identify the bubble function space and then only need to work on the dual of the trace space.

**4.1. Div operator.**

LEMMA 4.1. *Let $k \geq 0$. The operator* $\operatorname{div} : \operatorname{sym}(\mathbb{H}_k(D; \mathbb{R}^d)\boldsymbol{x}^\intercal) \to \mathbb{H}_k(D; \mathbb{R}^d)$ *is bijective and consequently* $\operatorname{div} : \mathbb{P}_{k+1}(D; \mathbb{S}) \to \mathbb{P}_k(D; \mathbb{R}^d)$ *is surjective.*

*Proof.* Noting that

$$\operatorname{div}(\operatorname{sym}(\mathbb{H}_k(D; \mathbb{R}^d)\boldsymbol{x}^\intercal)) \subseteq \mathbb{H}_k(K; \mathbb{R}^d),$$
$$\dim(\operatorname{sym}(\mathbb{H}_k(D; \mathbb{R}^d)\boldsymbol{x}^\intercal)) = \dim \mathbb{H}_k(K; \mathbb{R}^d),$$

it is sufficient to prove $\operatorname{sym}(\mathbb{H}_k(D; \mathbb{R}^d)\boldsymbol{x}^\intercal) \cap \ker(\operatorname{div}) = \{\boldsymbol{0}\}$. That is, for any $\boldsymbol{q} \in \mathbb{H}_k(D; \mathbb{R}^d)$ satisfying $\operatorname{div} \operatorname{sym}(\boldsymbol{q}\boldsymbol{x}^\intercal) = \boldsymbol{0}$, we are going to prove $\boldsymbol{q} = \boldsymbol{0}$.

By (2.2), we have

$$\begin{aligned} 2 \operatorname{div} \operatorname{sym}(\boldsymbol{q}\boldsymbol{x}^\intercal) &= \operatorname{div}(\boldsymbol{q}\boldsymbol{x}^\intercal) + \operatorname{div}(\boldsymbol{x}\boldsymbol{q}^\intercal) = (k+d)\boldsymbol{q} + (\operatorname{grad} \boldsymbol{x})\boldsymbol{q} + (\operatorname{div} \boldsymbol{q})\boldsymbol{x} \\ &= (k+d+1)\boldsymbol{q} + (\operatorname{div} \boldsymbol{q})\boldsymbol{x}. \end{aligned}$$

It follows from $\operatorname{div} \operatorname{sym}(\boldsymbol{q}\boldsymbol{x}^\intercal) = \boldsymbol{0}$ that

$$(k+d+1)\boldsymbol{q} + (\operatorname{div} \boldsymbol{q})\boldsymbol{x} = \boldsymbol{0}. \tag{4.1}$$

Applying the divergence operator div on both side of (4.1), we get from (2.2) that

$$2(k+d) \operatorname{div} \boldsymbol{q} = 0.$$

Hence $\operatorname{div} \boldsymbol{q} = 0$, which together with (4.1) gives $\boldsymbol{q} = \boldsymbol{0}$. ∎

**4.2. Bubble space.** Define an $\boldsymbol{H}(\operatorname{div}, K; \mathbb{S})$ bubble function space of polynomials of degree $k$ as

$$\mathbb{B}_k(\operatorname{div}, K; \mathbb{S}) := \{\boldsymbol{\tau} \in \mathbb{P}_k(K; \mathbb{S}) : \boldsymbol{\tau}\boldsymbol{n}|_{\partial K} = \boldsymbol{0}\}.$$

It is easy to check that $\mathbb{B}_1(\operatorname{div}, K; \mathbb{S})$ is merely the zero space. The following characterization of $\mathbb{B}_k(\operatorname{div}, K; \mathbb{S})$ is given in [21, Lemma 2.2].

LEMMA 4.2. *For $k \geq 2$, it holds that*

$$\mathbb{B}_k(\mathrm{div}, K; \mathbb{S}) = \sum_{0\leq i<j\leq d} \lambda_i\lambda_j \mathbb{P}_{k-2}(K)\boldsymbol{T}_{i,j}. \tag{4.2}$$

*Consequently*

$$\dim \mathbb{B}_k(\mathrm{div}, K; \mathbb{S}) = \dim \mathbb{P}_{k-2}(K; \mathbb{S}) = \frac{d(d+1)}{2}\binom{d+k-2}{d}.$$

LEMMA 4.3. *For $k \geq 2$, it holds that*

$$\mathbb{B}_k'(\mathrm{div}, K; \mathbb{S}) = \mathcal{N}(\mathbb{P}_{k-2}(K; \mathbb{S})).$$

*That is $\boldsymbol{\tau} \in \mathbb{B}_k(\mathrm{div}, K; \mathbb{S})$ is uniquely determined by*

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\ \boldsymbol{\varsigma} \in \mathbb{P}_{k-2}(K; \mathbb{S}).$$

*Proof.* Given $\boldsymbol{\tau} \in \mathbb{B}_k(\mathrm{div}, K; \mathbb{S})$, by (4.2), there exist $q_{ij} \in \mathbb{P}_{k-2}(K)$ with $0 \leq i < j \leq d$ such that

$$\boldsymbol{\tau} = \sum_{0\leq i<j\leq d} \lambda_i\lambda_j q_{ij} \boldsymbol{T}_{i,j}.$$

Note that symmetric tensors $\{\boldsymbol{N}_{i,j}\}_{0\leq i<j\leq d}$ are dual to $\{\boldsymbol{T}_{i,j}\}_{0\leq i<j\leq d}$ with respect to the Frobenius inner product (cf. [9, section 3.1] and also section 2.5). Choosing $\boldsymbol{\varsigma} = \sum_{0\leq i<j\leq d} q_{ij}\boldsymbol{N}_{i,j} \in \mathbb{P}_{k-2}(K; \mathbb{S})$, we get

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K = \sum_{0\leq i<j\leq d} (\lambda_i\lambda_j, q_{ij}^2)_K = 0.$$

Hence $q_{ij} = 0$ for all $i, j$, and then $\boldsymbol{\tau} = \boldsymbol{0}$. As the dimensions match, we conclude the result. ∎

Another characterization of $\mathbb{B}_k(\mathrm{div}, K; \mathbb{S})$ and $\mathbb{B}_k'(\mathrm{div}, K; \mathbb{S})$ is given in [13].

**4.3. Trace spaces.** The mapping $\mathrm{tr}^{\mathrm{div}} : \mathbb{P}_k(K; \mathbb{S}) \to \mathbb{P}_k(\mathcal{F}^1(K; \mathbb{R}^{d-1}))$ is not onto due to the symmetry. Some compatible conditions should be imposed on lower-dimensional simplexes. Fortunately, we only need its dimension.

LEMMA 4.4. *Let integer $k \geq 1$. It holds that*

$$\begin{aligned}\dim \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K; \mathbb{S})) &= \dim \mathbb{P}_k(K; \mathbb{S}) - \dim \mathbb{B}_k(\mathrm{div}, K; \mathbb{S})\\ &= \dim \mathbb{H}_k(K; \mathbb{S}) + \dim \mathbb{H}_{k-1}(K; \mathbb{S})\\ &= \frac{1}{2}d(d+1)\left[\binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}\right].\end{aligned}$$

We show the supersmoothness induced by the symmetry for the $H(\mathrm{div}; \mathbb{S})$ element. For a $(d-r)$-dimensional face $e \in \mathcal{F}^r(K)$ with $r = 2, \ldots, d$ shared by two $(d-1)$-dimensional faces $F, F' \in \mathcal{F}^1(K)$, by the symmetry of $\boldsymbol{\tau}$, $(\boldsymbol{n}_F^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F'})|_e$ is concurrently determined by $(\boldsymbol{\tau}\boldsymbol{n}_F)|_F$ and $(\boldsymbol{\tau}\boldsymbol{n}_{F'})|_{F'}$. This implies the DoFs $\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j$ on $e$ for all $i, j = 1, \ldots, r$. In particular, for a 0-dimensional vertex $\delta$, $(\boldsymbol{\tau}_{ij}(\delta))_{d\times d}$ is taken as a DoF.

The trace $\boldsymbol{\tau}\boldsymbol{n}$ restricted to a face $F \in \mathcal{F}^1(K)$ can be further split into two components: (1) the normal-normal component $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$ will be determined by $\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j$; (2) the tangential-normal component $\Pi_F\boldsymbol{\tau}\boldsymbol{n}$ will be determined by the interior moments relative to $F$ after the trace $\mathrm{tr}^{\mathrm{div}_F}(\Pi_F\boldsymbol{\tau}\boldsymbol{n}) = \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}$ has been determined.

