ON MINIMIZING THE LINEAR INTERPOLATION ERROR OF CONVEX QUADRATIC FUNCTIONS AND THE OPTIMAL SIMPLEX

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ABSTRACT. This paper shows that the optimal simplex, in the sense of minimizing the linear interpolation error of the quadratic function $u(\boldsymbol{x}) = \boldsymbol{x}^2$ measured in L^p-norm, is the equilateral simplex. It also contains several explicit formulae on the best interpolation error when the volume of the simplex is one.

1. INTRODUCTION AND MAIN RESULTS

Let τ be a d-simplex in $\mathbb{R}^d$ and $u(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^d$, be a convex quadratic function. The linear nodal interpolation $u_I(\boldsymbol{x})$ is the affine function such that $u_I(\boldsymbol{x}_i) = u(\boldsymbol{x}_i)$ for all vertices $\boldsymbol{x}_i$ of τ. Let $\mathrm{vol}(\tau)$ be the Lebesgue measure of τ in $\mathbb{R}^d$. We shall consider the following minimization problem:

$$\min_{\tau, \mathrm{vol}(\tau)=1} \|u - u_I\|_{L^p(\tau)}, \quad \text{where } 1 \le p \le \infty, \tag{1.1}$$

That is to find the best shape of a simplex with a fixed volume by minimizing the interpolation error in L^p norm. Since the linear interpolation will preserve the linear polynomial, we can assume $u(\boldsymbol{x}) = \boldsymbol{x}^t A \boldsymbol{x}$, where A is a symmetric positive definite $d \times d$ matrix. By the change of variables $\boldsymbol{y} = \sqrt{A}\boldsymbol{x}$, we can further assume the convex quadratic function as $u(\boldsymbol{x}) = \boldsymbol{x}^2 := \boldsymbol{x}^t \boldsymbol{x}$.

The problem (1.1) has been studied by many mathematicians in different context during the last three decades [27, 14, 3, 2, 25, 26, 21, 22, 16, 17, 4, 12, 13, 15, 19, 23, 5, 6, 18, 7, 8, 10, 9, 1, 11]. The answer to (1.1), when $u(\boldsymbol{x}) = \boldsymbol{x}^2$, is that the best simplex is the regular simplex, i.e., the equilateral simplex. In this paper we shall give a unified proof about this fact and present several explicit formulae on the best interpolation error when the volume of the simplex is one.

Theorem 1.1. *Let $u(\boldsymbol{x}) = \boldsymbol{x}^2$. For any $1 \le p \le \infty, d \ge 1$, there exists a constant $C_{d,p}$ such that*

$$\|u - u_I\|_{L^p(\tau)} \ge C_{d,p}\, \mathrm{vol}(\tau)^{1/p+2/d}, \tag{1.2}$$

and the equality holds if and only if τ is equilateral.

The constants $C_{d,p}$ are useful in other related problems. For example, in [9] we show that $C_{d,\infty}$ is closely related to the sphere covering problem and $C_{d,1}$ to optimal polytopes approximation of convex bodies. Therefore it is desirable to determine the constant $C_{d,p}$ explicitly.

Remark 1.2. *We have the following formulae:*

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- *for* $p = \infty$,

$$C_{d,\infty} = \frac{d}{d+1}\frac{d!^{2/d}}{(d+1)^{1/d}}; \tag{1.3}$$

- *for* $1 \le p < \infty$,

$$C_{d,p} = \left\{ \frac{1}{\mathrm{vol}(\tau)} \int_\tau \left[1 - \sum_{i=1}^{d+1} \lambda_i^2(\boldsymbol{x})\right]^p d\boldsymbol{x} \right\}^{1/p} \frac{d!^{2/d}}{(d+1)^{1/d}} \tag{1.4}$$

$$= \left\{ \sum_{j=0}^{\infty} (-1)^j \frac{d!(p)_j}{(2j+d)!} \sum_{|\boldsymbol{\alpha}|=j} \frac{(2\boldsymbol{\alpha})!}{\boldsymbol{\alpha}!} \right\}^{1/p} \frac{d!^{2/d}}{(d+1)^{1/d}}, \tag{1.5}$$

where $\{\lambda_i(\boldsymbol{x})\}_{i=1}^{d+1}$ *is the barycentric coordinates of* $\boldsymbol{x}$*,* $(p)_j = p(p-1)\cdots(p-j+1)$ *is the Pochhammer symbol, and* $\boldsymbol{\alpha}$ *is a* $d+1$*-multi-index.*