LEMMA 4.5. *A basis of*

$$(\mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{S})))'$$

*is given by the DoFs*

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{4.3}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1, \tag{4.4}$$

$$(\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K). \tag{4.5}$$

*Proof.* We first prove that if all the DoFs (4.3)–(4.5) vanish, then $\boldsymbol{\tau} = \boldsymbol{0}$. As $\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j|_f \in \mathbb{P}_k(f)$, by the vanishing DoFs (4.3)–(4.4) and the uni-solvence of the Lagrange element, we get

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j|_f = 0 \quad \forall\, f \in \mathcal{F}^r(K),\ i,j = 1,\ldots,d-r, \text{ and } r = 1,\ldots,d-1.$$

This implies

$$\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_F = 0,\ \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_e = 0 \quad \forall\, F \in \mathcal{F}^1(K), e \in \mathcal{F}^1(F). \tag{4.6}$$

Notice that $\Pi_F\boldsymbol{\tau}\boldsymbol{n}|_F \in \mathbb{P}_k(F;\mathbb{R}^{d-1})$. Due to the uni-solvence of the BDM element on $F$ (cf. Theorem 3.14), we acquire from the second identity in (4.6) and the vanishing DoFs (4.5) that $\Pi_F\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$, which together with the first identity in (4.6) yields $\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$.

We then count the dimension to finish the proof. By comparing DoFs of the Hu element (cf. Remark 4.6) and DoFs (4.3)–(4.5), it follows from the DoFs of the first kind of Nédélec element (cf. [25, 4]) that the number of DoFs (4.3)–(4.5) is equal to the number of DoFs of the Hu element, thus equal to $\dim \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{S}))$. ∎

*Remark* 4.6. As a comparison, the DoFs of the Hu element on the boundary in [21] are

$$\begin{aligned}
\boldsymbol{\tau}(\delta) &\quad \forall\, \delta \in \mathcal{V}(K),\\
(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f &\quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K),\\
&\qquad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1,\\
(\boldsymbol{t}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f &\quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K),\\
&\qquad i = 1,\ldots,d-r, j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1.
\end{aligned}$$

The difference is the way to impose the tangential-normal component. ∎

**4.4. Split of the bubble space.** To construct $\boldsymbol{H}(\mathrm{div}, K;\mathbb{S})$ elements, the interior DoFs given by $\mathcal{N}(\mathbb{P}_{k-2}(K;\mathbb{S}))$ are enough. For the construction of the $\boldsymbol{H}(\mathrm{div}\,\mathrm{div}, K;\mathbb{S})$ element, we use div operator to decompose $\mathbb{B}_k(\mathrm{div}, K;\mathbb{S})$ into

$$E_{0,k}(\mathbb{S}) := \mathbb{B}_k(\mathrm{div}, K;\mathbb{S}) \cap \ker(\mathrm{div}), \quad E_{0,k}^{\perp}(\mathbb{S}) := \mathbb{B}_k(\mathrm{div}, K;\mathbb{S})/E_{0,k}(\mathbb{S}).$$

We will abbreviate $E_{0,k}(\mathbb{S})$ and $E_{0,k}^{\perp}(\mathbb{S})$ as $E_0(\mathbb{S})$ and $E_0^{\perp}(\mathbb{S})$, respectively, if this will not cause any confusion. As before we can characterize the dual space of $E_{0,k}^{\perp}(\mathbb{S})$ through $\mathrm{div}^*$, which is $-\,\mathrm{def} := -\,\mathrm{sym}\,\mathrm{grad}$ restricted to the bubble space and can be extended to $H^1(K;\mathbb{R}^d)$.

LEMMA 4.7. *Let integer $k \geq 2$. The mapping*

$$\operatorname{div}: E_{0,k}^{\perp}(\mathbb{S}) \to \mathbb{P}_{k-1,\mathrm{RM}}^{\perp} := \mathbb{P}_{k-1}(K, \mathbb{R}^d)/\ker(\operatorname{def})$$

*is a bijection and consequently*

$$(E_{0,k}^{\perp}(\mathbb{S}))' = \mathcal{N}(\operatorname{def} \mathbb{P}_{k-1}(K, \mathbb{R}^d)),$$
$$\dim E_{0,k}^{\perp}(\mathbb{S}) = d\binom{k+d-1}{k-1} - \frac{1}{2}(d^2+d).$$

*Proof.* The fact $\operatorname{div} \mathbb{B}_k(\operatorname{div}, K; \mathbb{S}) = \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$ was proved in [21, Theorem 2.2]. Here we recall it for completeness.

The inclusion $\operatorname{div}(\mathbb{B}_k(\operatorname{div}, K; \mathbb{S})) \subseteq \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$ can be proved through integration by parts,

$$(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{v})_K = -(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{v})_K = 0 \quad \forall\, \boldsymbol{v} \in \ker(\operatorname{def}).$$

If $\operatorname{div}(\mathbb{B}_k(\operatorname{div}, K; \mathbb{S})) \neq \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$, then there exists a function $\boldsymbol{v} \in \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$ satisfying $\boldsymbol{v} \perp \operatorname{div}(\mathbb{B}_k(\operatorname{div}, K; \mathbb{S}))$, which is equivalent to $\operatorname{def} \boldsymbol{v} \perp \mathbb{B}_k(\operatorname{div}, K; \mathbb{S})$. Expand the symmetric matrix $\operatorname{def} \boldsymbol{v}$ in the basis $\{\boldsymbol{N}_{i,j}, 0 \leq i < j \leq d\}$ as $\operatorname{def} \boldsymbol{v} = \sum_{0 \leq i < j \leq d} q_{ij} \boldsymbol{N}_{i,j}$ with $q_{ij} \in \mathbb{P}_{k-2}(K)$. Then set $\boldsymbol{\tau}_v = \sum_{0 \leq i < j \leq d} q_{ij} \lambda_i \lambda_j \boldsymbol{T}_{i,j} \in \mathbb{B}_k(\operatorname{div}, K; \mathbb{S})$. We have

$$(\operatorname{def} \boldsymbol{v}, \boldsymbol{\tau}_v)_K = \sum_{0 \leq i < j \leq d} \int_K q_{ij}^2 \lambda_i \lambda_j \,\mathrm{d}x = 0,$$

which implies $q_{ij} = 0$ for all $0 \leq i < j \leq d$, i.e., $\operatorname{def} \boldsymbol{v} = 0$ and $\boldsymbol{v} = 0$ as $\boldsymbol{v} \in \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$. Since $\operatorname{div} E_{0,k}^{\perp}(\mathbb{S}) = \operatorname{div} \mathbb{B}_k(\operatorname{div}, K; \mathbb{S})$, the mapping $\operatorname{div}: E_{0,k}^{\perp}(\mathbb{S}) \to \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$ is a bijection.

For $\boldsymbol{v} \in E_{0,k}^{\perp}(\mathbb{S})$, $(\boldsymbol{v}, \operatorname{def} \boldsymbol{q})_K = 0$ for all $\boldsymbol{q} \in \mathbb{P}_{k-1}(K, \mathbb{R}^d)$ implies $\operatorname{div} \boldsymbol{v} = \boldsymbol{0}$, i.e., $\boldsymbol{v} \in E_{0,k}(\mathbb{S})$. Then $\boldsymbol{v} \in E_{0,k}(\mathbb{S}) \cap E_{0,k}^{\perp}(\mathbb{S}) = \{\boldsymbol{0}\}$. Hence $(E_{0,k}^{\perp}(\mathbb{S}))' = \mathcal{I}'\mathcal{N}(\operatorname{def} \mathbb{P}_{k-1}(K, \mathbb{R}^d))$. As the dimensions match, $\mathcal{I}'$ is a bijection. ∎

We then move to the space $E_{0,k}(\mathbb{S})$. Using the primary approach, we need the bubble space in the previous space and the differential operator. For example, we have $E_{0,k}(\mathbb{S}) = \operatorname{curl} \operatorname{curl}(\mathbb{P}_{k+2}(K) \cap H_0^2(K))$ in two dimensions [6], and in three dimensions [3, 14]

$$E_{0,k}(\mathbb{S}) = \operatorname{inc} \mathbb{B}_{k+2}(\operatorname{inc}, K; \mathbb{S})$$

with

$$\mathbb{B}_{k+2}(\operatorname{inc}, K; \mathbb{S}) := \{\boldsymbol{\tau} \in \mathbb{P}_{k+2}(K; \mathbb{S}) : \boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n} = \boldsymbol{0},$$
$$2 \operatorname{def}_F(\boldsymbol{n} \cdot \boldsymbol{\tau} \Pi_F) - \Pi_F \partial_n \boldsymbol{\tau} \Pi_F = \boldsymbol{0} \ \ \forall\, F \in \mathcal{F}^1(K)\}.$$

Such characterization is hard to generalize to arbitrary dimension.

Instead we use the dual approach to identify $E_{0,k}'(\mathbb{S})$. To this end, denote the space of rigid motions by $\boldsymbol{RM} := \mathrm{ND}_0(K) = \{\boldsymbol{c} + \boldsymbol{N}\boldsymbol{x} : \ \boldsymbol{c} \in \mathbb{R}^d, \ \boldsymbol{N} \in \mathbb{K}\}$. Define operator $\boldsymbol{\pi}_{RM} : \mathcal{C}^1(D; \mathbb{R}^d) \to \boldsymbol{RM}$ as

$$\boldsymbol{\pi}_{RM} \boldsymbol{v} := \boldsymbol{v}(\boldsymbol{0}) + (\operatorname{skw}(\nabla \boldsymbol{v}))(\boldsymbol{0})\boldsymbol{x}.$$

Clearly it holds that $\boldsymbol{\pi}_{RM} \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \boldsymbol{RM}$. We denote by $\cdot\boldsymbol{x} : \mathbb{P}_k(D; \mathbb{S}) \to \mathbb{P}_{k+1}(D; \mathbb{R}^d)$ the mapping $\boldsymbol{\tau} \to \boldsymbol{\tau}\boldsymbol{x}$ as the matrix-vector product $\boldsymbol{\tau}\boldsymbol{x}$ is applying row-wise inner product with vector $\boldsymbol{x}$.

We shall establish the short exact sequence

$$\boldsymbol{RM} \underset{\boldsymbol{\pi}_{RM}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(D;\mathbb{R}^d) \underset{\cdot\boldsymbol{x}}{\overset{\mathrm{def}}{\rightleftarrows}} \operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d) \rightleftarrows \boldsymbol{0}$$

and derive a space decomposition from it.

Lemma 4.8. *Let integer $k \geq 0$. If $\boldsymbol{q} \in \mathbb{P}_{k+1}(D;\mathbb{R}^d)$ satisfying $(\operatorname{def}\boldsymbol{q})\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{q} \in \boldsymbol{RM}$.*

*Proof.* Since $\boldsymbol{x}^\intercal(\boldsymbol{x}\cdot\nabla)\boldsymbol{q} = \boldsymbol{x}^\intercal(\nabla\boldsymbol{q})\boldsymbol{x} = \boldsymbol{x}^\intercal(\operatorname{def}\boldsymbol{q})\boldsymbol{x} = 0$, we get

$$(\boldsymbol{x}\cdot\nabla)(\boldsymbol{x}^\intercal\boldsymbol{q}) = \boldsymbol{x}^\intercal(\boldsymbol{x}\cdot\nabla)\boldsymbol{q} + \boldsymbol{x}^\intercal\boldsymbol{q} = \boldsymbol{x}^\intercal\boldsymbol{q}.$$

By (2.1), this indicates $\boldsymbol{x}^\intercal\boldsymbol{q} \in \mathbb{P}_1(D)$. Noting that $(\nabla\boldsymbol{q})\boldsymbol{x} = \nabla(\boldsymbol{x}^\intercal\boldsymbol{q}) - \boldsymbol{q}$, we obtain

$$(\boldsymbol{x}\cdot\nabla)\boldsymbol{q} + (\nabla(\boldsymbol{x}^\intercal\boldsymbol{q}) - \boldsymbol{q}) = (\nabla\boldsymbol{q})^\intercal\boldsymbol{x} + (\nabla\boldsymbol{q})\boldsymbol{x} = 2(\operatorname{def}\boldsymbol{q})\boldsymbol{x} = \boldsymbol{0},$$

which implies $(\boldsymbol{x}\cdot\nabla)\boldsymbol{q} - \boldsymbol{q} = -\nabla(\boldsymbol{x}^\intercal\boldsymbol{q}) \in \mathbb{P}_0(D;\mathbb{R}^d)$. Hence $\boldsymbol{q} \in \mathbb{P}_1(D;\mathbb{R}^d)$. Assume $\boldsymbol{q} = \boldsymbol{N}\boldsymbol{x} + \boldsymbol{C}$ with $\boldsymbol{N} \in \mathbb{M}$ and $\boldsymbol{C} \in \mathbb{R}^d$. Then

$$\boldsymbol{x}^\intercal(\operatorname{sym}\boldsymbol{N})\boldsymbol{x} + \boldsymbol{x}^\intercal\boldsymbol{C} = \boldsymbol{x}^\intercal\boldsymbol{N}\boldsymbol{x} + \boldsymbol{x}^\intercal\boldsymbol{C} = \boldsymbol{x}^\intercal\boldsymbol{q} \in \mathbb{P}_1(D),$$

which implies $\operatorname{sym}\boldsymbol{N} = \boldsymbol{0}$. Therefore $\boldsymbol{N} \in \mathbb{K}$ and $\boldsymbol{q} \in \boldsymbol{RM}$. ∎

Lemma 4.9. *Let integer $k \geq 0$. We have*

$$\left(\operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d)\right)\boldsymbol{x} = \mathbb{P}_k(D;\mathbb{S})\boldsymbol{x} = \mathbb{P}_{k+1}(D;\mathbb{R}^d) \cap \ker(\boldsymbol{\pi}_{RM}). \tag{4.7}$$

*Proof.* For any $\boldsymbol{\tau} \in \mathbb{P}_k(D;\mathbb{S})$, it follows that

$$\boldsymbol{\pi}_{RM}(\boldsymbol{\tau}\boldsymbol{x}) = (\operatorname{skw}(\nabla(\boldsymbol{\tau}\boldsymbol{x})))(\boldsymbol{0})\boldsymbol{x} = \operatorname{skw}(\boldsymbol{\tau}(\boldsymbol{0}))\boldsymbol{x} = \boldsymbol{0}.$$

Thus $\mathbb{P}_k(D;\mathbb{S})\boldsymbol{x} \subseteq \mathbb{P}_{k+1}(D;\mathbb{R}^d) \cap \ker(\boldsymbol{\pi}_{RM})$. On the other hand, we obtain from Lemma 4.8 that

$$\dim\left(\left(\operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d)\right)\boldsymbol{x}\right) = \dim\mathbb{P}_{k+1}(D;\mathbb{R}^d) - \dim\boldsymbol{RM},$$

which equals the dimension of $\mathbb{P}_{k+1}(D;\mathbb{R}^d) \cap \ker(\boldsymbol{\pi}_{RM})$. Thus (4.7) follows. ∎

Corollary 4.10. *Let integer $k \geq 0$. We have the space decomposition*

$$\mathbb{P}_k(D;\mathbb{S}) = \operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d) \oplus (\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_k(D;\mathbb{S})). \tag{4.8}$$

*Proof.* It follows from Lemma 4.8 that $\operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d) \cap (\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_k(D;\mathbb{S})) = \{\boldsymbol{0}\}$. Due to (4.7),

$$\begin{aligned}
&\dim\operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d) + \dim(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_k(D;\mathbb{S}))\\
&= \dim\operatorname{def}\mathbb{P}_{k+1}(D;\mathbb{R}^d) + \dim\mathbb{P}_k(D;\mathbb{S}) - \dim(\mathbb{P}_k(D;\mathbb{S})\boldsymbol{x})\\
&= \dim\mathbb{P}_{k+1}(D;\mathbb{R}^d) - \dim\boldsymbol{RM} + \dim\mathbb{P}_k(D;\mathbb{S}) - \dim(\mathbb{P}_k(D;\mathbb{S})\boldsymbol{x})\\
&= \dim\mathbb{P}_k(D;\mathbb{S}),
\end{aligned}$$

which means (4.8). ∎

LEMMA 4.11. *Let integer $k \geq 2$. We have*

$$E'_{0,k}(\mathbb{S}) = \mathcal{N}(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})). \tag{4.9}$$

*That is, a function $\boldsymbol{\tau} \in E_{0,k}(\mathbb{S})$ is uniquely determined by*

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}).$$

*And*

$$\dim E_{0,k}(\mathbb{S}) = \frac{d(d+1)}{2}\binom{k-2+d}{d} - d\binom{d+k-1}{d} + \frac{d(d+1)}{2}.$$

*Proof.* We apply Lemma 2.2 with $V = \mathbb{P}_k(K;\mathbb{S}), U = \mathbb{P}_{k-2}(K;\mathbb{S})$, and $\kappa = \cdot\boldsymbol{x}$. Lemma 4.3 and (4.7)–(4.8) verify the assumptions (B1)–(B2), and we only need to count the dimension.