The formula (1.4) can be simplified in several ways. For example, in two dimensions, by choosing the equilateral triangle inscribed to the unit circle with a vertex being $(0,1)$, one can obtain

$$C_{2,p} = \left(\frac{4}{3\sqrt{3}}\right)^{1+1/p} \left[\frac{\pi}{p+1} - 6\int_{1/2}^{1} x(1-x^2)^p \arccos\frac{1}{2x}\,\mathrm{dx}\right]^{1/p}. \tag{1.6}$$

The formula (1.5) is obtained by the Taylor expansion of the integrand in (1.4) and an integral formula of barycentric coordinates. Another formula for integers p involving only finite summation will be presented in Section 3.

By the relation between L^p-norms, we have the following properties of $C_{d,p}$.

(1) $C_{d,p}$ is monotone increasing with respect to p:

$$C_{d,p} < C_{d,q}, \quad \text{if } 1 \le p < q \le \infty.$$

(2) $\lim_{p\to\infty} C_{d,p} = C_{d,\infty}$. We choose $p \ge 1$ as integers and use (1.5) to get

$$\lim_{p\to\infty} \left[\sum_{j=0}^{p} \frac{(-1)^j p! d!}{j!(2j+d)!} \sum_{|\boldsymbol{\alpha}|=j} \frac{(2\boldsymbol{\alpha})!}{\boldsymbol{\alpha}!}\right]^{1/p} = \frac{d}{d+1}. \tag{1.7}$$

The rest of this paper is organized as follows. In Section 2, we shall introduce barycentric coordinates and derive an error formula for the linear interpolation of quadratic functions. In Section 3, we shall prove the Theorem 1.1 and determine the constant $C_{d,p}$.

2. Simplex and barycentric calculus

2.1. **Simplex and barycentric coordinate.** Let $\boldsymbol{x}_i = (x_{1,i}, \cdots, x_{d,i})^t, i = 1, \cdots, d+1$ be $d+1$ points in $\mathbb{R}^d$. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $\overrightarrow{\boldsymbol{x}\boldsymbol{y}}$ represents the vector pointing from $\boldsymbol{x}$ to $\boldsymbol{y}$. We say $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ do not all lie in one hyperplane if the d-vectors $\overrightarrow{\boldsymbol{x}_1\boldsymbol{x}_2}, \ldots, \overrightarrow{\boldsymbol{x}_1\boldsymbol{x}_{d+1}}$ are linear independent. For any $\boldsymbol{x} \in \mathbb{R}^d$, we can find unique $d+1$ real numbers $\lambda_i(\boldsymbol{x}), i = 1, \ldots, d+1$, such that

$$\boldsymbol{x} = \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i, \quad \text{and} \quad \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x}) = 1. \tag{2.1}$$

The numbers $\lambda_i(\boldsymbol{x})$ are called *barycentric coordinates* of $\boldsymbol{x}$ with respect to the $d+1$ points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$. The *convex hull* of $d+1$ points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$ is the set of points of $\mathbb{R}^d$ with barycentric coordinates satisfying $0 \le \lambda_i(\boldsymbol{x}) \le 1, i = 1, \dots, d+1$. Namely

$$\tau := \{\boldsymbol{x} = \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i \,|\, 0 \le \lambda_i \le 1, i = 1, \dots, d+1, \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x}) = 1\} \tag{2.2}$$

We call τ the d-*simplex* generated by the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{d+1}$, which are called the *vertices* of τ. For an integer $1 \le m \le d-1$, an m-dimensional face of τ is any m-simplex generated by $m+1$ of the vertices of τ. A one-dimensional face is an edge with two ending vertices.