By the space decomposition (4.8), Lemma 4.3, and Lemma 4.7,

$$\begin{aligned}\dim(\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})) &= \dim \mathbb{P}_{k-2}(K;\mathbb{S}) - \dim \operatorname{def} \mathbb{P}_{k-1}(K;\mathbb{R}^d)\\ &= \dim \mathbb{B}_k(\operatorname{div}, K;\mathbb{S}) - \dim E^{\perp}_{0,k}(\mathbb{S}) = \dim E_{0,k}(\mathbb{S}),\end{aligned}$$

as required. ∎

*Remark* 4.12. In two and three dimensions, we have (cf. [11, 14])

$$\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_k(D;\mathbb{S}) = \begin{cases} \boldsymbol{x}^{\perp}(\boldsymbol{x}^{\perp})^{\intercal}\mathbb{P}_{k-2}(D) & \text{for } d = 2,\\ \boldsymbol{x} \times \mathbb{P}_{k-2}(D;\mathbb{S}) \times \boldsymbol{x} & \text{for } d = 3,\end{cases}$$

where $\boldsymbol{x}^{\perp} := \left(\begin{smallmatrix} x_2 \\ -x_1\end{smallmatrix}\right)$, but generalization to arbitrary dimension is not easy and not necessary. A computation approach to find an explicit basis of $\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(K;\mathbb{S})$ is as follows. Find a basis for $\mathbb{P}_{k-1}(K;\mathbb{S})$ and one for $\mathbb{P}_k(K;\mathbb{R}^d)$. Then form the matrix representation $X$ of the operator $\cdot\boldsymbol{x}$. Afterward the null space $\ker(X)$ can be found algebraically.

**4.5. $\boldsymbol{H}(\operatorname{div};\mathbb{S})$-conforming elements.** Combining Lemmas 4.5, 4.7, and 4.11 and space decomposition (4.8) yields the DoFs of $H(\operatorname{div};\mathbb{S})$-conforming elements.

THEOREM 4.13 ($\mathbb{P}_k(K;\mathbb{S})$-type $H(\operatorname{div};\mathbb{S})$-conforming elements). *Take the shape function space $V(\mathbb{S}) = \mathbb{P}_k(K;\mathbb{S})$ with $k \geq d+1$. The DoFs*

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{4.10}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K),\quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1, \tag{4.11}$$

$$(\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K), \tag{4.12}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(K;\mathbb{S}) \tag{4.13}$$

*are uni-solvent for $\mathbb{P}_k(K;\mathbb{S})$. The last DoF* (4.13) *can be replaced by*

$$(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(K;\mathbb{R}^d)/\boldsymbol{RM}, \tag{4.14}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}). \tag{4.15}$$

The global finite element space $\boldsymbol{V}_h(\operatorname{div};\mathbb{S}) \subset H(\operatorname{div},\Omega;\mathbb{S})$, where

$$\begin{aligned}\boldsymbol{V}_h(\operatorname{div};\mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K;\mathbb{S}) &\text{ for each } K \in \mathcal{T}_h,\\ &\text{the DoFs (4.10)–(4.12) are single-valued}\}.\end{aligned}$$

Clearly $\boldsymbol{V}_h(\operatorname{div};\mathbb{S}) \subset H(\operatorname{div},\Omega;\mathbb{S})$ follows from the proof of Lemma 4.5.

For the most important three dimensional case, the DoFs (4.10)–(4.13) become

$$
\begin{aligned}
\boldsymbol{\tau}(\delta) &\quad \forall\, \delta \in \mathcal{V}(K),\\
(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_e &\quad \forall\, q \in \mathbb{P}_{k-2}(e), e \in \mathcal{F}^2(K), i,j=1,2,\\
(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F &\quad \forall\, q \in \mathbb{P}_{k-3}(F), F \in \mathcal{F}^1(K),\\
(\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F &\quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K),\\
(\boldsymbol{\tau}, \boldsymbol{q})_K &\quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(K;\mathbb{S}),
\end{aligned}
$$

which are slightly different from the Hu–Zhang element in three dimensions [23].

Uni-solvence holds for $k \geq 1$. The requirement $k \geq d+1$ contains the DoFs $(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F$ for all $\boldsymbol{q} \in \mathbb{P}_1(F;\mathbb{R}^{d-1})$ on each face $F \in \mathcal{F}^1(K)$, by which the divergence of the global $H(\mathrm{div};\mathbb{S})$-conforming element space will include the piecewise $\boldsymbol{RM}$ space and combining with $\operatorname{div}\mathbb{B}_k(\mathrm{div}, K;\mathbb{S}) = \mathbb{P}_{k-1,\mathrm{RM}}^{\perp}$ will imply the following discrete inf-sup condition.

Lemma 4.14. *Let $k \geq d+1$. The inf-sup condition*

$$
\|\boldsymbol{p}_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h \in \boldsymbol{V}_h(\mathrm{div};\mathbb{S})} \frac{(\operatorname{div}\boldsymbol{\tau}_h, \boldsymbol{p}_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div})}} \qquad \forall\, \boldsymbol{p}_h \in \mathbb{P}_{k-1}(\mathcal{T}_h;\mathbb{R}^d)
$$

*holds, where $\mathbb{P}_{k-1}(\mathcal{T}_h;\mathbb{R}^d) := \{\boldsymbol{p}_h \in \boldsymbol{L}^2(\Omega;\mathbb{R}^d) : \boldsymbol{p}_h|_K \in \mathbb{P}_{k-1}(K;\mathbb{R}^d)$ for each $K \in \mathcal{T}_h\}$.*

*Proof.* For $\boldsymbol{p}_h \in \mathbb{P}_{k-1}(\mathcal{T}_h;\mathbb{R}^d)$, there exists $\boldsymbol{\tau} \in \boldsymbol{H}^1(\Omega;\mathbb{S})$ such that [19]

$$
\operatorname{div}\boldsymbol{\tau} = \boldsymbol{p}_h, \quad \|\boldsymbol{\tau}\|_1 \lesssim \|\boldsymbol{p}_h\|_0.
$$

Let $\boldsymbol{\tau}_h \in \boldsymbol{V}_h(\mathrm{div};\mathbb{S})$ such that all the DoFs (4.10)–(4.12) and (4.14)–(4.15) vanish except

$$
\begin{aligned}
(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}_h\boldsymbol{n}, q)_F &= (\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_1(F), F \in \mathcal{F}^1(K),\\
(\Pi_F\boldsymbol{\tau}_h\boldsymbol{n}, \boldsymbol{q})_F &= (\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_1(F;\mathbb{R}^{d-1}), F \in \mathcal{F}^1(K),\\
(\operatorname{div}\boldsymbol{\tau}_h, \boldsymbol{q})_K &= (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K = (\boldsymbol{p}_h, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(K;\mathbb{R}^d)/\boldsymbol{RM}
\end{aligned}
$$

for all $K \in \mathcal{T}_h$. By the scaling argument, we have

$$
\|\boldsymbol{\tau}_h\|_0 \lesssim \|\boldsymbol{\tau}\|_1 \lesssim \|\boldsymbol{p}_h\|_0. \tag{4.16}
$$

Applying the integration by parts,

$$
(\operatorname{div}\boldsymbol{\tau}_h, \boldsymbol{q})_K = (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K = (\boldsymbol{p}_h, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \boldsymbol{RM}.
$$

Hence

$$
(\operatorname{div}\boldsymbol{\tau}_h, \boldsymbol{q})_K = (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K = (\boldsymbol{p}_h, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(K;\mathbb{R}^d),
$$

which implies $\operatorname{div}\boldsymbol{\tau}_h = \boldsymbol{p}_h$. Therefore we derive the inf-sup condition from (4.16). ∎

**4.6. $\mathbb{P}_{k+1}^{-}(K;\mathbb{S})$-type elements.** Let $k \geq d+1$. The space of shape functions is taken as

$$
\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S}) + E_{0,k+1}^{\perp}(\mathbb{S}).
$$