2.2. **Multi-index and an integral formula.** We shall introduce some short-hand notation for multiple indices. A k- multi-index vector $\boldsymbol{\alpha}$ is a k-tuple of non-negative integers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$. The length of $\boldsymbol{\alpha}$ is defined by $|\boldsymbol{\alpha}| = \sum_{i=1}^k \alpha_i$, and the factorial of $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha}! = \alpha_1!\alpha_2! \cdots \alpha_k!$. For a vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_k)$, we define $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$. Note that in the notation $\boldsymbol{x}^{\boldsymbol{\alpha}}$, the power $\boldsymbol{\alpha}$ is a vector having the same length as that of the vector $\boldsymbol{x}$. It should not be confused with the notation $\boldsymbol{x}^2 := \boldsymbol{x}^t\boldsymbol{x} = \sum_{i=1}^d x_i^2$.

Thanks to the multi-index notation, when $j \ge 1$ is an integer, we have a compact formula on the power of a sum

$$\Big(\sum_{i=1}^k x_i\Big)^j = \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!}\boldsymbol{x}^{\boldsymbol{\alpha}}, \tag{2.3}$$

where $\boldsymbol{\alpha}$ is a k-multi-index. The following integral formulae can be easily proved by the induction of d.

Lemma 2.1. *Let τ be a d-simplex and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{d+1})$. For a $d+1$ multi-index $\boldsymbol{\alpha}$, one has*

$$\int_\tau \boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\boldsymbol{x})d\boldsymbol{x} = \frac{\boldsymbol{\alpha}!d!}{(|\boldsymbol{\alpha}|+d)!}\,\mathrm{vol}(\tau), \tag{2.4}$$

2.3. **Linear interpolation and error formula.** Given a d-simplex $\tau \in \mathbb{R}^d$ and a continuous function $u(\boldsymbol{x})$ defined over τ, we define the linear nodal interpolation of u by setting

$$u_I(\boldsymbol{x}) = \sum_{i=1}^{d+1} u(\boldsymbol{x}_i)\lambda_i(\boldsymbol{x}). \tag{2.5}$$

We shall give an explicit formula for $u_I - u$ when u is a quadratic function. Throughout this paper, ∇u denotes the gradient of u, $\nabla^2 u$ the Hessian matrix, and $H = \frac{1}{2}\nabla^2 u$. Note that the error formula presented below holds for any quadratic function. It is not restricted to the convex case.

Lemma 2.2. *Let τ be a d-simplex with vertices $(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_{d+1})$, and u be a quadratic function. Let $l_{ij}^2 = (\boldsymbol{x}_i - \boldsymbol{x}_j)^t H(\boldsymbol{x}_i - \boldsymbol{x}_j)$ with $H = \frac{1}{2}\nabla^2 u$. Then*

$$(u_I - u)(\boldsymbol{x}) = \sum_{i,j=1, j>i}^{d+1} (\lambda_i\lambda_j)(\boldsymbol{x})l_{ij}^2. \tag{2.6}$$

Proof. By Taylor expansion,

$$u(\boldsymbol{x}_i) = u(\boldsymbol{x}) + \nabla u(\boldsymbol{x})(\boldsymbol{x}_i - \boldsymbol{x}) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_i)^t\nabla^2 u(\boldsymbol{x} - \boldsymbol{x}_i). \tag{2.7}$$