Since $E_{0,k+1}^{\perp}(\mathbb{S}) \subseteq \mathbb{B}_{k+1}(\mathrm{div}, K;\mathbb{S})$ and $\operatorname{div} E_{0,k+1}^{\perp}(\mathbb{S}) = \mathbb{P}_{k,\mathrm{RM}}^{\perp}$, we have

$$
\mathrm{tr}^{\mathrm{div}}\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) = \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_k(K;\mathbb{S})), \quad \operatorname{div}\mathbb{P}_{k+1}^{-}(K;\mathbb{S}) = \mathbb{P}_k(K;\mathbb{R}^d).
$$

By applying Lemma 2.3 with $\mathrm{d} = \operatorname{div}$, $V = \mathbb{P}_k(K;\mathbb{S})$, $\mathbb{H} = E^{\perp}_{0,k+1}(\mathbb{S})\backslash E^{\perp}_{0,k}(\mathbb{S})$, $\mathbb{P} = \mathbb{P}_{k-1}(K,\mathbb{R}^d)$, and $\mathbb{Q} = \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})$, we get the uni-solvent DoFs

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{4.17}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1, \tag{4.18}$$

$$\begin{aligned} (\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F &\quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K),\\ (\boldsymbol{\tau}, \boldsymbol{q})_K &\quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}),\\ (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K &\quad \forall\, \boldsymbol{q} \in \mathbb{P}_k(K;\mathbb{R}^d)/\boldsymbol{RM}. \end{aligned} \tag{4.19}$$

Since $\operatorname{div}\mathbb{P}^{-}_{k+1}(K;\mathbb{S}) = \mathbb{P}_k(K;\mathbb{R}^d)$ and $\operatorname{div}\mathbb{P}_k(K;\mathbb{S}) = \mathbb{P}_{k-1}(K;\mathbb{R}^d)$, it is expected that using $\mathbb{P}^{-}_{k+1}(K;\mathbb{S})$ to discretize the mixed elasticity problem will possess one-order higher convergence rate of the divergence of the discrete stress than that of $\mathbb{P}_k(K;\mathbb{S})$ symmetric element.

*Remark* 4.15. By the DoFs (4.10)–(4.13), we can find a basis $\{\phi_i\}_{i=1}^{N_1}$ of the bubble function space $\mathbb{B}_k(\operatorname{div},K;\mathbb{S})$. Let $\{\psi_i\}_{i=1}^{N_2}$ be a basis of $\mathbb{P}_{k-1}(K;\mathbb{R}^d)\backslash\boldsymbol{RM}$. Then form the matrix $\big((\operatorname{div}\phi_i,\psi_j)_K\big)_{N_1\times N_2}$, whose kernel space combined with $\{\phi_i\}_{i=1}^{N_1}$ yields the basis of $E_{0,k}(\mathbb{S})$. Finally, a basis of $E^{\perp}_{0,k}(\mathbb{S})$ is achieved by finding the orthogonal complement of the basis of $E_{0,k}(\mathbb{S})$ under the inner product $(\cdot,\cdot)_K$.

The global finite element space $\boldsymbol{V}^{-}_h(\operatorname{div};\mathbb{S}) \subset H(\operatorname{div},\Omega;\mathbb{S})$, where

$$\boldsymbol{V}^{-}_h(\operatorname{div};\mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K \in \mathbb{P}^{-}_{k+1}(K;\mathbb{S}) \text{ for each } K \in \mathcal{T}_h, \text{ the DoFs (4.17)–(4.19) are single-valued}\}.$$

Similarly as Lemma 4.14, we have the following inf-sup condition.

LEMMA 4.16. *Let $k \geq d+1$. The inf-sup condition*

$$\|\boldsymbol{p}_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h\in\boldsymbol{V}^{-}_h(\operatorname{div};\mathbb{S})} \frac{(\operatorname{div}\boldsymbol{\tau}_h,\boldsymbol{p}_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div})}} \qquad \forall\, \boldsymbol{p}_h \in \mathbb{P}_k(\mathcal{T}_h;\mathbb{R}^d)$$

*holds.*

As with the RT element, it is natural to enrich $\mathbb{P}_k(K;\mathbb{S})$ to $\mathbb{P}_k(K;\mathbb{S})\oplus\operatorname{sym}(\mathbb{H}_k(K;\mathbb{R}^d)\boldsymbol{x}^{\intercal})$. Unfortunately, $\operatorname{tr}^{\operatorname{div}}(\operatorname{sym}(\mathbb{H}_k(K;\mathbb{R}^d)\boldsymbol{x}^{\intercal})) \not\subseteq \operatorname{tr}^{\operatorname{div}}(\mathbb{P}_k(K;\mathbb{S}))$, i.e., assumption (H2) in Lemma 2.3 does not hold, which ruins the discrete inf-sup condition.

## 5. Symmetric $\boldsymbol{H}(\operatorname{div}\operatorname{div})$-conforming finite elements.

We use the previous building blocks to construct $H(\operatorname{div}\operatorname{div})$-conforming finite elements in arbitrary dimension. Motivated by the recent construction [22] in two and three dimensions, we first construct $H(\operatorname{div}\operatorname{div}) \cap H(\operatorname{div})$-conforming finite elements for symmetric tensors and then apply a simple modification to construct $H(\operatorname{div}\operatorname{div})$-conforming finite elements. We then extend the construction to obtain a new family of $\mathbb{P}^{-}_{k+1}(\mathbb{S})$-type elements.

### 5.1. Divdiv operator and Green's identity.

LEMMA 5.1. *For integer $k \geq 1$, the operator*

$$\operatorname{div}\operatorname{div} : \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{H}_{k-1}(D) \to \mathbb{H}_{k-1}(D)$$

*is bijective. Consequently* $\operatorname{div}\operatorname{div} : \mathbb{P}_{k+1}(D;\mathbb{S}) \to \mathbb{P}_{k-1}(D)$ *is surjective.*

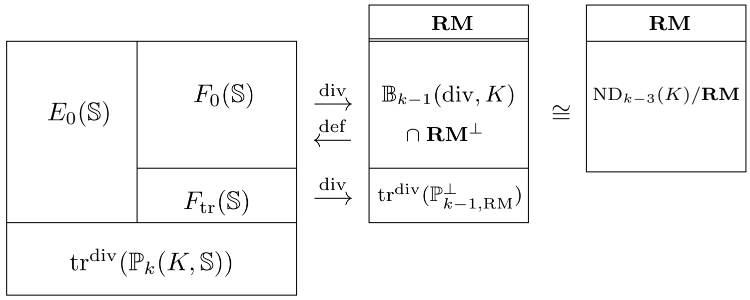


FIG. 5.1. *Decomposition of $\mathbb{P}_k(K,\mathbb{S})$ for an $H(\operatorname{div}\operatorname{div})\cap H(\operatorname{div})$-conforming finite element.*

*Proof.* By (2.2), it follows that

$$\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^\intercal q) = \operatorname{div}((k+d)\boldsymbol{x}q) = (k+d)(k+d-1)q \quad \forall\, q\in\mathbb{H}_{k-1}(D),$$

which ends the proof. ∎

Next recall the Green's identity for operator divdiv in [12].

LEMMA 5.2. *We have for any $\boldsymbol{\tau}\in\mathcal{C}^2(K;\mathbb{S})$ and $v\in H^2(K)$ that*

$$\begin{aligned}(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K &= (\boldsymbol{\tau},\nabla^2 v)_K - \sum_{F\in\mathcal{F}^1(K)}\sum_{e\in\mathcal{F}^1(F)}(\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\tau}\boldsymbol{n}, v)_e\\ &\quad - \sum_{F\in\mathcal{F}^1(K)}\left[(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n},\partial_n v)_F - (\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau} + \operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), v)_F\right]. \qquad (5.1)\end{aligned}$$

*Proof.* We start from the standard integration by parts

$$\begin{aligned}(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K &= -(\operatorname{div}\boldsymbol{\tau},\nabla v)_K + \sum_{F\in\mathcal{F}^1(K)}(\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}, v)_F\\ &= \left(\boldsymbol{\tau},\nabla^2 v\right)_K - \sum_{F\in\mathcal{F}^1(K)}(\boldsymbol{\tau}\boldsymbol{n},\nabla v)_F + \sum_{F\in\mathcal{F}^1(K)}(\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}, v)_F.\end{aligned}$$

We then decompose $\nabla v = \partial_n v\boldsymbol{n} + \nabla_F v$ and apply the Stokes theorem to get

$$\begin{aligned}(\boldsymbol{\tau}\boldsymbol{n},\nabla v)_F &= (\boldsymbol{\tau}\boldsymbol{n},\partial_n v\boldsymbol{n} + \nabla_F v)_F\\ &= (\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n},\partial_n v)_F - (\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), v)_F + \sum_{e\in\mathcal{F}^1(F)}(\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\tau}\boldsymbol{n}, v)_e.\end{aligned}$$

Thus the Green's identity (5.1) follows from the last two identities. ∎

**5.2. $\boldsymbol{H}(\operatorname{div}\operatorname{div};\mathbb{S})\cap\boldsymbol{H}(\operatorname{div};\mathbb{S})$-conforming elements.** Based on (5.1), it suffices to enforce the continuity of both $\boldsymbol{\tau}\boldsymbol{n}$ and $\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}$ so that the constructed finite element space is $H(\operatorname{div},\mathbb{S})\cap H(\operatorname{div}\operatorname{div},\mathbb{S})$-conforming. Such an approach has been recently proposed in [22] to construct two- and three-dimensional $H(\operatorname{div},\mathbb{S})\cap H(\operatorname{div}\operatorname{div},\mathbb{S})$-conforming finite elements. The readers are referred to Figure 5.1 for an illustration of the space decomposition.

The subspaces $\operatorname{tr}^{\rm div}(\mathbb{P}_k(K,\mathbb{S}))$ and $E_0(\mathbb{S})$ are unchanged. The space $\operatorname{div}E_0^\perp(\mathbb{S}) = \mathbb{P}^\perp_{k-1,\rm RM}$ will be further split by the trace operator. Define

$$F_0(\mathbb{S})\subseteq E_0^\perp(\mathbb{S}),\ \text{satisfying}\ \operatorname{div}F_0(\mathbb{S}) = \mathbb{B}_{k-1}(\operatorname{div},K)\cap\boldsymbol{RM}^\perp,$$

and $F_{\mathrm{tr}}(\mathbb{S}) \subseteq E_0^{\perp}(\mathbb{S})$ with $\mathrm{tr}^{\mathrm{div}}(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})) = \mathrm{tr}^{\mathrm{div}}(\operatorname{div} E_0^{\perp}) = \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_{k-1,\mathrm{RM}}^{\perp})$, which is well defined as div restricted to $E_0^{\perp}(\mathbb{S})$ is a bijection. Here

$$\mathbb{B}_{k-1}(\mathrm{div}, K) \cap \boldsymbol{RM}^{\perp} := \{\boldsymbol{v} \in \mathbb{B}_{k-1}(\mathrm{div}, K) : (\boldsymbol{v}, \boldsymbol{q})_K = 0 \ \forall \ \boldsymbol{q} \in \boldsymbol{RM}\}.$$

LEMMA 5.3. *For integer $k \geq 3$, it holds that*

$$\mathrm{tr}^{\mathrm{div}}(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})) = \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_{k-1}(K; \mathbb{R}^d)).$$

*Consequently* $(\mathrm{tr}^{\mathrm{div}}(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})))' = \mathcal{N}(\mathbb{P}_{k-1}(\mathcal{F}^1(K)))$.

*Proof.* By definition, $\mathrm{tr}^{\mathrm{div}}(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})) = \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_{k-1,\mathrm{RM}}^{\perp}) \subseteq \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_{k-1}(K; \mathbb{R}^d))$. On the other hand, given a trace $p \in \mathrm{tr}^{\mathrm{div}}(\mathbb{P}_{k-1}(K; \mathbb{R}^d))$, by the uni-solvence of the BDM element (cf. Theorem 3.14), we can find a $\boldsymbol{v} \in \mathbb{P}_{k-1}(K; \mathbb{R}^d)$ such that $\boldsymbol{v} \cdot \boldsymbol{n} = p$ on $\partial K$ and $\boldsymbol{v} \perp \boldsymbol{RM}$ as $\boldsymbol{RM} = \mathrm{ND}_0(K) \subseteq \mathrm{ND}_{k-3}(K)$ when $k \geq 3$. ∎

LEMMA 5.4. *For integer $k \geq 3$, we have*

$$F_0'(\mathbb{S}) = \mathcal{N}(\mathrm{def}(\mathrm{ND}_{k-3}(K))).$$

*Proof.* We pick a $\boldsymbol{\tau} \in F_0(\mathbb{S})$, i.e., $\boldsymbol{\tau}$ satisfies

$$(\boldsymbol{\tau}\boldsymbol{n})|_{\partial K} = 0, \quad \boldsymbol{n}^{\intercal} \operatorname{div} \boldsymbol{\tau}|_{\partial K} = 0, \quad \boldsymbol{\tau} \perp E_0(\mathbb{S}).$$

Assume

$$(\boldsymbol{\tau}, \mathrm{def}\, \boldsymbol{q})_K = 0 \quad \forall \ \boldsymbol{q} \in \mathrm{ND}_{k-3}(K).$$

Note that $\boldsymbol{v} = \operatorname{div} \boldsymbol{\tau} \in \mathbb{B}_{k-1}(\mathrm{div}, K)$, and $(\boldsymbol{v}, \boldsymbol{q})_K = 0$ for all $\boldsymbol{q} \in \mathrm{ND}_{k-3}(K)$; then $\boldsymbol{v} = \boldsymbol{0}$ by Theorem 3.14. Therefore $\operatorname{div} \boldsymbol{\tau} = \boldsymbol{0}$, i.e., $\boldsymbol{\tau} \in E_0(\mathbb{S})$. As $\boldsymbol{\tau} \perp E_0(\mathbb{S})$, the only possibility is $\boldsymbol{\tau} = \boldsymbol{0}$.

Then the dimension count

$$\dim F_0(\mathbb{S}) = \dim \mathbb{B}_{k-1}(\mathrm{div}, K) - \dim \boldsymbol{RM} = \dim \mathrm{ND}_{k-3}(K) - \dim \ker(\mathrm{def})$$

will finish the proof. ∎

We summarize the construction in the following theorem.

THEOREM 5.5. *Let $V(\operatorname{div}\operatorname{div}^+; \mathbb{S}) := \mathbb{P}_k(K, \mathbb{S})$ with $k \geq \max\{d, 3\}$. Then the following set of DoFs determines an $H(\operatorname{div}\operatorname{div}; \mathbb{S}) \cap H(\operatorname{div}; \mathbb{S})$-conforming finite element*

$$\boldsymbol{\tau}(\delta) \quad \forall \ \delta \in \mathcal{V}(K), \tag{5.2}$$

$$(\boldsymbol{n}_i^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_j, q)_f \quad \forall \ q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i, j = 1, \ldots, r, \ \textit{and}\ r = 1, \ldots, d-1, \tag{5.3}$$

$$(\Pi_F \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q})_F \quad \forall \ \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K), \tag{5.4}$$

$$(\boldsymbol{n}^{\intercal} \operatorname{div} \boldsymbol{\tau}, q)_F \quad \forall \ q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^1(K), \tag{5.5}$$

$$(\operatorname{div}\operatorname{div} \boldsymbol{\tau}, q)_K \quad \forall \ q \in \mathbb{P}_{k-2}(K)/\mathbb{P}_1(K), \tag{5.6}$$

$$(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall \ \boldsymbol{q} \in \big(\mathbb{P}_{k-3}(K; \mathbb{K})/\mathbb{P}_0(K; \mathbb{K})\big)\boldsymbol{x}, \tag{5.7}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall \ \boldsymbol{q} \in \ker(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K; \mathbb{S}). \tag{5.8}$$

*Proof.* By Lemma 4.5, the vanishing DoFs (5.2)–(5.4) imply $\boldsymbol{\tau}\boldsymbol{n}|_{\partial K} = \boldsymbol{0}$. Then applying Lemmas 5.3–5.4, we get from the vanishing DoFs (5.5)–(5.7) that $\boldsymbol{\tau} \in E_0(\mathbb{S})$. Finally combining (4.9) and (5.8) implies $\boldsymbol{\tau} = \boldsymbol{0}$.