Multiplying both sides of (2.7) by λ_i and summing up for all i, we obtain

$$\sum_{i=1}^{d+1}\lambda_i u(\boldsymbol{x}_i) = u(\boldsymbol{x})\sum_{i=1}^{d+1}\lambda_i + \nabla u(\boldsymbol{x})\sum_{i=1}^{d+1}\lambda_i(\boldsymbol{x}_i-\boldsymbol{x}) + \sum_{i=1}^{d+1}\lambda_i(\boldsymbol{x}-\boldsymbol{x}_i)^t H(\boldsymbol{x}-\boldsymbol{x}_i).$$

By the definition (2.5), the left hand side is $u_I(\boldsymbol{x})$. By the property (2.1), the first term in the right hand side is $u(\boldsymbol{x})$ and the second term is vanished. We thus obtain

$$u_I(\boldsymbol{x}) - u(\boldsymbol{x}) = \sum_{i,j=1}^{d+1}\lambda_i\lambda_j(\boldsymbol{x}_j-\boldsymbol{x}_i)^t H(\boldsymbol{x}-\boldsymbol{x}_i). \tag{2.8}$$

Switching the index i, j, we obtain an equivalent formula

$$u_I(\boldsymbol{x}) - u(\boldsymbol{x}) = \sum_{i,j=1}^{d+1}\lambda_i\lambda_j(\boldsymbol{x}_i-\boldsymbol{x}_j)^t H(\boldsymbol{x}-\boldsymbol{x}_j). \tag{2.9}$$

Summing (2.8) and (2.9), we obtain

$$u_I(\boldsymbol{x}) - u(\boldsymbol{x}) = \frac{1}{2}\sum_{i,j=1}^{d+1}\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{x})l_{ij}^2 = \sum_{i,j=1,j>i}^{d+1}\lambda_i(\boldsymbol{x})\lambda_j(\boldsymbol{x})l_{ij}^2.$$

□

When u is convex, the quantity l_{ij}^2 represents the square of the edge length of $\overrightarrow{\boldsymbol{x}_i\boldsymbol{x}_j}$ under the metric H. We can then use well known geometric inequalities to study the optimization problem (1.1). For general quadratic functions, l_{ij}^2 could be negative which makes the optimization problem more complicated.

3. Optimal simplex

3.1. **L^p-norm for $p \geq 1$.** In the formula (2.6), there are two summation indices i and j. It would be more convenient to switch to one single index, say lexigraphically, i.e. $l_k^2, k = 1, 2, \ldots, N_d$, where $N_d = d(d+1)/2$ is the number of edges of a d-simplex. For a given index $k = 1, 2, \ldots, N_d$, we shall use i and j to denote the indices of two ending vertices of the k-th edge. Let $\boldsymbol{b} = (b_1, b_2, \cdots, b_{N_d}), b_k = \lambda_i\lambda_j$ and $\boldsymbol{t} = (t_1, t_2, \cdots, t_{N_d}), t_k = l_{ij}^2, k = 1, \ldots, N_d$. We define the following function:

$$E_p(\boldsymbol{t}) = \int_\tau \Big[\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}\Big]^p d\boldsymbol{x}.$$

Let $D = \{\boldsymbol{t} \in \mathbb{R}^{N_d}, \sum_{k=1}^{N_d} t_k = 1, t_k > 0, k = 1, 2, \ldots, N_d\}$. The sequence $\{\boldsymbol{t}^n\}$ generated by the following procedure is useful.

```
subroutine {t_n} = equidistribution(t)
Starting from t^0 = t. Set n = 0.
while max_k t_k^n > min_k t_k^n
    Choose t_i^n = max_k t_k^n and t_j^n = min_k t_k^n;
    Update t_i^{n+1} = (t_i^n + t_j^n)/2, t_j^{n+1} = (t_i^n + t_j^n)/2;
    Set n = n + 1;
end
```

Lemma 3.1. *For any $\boldsymbol{t} \in D, \boldsymbol{t} \neq \boldsymbol{t}^*$, let $\{\boldsymbol{t}^n\}$ be the sequence generated by the subroutine* `equidistribution(t)`. *Then*

(1) $\lim_{n\to\infty} \|\boldsymbol{t}^n - \boldsymbol{t}^*\|_\infty = 0.$

(2) *when* $p > 1$, $E_p(\boldsymbol{t}^{n+1}) < E_p(\boldsymbol{t}^n)$.