We then count the dimensions. Compared to the DoFs of the BDM-type $H(\operatorname{div},\mathbb{S})$ element (cf. Theorem 4.13), the difference is (4.14) versus (5.5)–(5.7). Then from the uni-solvence of the BDM $H(\operatorname{div})$-conforming element (cf. Theorem 3.14), we have

$$\dim \mathbb{P}_{k-1}(K;\mathbb{R}^d) = \dim \mathrm{ND}_{k-3}(K) + \sum_{F\in\mathcal{F}^1(K)} \dim \mathbb{P}_{k-1}(F),$$

and consequently the number of DoFs (5.2)–(5.8) is $\dim \mathbb{P}_k(K;\mathbb{S})$. ∎

The global finite element space $\boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S}) \subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) \cap H(\operatorname{div},\Omega;\mathbb{S})$ is defined as follows:

$$\begin{aligned}\boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K,\mathbb{S}) \text{ for each } K \in \mathcal{T}_h,\\ \text{the DoFs (5.2)–(5.5) are single-valued}\}.\end{aligned}$$

The requirement $k \geq d$ ensures the DoFs $(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}, q)_F$ for all $q \in \mathbb{P}_0(F)$ on each face $F \in \mathcal{F}^1(K)$, by which space $\operatorname{div}\operatorname{div}\boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S})$ will include all the piecewise linear functions.

LEMMA 5.6. *Let $k \geq \max\{d,3\}$. The inf-sup condition*

$$\|p_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h\in\boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S})} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div})} + \|\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h\|_0} \qquad \forall\, p_h \in \mathbb{P}_{k-2}(\mathcal{T}_h)$$

*holds, where $\mathbb{P}_{k-2}(\mathcal{T}_h) := \{p_h \in L^2(\Omega) : p_h|_K \in \mathbb{P}_{k-2}(K)$ for each $K \in \mathcal{T}_h\}$.*

*Proof.* For $p_h \in \mathbb{P}_{k-2}(\mathcal{T}_h)$, there exists $\boldsymbol{\tau} \in \boldsymbol{H}^2(\Omega;\mathbb{S})$ such that [19]

$$\operatorname{div}\boldsymbol{\tau} = p_h, \quad \|\boldsymbol{\tau}\|_2 \lesssim \|p_h\|_0.$$

Let $\boldsymbol{\tau}_h \in \boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S})$ such that all the DoFs (5.2)–(5.8) vanish except

$$\begin{aligned}(\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n}, q)_F &= (\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}, q)_F && \forall\, q \in \mathbb{P}_0(F), F \in \mathcal{F}^1(K),\\ (\Pi_F\boldsymbol{\tau}_h\boldsymbol{n}, \boldsymbol{q})_F &= (\Pi_F\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F && \forall\, \boldsymbol{q} \in \mathbb{P}_0(F;\mathbb{R}^{d-1}), F \in \mathcal{F}^1(K),\\ (\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}_h, q)_F &= (\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}, q)_F && \forall\, q \in \mathbb{P}_1(F;\mathbb{R}^d), F \in \mathcal{F}^1(K),\\ (\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, q)_K &= (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, q)_K = (p_h, q)_K && \forall\, q \in \mathbb{P}_{k-2}(K)/\mathbb{P}_1(K)\end{aligned}$$

for all $K \in \mathcal{T}_h$. By the scaling argument, we have

$$\|\boldsymbol{\tau}_h\|_{H(\operatorname{div})} \lesssim \|\boldsymbol{\tau}\|_2 \lesssim \|p_h\|_0. \tag{5.9}$$

Applying the integration by parts,

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, q)_K = (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, q)_K = (p_h, q)_K \quad \forall\, q \in \mathbb{P}_1(K).$$

Hence

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, q)_K = (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, q)_K = (p_h, q)_K \quad \forall\, q \in \mathbb{P}_{k-2}(K),$$

which implies $\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h = p_h$. Therefore we derive the inf-sup condition from (5.9). ∎

**5.3. $\mathbb{P}_{k+1}^-(\mathbb{S})$-type $\boldsymbol{H}(\operatorname{div}\operatorname{div};\mathbb{S}) \cap \boldsymbol{H}(\operatorname{div};\mathbb{S})$-conforming elements.** The space of shape functions is taken as

$$V^-(\operatorname{div}\operatorname{div}^+;\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^\intercal \mathbb{H}_{k-1}(K)$$

with $k \geq \max\{d, 3\}$. The DoFs are

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{5.10}$$

$$(\boldsymbol{n}_i^\intercal \boldsymbol{\tau} \boldsymbol{n}_j, q)_f \quad \forall\, q \in \mathbb{P}_{k+r-d-1}(f), f \in \mathcal{F}^r(K), \quad i,j = 1,\ldots,r, \text{ and } r = 1,\ldots,d-1, \tag{5.11}$$

$$(\Pi_F \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^1(K), \tag{5.12}$$

$$(\boldsymbol{n}^\intercal \operatorname{div}\boldsymbol{\tau}, p)_F \quad \forall\, p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^1(K), \tag{5.13}$$

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, q)_K \quad \forall\, q \in \mathbb{P}_{k-1}(K)/\mathbb{P}_1(K), \tag{5.14}$$

$$(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \big(\mathbb{P}_{k-3}(K;\mathbb{K})/\mathbb{P}_0(K;\mathbb{K})\big)\boldsymbol{x}, \tag{5.15}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S}). \tag{5.16}$$

We can see that $\mathbb{P}_{k+1}^-(\mathbb{S})$-type $H(\operatorname{div}\operatorname{div};\mathbb{S}) \cap H(\operatorname{div};\mathbb{S})$-conforming elements follow from Lemma 2.3 with $\mathrm{d} = \operatorname{div}\operatorname{div}$, $V = \mathbb{P}_k(K;\mathbb{S})$, $\mathbb{H} = \boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K)$, $\mathbb{P} = \mathbb{P}_{k-1}(K)/\mathbb{P}_1(K)$, and $\mathbb{Q} = \ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-2}(K;\mathbb{S})$. The assumption (H5) holds from the fact $\operatorname{div}\operatorname{div}\mathbb{B}^+ = \mathbb{P}_{k-1}(K)/\mathbb{P}_1(K)$ and $\nabla^2(\mathbb{P} + \mathrm{d}\mathbb{H}) = \nabla^2\mathbb{P}_{k-1}(K)$.

Due to the added component $\boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K)$, the range of $\operatorname{div}\operatorname{div}$ operator is increased to $\mathbb{P}_{k-1}(K)$ instead of $\mathbb{P}_{k-2}(K)$. The DoF $(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_K$ is increased from $\boldsymbol{q} \in \mathrm{ND}_{k-3}(K) = \operatorname{grad}\mathbb{P}_{k-2}(K) \oplus \mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}$ to $\mathbb{P}_{k-2}(K;\mathbb{R}^d) = \operatorname{grad}\mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}$. Hence the number of DoFs (5.10)–(5.16) equals to $\dim V^-(\operatorname{div}\operatorname{div}^+;\mathbb{S})$. The boundary DoFs, however, remain the same as $(\boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K))\boldsymbol{n}|_F \in \mathbb{P}_k(F;\mathbb{R}^d)$.

It is expected that using the $\mathbb{P}_{k+1}^-(\mathbb{S})$-type symmetric element to discretize the biharmonic problem will possess one-order higher convergence rate of the $\operatorname{div}\operatorname{div}$ of the discrete bending moment than that of the $\mathbb{P}_k(\mathbb{S})$-type symmetric element while the computational cost is not increased significantly; see [11, section 4]. When solving the linear algebraic equation, all interior DoFs can be eliminated element-wise.

LEMMA 5.7. *Let* $\boldsymbol{\tau} \in V^-(\operatorname{div}\operatorname{div}^+;\mathbb{S})$. *If the DoFs* (5.10)–(5.15) *vanish, then* $\boldsymbol{\tau} \in E_0(\mathbb{S})$.

*Proof.* Since $\boldsymbol{x}\cdot\boldsymbol{n}$ is constant on each $(d-1)$-dimensional face, the trace $\boldsymbol{\tau}\boldsymbol{n}|_F \in \mathbb{P}_k(F;\mathbb{R}^d)$ and $(\boldsymbol{n}^\intercal \operatorname{div}\boldsymbol{\tau})|_F \in \mathbb{P}_{k-1}(F)$ remain unchanged. Then we conclude $\operatorname{tr}^{\operatorname{div}}\boldsymbol{\tau} = \boldsymbol{0}$ and $\operatorname{tr}^{\operatorname{div}}(\operatorname{div}\boldsymbol{\tau}) = 0$ from Theorem 5.5.

Applying the Green's identity (5.1), we get

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K = (\boldsymbol{\tau}, \nabla^2 v)_K = 0 \quad \forall\, v \in \mathbb{P}_1(K).$$

Hence it follows from the vanishing DoF (5.14) that $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$, which combined with Lemma 5.1 implies $\boldsymbol{\tau} \in \mathbb{P}_k(K;\mathbb{S})$. Finally we achieve from Lemma 5.4 and the vanishing DoF (5.15) that $\boldsymbol{\tau} \in E_0(\mathbb{S})$. ∎

Combining Lemma 5.7, (4.9), and the DoF (5.16) shows the uni-solvence of the $\mathbb{P}_{k+1}^-(\mathbb{S})$-type $H(\operatorname{div}\operatorname{div};\mathbb{S}) \cap H(\operatorname{div};\mathbb{S})$-conforming elements.

THEOREM 5.8. *The DoFs* (5.10)–(5.16) *are uni-solvent for the space* $V^-(\operatorname{div}\operatorname{div}^+;\mathbb{S}) = \mathbb{P}_k(K;\mathbb{S}) \oplus \boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K)$.

The finite element space $\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div}^+;\mathbb{S}) \subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) \cap H(\operatorname{div},\Omega;\mathbb{S})$ is then defined as follows:

$$\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div}^+;\mathbb{S}) := \{\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K,\mathbb{S})\oplus \boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K) \text{ for each } K\in\mathcal{T}_h, \text{ the DoFs (5.10)–(5.13) are single-valued}\}.$$

Similarly as Lemma 5.6, we have the following inf-sup condition.

LEMMA 5.9. *Let $k\geq \max\{d,3\}$. It holds that*

$$\|p_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h\in\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div}^+;\mathbb{S})} \frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h, p_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div})}+\|\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h\|_0} \qquad \forall\, p_h\in\mathbb{P}_{k-1}(\mathcal{T}_h).$$

**5.4. $\boldsymbol{H}(\operatorname{div}\operatorname{div})$-conforming elements.** The requirement that both $\boldsymbol{\tau n}$ and $\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}$ are continuous is sufficient but not necessary for a function to be in $H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. In addition to $\boldsymbol{n}^\intercal\boldsymbol{\tau n}$, the combination $\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}+\operatorname{div}_F(\boldsymbol{\tau n})$ to be continuous is enough due to the Green's identity (5.1).

THEOREM 5.10. *Take $V(\operatorname{div}\operatorname{div};\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S})$ with $k\geq\max\{d,3\}$, as the space of shape functions. The DoFs are given by*

$$\begin{aligned}
&\boldsymbol{\tau}(\delta) && \forall\,\delta\in\mathcal{V}(K), && (5.17)\\
&(\boldsymbol{n}_i^\intercal\boldsymbol{\tau}\boldsymbol{n}_j, q)_f && \forall\, q\in\mathbb{P}_{k+r-d-1}(f), f\in\mathcal{F}^r(K), && (5.18)\\
& && \quad i,j=1,\dots,r, \text{ and } r=1,\dots,d-1, &&\\
&(\Pi_F\boldsymbol{\tau n},\boldsymbol{q})_F && \forall\,\boldsymbol{q}\in \mathrm{ND}_{k-2}(F), F\in\mathcal{F}^1(K), && (5.19)\\
&(\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}+\operatorname{div}_F(\boldsymbol{\tau n}), p)_F && \forall\, p\in\mathbb{P}_{k-1}(F), F\in\mathcal{F}^1(K), && (5.20)\\
&(\boldsymbol{\tau},\operatorname{def}\boldsymbol{q})_K && \forall\,\boldsymbol{q}\in\mathrm{ND}_{k-3}(K), &&\\
&(\boldsymbol{\tau},\boldsymbol{q})_K && \forall\,\boldsymbol{q}\in\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-2}(K;\mathbb{S}). &&
\end{aligned}$$

*The DoF (5.19) is considered as interior to $K$, i.e., it is not single-valued across elements.*

*Proof.* By Lemma 4.5, the $\operatorname{div}_F(\boldsymbol{\tau n})$ can be determined by (5.17), (5.18), and (5.19). A linear combination with (5.20), the trace $\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}$ can be determined. Then the uni-solvence is obtained from Theorem 5.5. ∎

The finite element space $\boldsymbol{V}_h(\operatorname{div}\operatorname{div})$ is defined as follows:

$$\boldsymbol{V}_h(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) := \{\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K\in\mathbb{P}_k(K;\mathbb{S}) \text{ for each } K\in\mathcal{T}_h, \text{ the DoFs (5.17)–(5.18) and (5.20) are single-valued}\}.$$

As $\boldsymbol{n}^\intercal\boldsymbol{\tau n}$ and $\boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}+\operatorname{div}_F(\boldsymbol{\tau n})$ are continuous, $\boldsymbol{V}_h(\operatorname{div}\operatorname{div})\subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$; see [12, Lemma 4.4].

Finally we present a $\mathbb{P}_{k+1}^-(\mathbb{S})$-type $H(\operatorname{div}\operatorname{div};\mathbb{S})$-conforming element.

THEOREM 5.11. *Let integer $k\geq\max\{d,3\}$. Take the space of shape functions as*

$$V^-(\operatorname{div}\operatorname{div};\mathbb{S}) := \mathbb{P}_k(K;\mathbb{S})\oplus\boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{H}_{k-1}(K).$$

*The DoFs are*

$$\begin{aligned}
&\boldsymbol{\tau}(\delta) && \forall\,\delta\in\mathcal{V}(K), && (5.21)\\
&(\boldsymbol{n}_i^\intercal\boldsymbol{\tau}\boldsymbol{n}_j, q)_f && \forall\, q\in\mathbb{P}_{k+r-d-1}(f), f\in\mathcal{F}^r(K), && (5.22)\\
& && \quad i,j=1,\dots,r, \text{ and } r=1,\dots,d-1, &&\\
&(\Pi_F\boldsymbol{\tau n},\boldsymbol{q})_F && \forall\,\boldsymbol{q}\in \mathrm{ND}_{k-2}(F), F\in\mathcal{F}^1(K), && (5.23)
\end{aligned}$$

$$(5.24)\qquad (\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}+\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}),p)_F \quad \forall\ p\in\mathbb{P}_{k-1}(F), F\in\mathcal{F}^1(K),$$
$$(\boldsymbol{\tau},\operatorname{def}\boldsymbol{q})_K \quad \forall\ \boldsymbol{q}\in\mathbb{P}_{k-2}(K;\mathbb{R}^d),$$
$$(\boldsymbol{\tau},\boldsymbol{q})_K \quad \forall\ \boldsymbol{q}\in\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-2}(K;\mathbb{S}).$$

*Again the DoF* (5.23) *is considered as interior to $K$, i.e., it is not single-valued across elements.*

*Proof.* By Lemma 4.5, the $\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n})$ can be determined by (5.21), (5.22), and (5.23). A linear combination with (5.24), the trace $\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}$ can be determined. Then the uni-solvence is obtained from Theorem 5.8. ∎

The global finite element space $\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div})\subset H(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$, where

$$\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}):=\{\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega;\mathbb{S}):\boldsymbol{\tau}|_K\in V^-(\operatorname{div}\operatorname{div};\mathbb{S})\text{ for each }K\in\mathcal{T}_h,\text{ the DoFs (5.21)–(5.22) and (5.24) are single-valued}\}.$$

Finally we list inf-sup conditions for divdiv conforming elements.

LEMMA 5.12. *Let $k\geq\max\{d,3\}$. We have*

$$(5.25)\qquad \|p_h\|_0\lesssim\sup_{\boldsymbol{\tau}_h\in\boldsymbol{V}_h(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})}\frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h,p_h)}{\|\boldsymbol{\tau}_h\|_0+\|\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h\|_0}\qquad\forall\ p_h\in\mathbb{P}_{k-2}(\mathcal{T}_h),$$

$$(5.26)\qquad \|p_h\|_0\lesssim\sup_{\boldsymbol{\tau}_h\in\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})}\frac{(\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h,p_h)}{\|\boldsymbol{\tau}_h\|_0+\|\operatorname{div}\operatorname{div}\boldsymbol{\tau}_h\|_0}\qquad\forall\ p_h\in\mathbb{P}_{k-1}(\mathcal{T}_h).$$

*Proof.* Since $\|\boldsymbol{\tau}_h\|_0\leq\|\boldsymbol{\tau}_h\|_{H(\operatorname{div})}$ and $\boldsymbol{V}_h(\operatorname{div}\operatorname{div}^+;\mathbb{S})\subseteq\boldsymbol{V}_h(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$, the inf-sup condition (5.25) follows from Lemma 5.6. Similarly, the inf-sup condition (5.26) follows from Lemma 5.9 and $\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div}^+;\mathbb{S})\subseteq\boldsymbol{V}_h^-(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. ∎

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