Proof. Obviously $\{\max_k t_k^n\}$ is a strictly decreasing sequence bounded below by $1/N_d$. Therefore $\lim_{n\to\infty} \max_k t_k^n$ exists. Suppose $\max_k t_k^n = 1/N_d + \delta$ with $\delta > 0$. Since $\min_k t_k^n < 1/N_d < \max_k t_k^n$, $\max_k t_k^{n+N_d} \le 1/N_d + \delta/2$. This implies that

$$\lim_{n\to\infty} \max_k t_k^n = 1/N_d.$$

Similarly one can prove $\lim_{n\to\infty} \min_k t_k^n = 1/N_d$. Then (1) follows.

To prove (2), without loss of generality, we assume $t_1^n = \max_k t_k^n, t_2^n = \min_k t_k^n$, and define

$$\boldsymbol{t}(s) = (a+s, a-s, t_3^n, \cdots, t_{N_d}^n), \quad \text{for } s \in (-a, a),$$

where $a = (t_1^n + t_2^n)/2$. Let us consider the 1-dimensional function

$$f(s) := E_p(\boldsymbol{t}(s)). \tag{3.1}$$

Then

$$\boldsymbol{t}^n = \boldsymbol{t}(s^*), E_p(\boldsymbol{t}^n) = f(s^*) \quad \text{for } s^* = (t_1^n - t_2^n)/2,$$
$$\text{and } \boldsymbol{t}^{n+1} = \boldsymbol{t}(0), E_p(\boldsymbol{t}^{n+1}) = f(0).$$

By the direct computation, we obtain

$$f'(0) = p\int_\tau [\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(0)]^{p-1}(b_1 - b_2)(\boldsymbol{x})d\boldsymbol{x}, \text{ and} \tag{3.2}$$

$$f''(s) = p(p-1)\int_\tau [\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(s)]^{p-1}(b_1 - b_2)^2(\boldsymbol{x})d\boldsymbol{x}. \tag{3.3}$$

By the symmetric of the function $\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(0)$, we have

$$\int_\tau [\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(0)]^{p-1} b_1(\boldsymbol{x})d\boldsymbol{x} = \int_\tau [\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(0)]^{p-1} b_2(\boldsymbol{x})d\boldsymbol{x},$$

and thus $f'(0) = 0$. Since $p > 1$ and $\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{t}(s) > 0$, we also have $f''(s) > 0$ for all $s \in (-a, a)$, i.e., $f(s)$ is strictly convex in the interval $(-a, a)$. We conclude that $s = 0$ is the minimal point of $f(s)$ in $(-a, a)$. Noting that $s^* \in (-a, a)$ and $s^* \neq 0$, we obtain

$$E_p(\boldsymbol{t}^{n+1}) = f(0) < f(s^*) = E_p(\boldsymbol{t}^n).$$

□

Lemma 3.2. *For any* $\boldsymbol{t} = (t_1, t_2, \cdots t_{N_d})$ *with* $t_k > 0$, $k = 1, 2, \cdots, N_d$,

$$E_p(\boldsymbol{t}) \ge E_p(\boldsymbol{t}^*)\left(\sum_{i=1}^{N_d} t_i\right)^p, \tag{3.4}$$

where $\boldsymbol{t}^* = (1/N_d, 1/N_d, \cdots, 1/N_d)$. *When* $p > 1$ *the equality holds if and only if* $\boldsymbol{t} = \boldsymbol{t}^*$.

Proof. When $p = 1$, by (2.4),

$$E_1(\boldsymbol{t}) = \sum_{k=1}^{N_d} t_k \int_\tau b_k(\boldsymbol{x})d\boldsymbol{x} = \frac{d!\,\mathrm{vol}(\tau)}{(d+2)!}\sum_{k=1}^{N_d} t_k = E_1(\boldsymbol{t}^*)\sum_{k=1}^{N_d} t_k.$$

Namely (3.4) holds for $p = 1$.

We then prove (3.4) for $p > 1$. By the following obvious identity

$$E_p\Big(\frac{\boldsymbol{t}}{\sum_{k=1}^{N_d} t_k}\Big) = \frac{E_p(\boldsymbol{t})}{\left(\sum_{k=1}^{N_d} t_k\right)^p},$$

it suffices to prove that for $p > 1$

$$E_p(\boldsymbol{t}) \geq E_p(\boldsymbol{t}^*), \quad \forall\, \boldsymbol{t} \in D, \tag{3.5}$$

and the equality holds if and only if $t_k = 1/N_d$ for all $k = 1, \cdots, N_d$.

For any $\boldsymbol{t} \in D, \boldsymbol{t} \neq \boldsymbol{t}^*$, we consider the sequence $\{\boldsymbol{t}^n\}$ generated by the subroutine `equidistribution(`$\boldsymbol{t}$`)`. By Lemma 3.1 (1), $\boldsymbol{t}^n \to \boldsymbol{t}^*$ in the maximum norm. Since $E_p(\boldsymbol{t})$ is a continuous function of $\boldsymbol{t}$ in the maximum norm, $E_p(\boldsymbol{t}^*) = \lim_{n\to\infty} E_p(\boldsymbol{t}^n)$. By Lemma 3.1 (2)

$$E_p(\boldsymbol{t}) = E_p(\boldsymbol{t}^0) > E_p(\boldsymbol{t}^1) > \cdots > E_p(\boldsymbol{t}^*).$$

The inequality (3.5) then follows. □

The next step is to connect the edge length with the volume of a simplex. The following lemma can be found at [20] (p.517).

Lemma 3.3. *Let τ be a d-simplex and l_k the edge length of τ, $k = 1, \cdots, d(d+1)/2$. One has*

$$\sum_{k=1}^{d(d+1)/2} l_k^2 \geq \frac{d(d+1)d!^{2/d}}{(d+1)^{1/d}} \operatorname{vol}(\tau)^{2/d},$$

the equality holds if and only if τ is equilateral.

We are in a position to prove Theorem 1.1 for $1 \leq p < \infty$.

Proof of Theorem 1.1 for $1 \leq p < \infty$. For $u(\boldsymbol{x}) = \boldsymbol{x}^2$, $|(u_I - u)(\boldsymbol{x})| = (u_I - u)(\boldsymbol{x})$, and $t_k = l_k^2 > 0, k = 1, \ldots, N_d$. Therefore

$$\|u - u_I\|_{L^p(\tau)} = E_p(\boldsymbol{t})^{1/p} \geq E_p(\boldsymbol{t}^*)^{1/p} \sum_{k=1}^{N_d} l_k^2 \geq E_p(\boldsymbol{t}^*)^{1/p} \frac{d(d+1)d!^{2/d}}{(d+1)^{1/d}} \operatorname{vol}(\tau)^{2/d}.$$

By Lemma 3.2 and Lemma 3.3, all equalities hold if and only if all edge lengths are equal. We thus proved that the optimal simplex is the equilateral simplex. □

The following identity is useful for the determination of $C_{d,p}$.

Lemma 3.4. *One has*

$$\sum_{k=1}^{N_d} b_k(\boldsymbol{x}) = \frac{1}{2}\left[1 - \sum_{i=1}^{d+1} \lambda_i^2(\boldsymbol{x})\right]. \tag{3.6}$$

Proof.

$$\sum_{k=1}^{N_d} b_k = \frac{1}{2} \sum_{i,j=1, i\neq j}^{d+1} \lambda_i\lambda_j = \frac{1}{2}\left[\Big(\sum_{i=1}^{d+1} \lambda_i\Big)^2 - \sum_{i=1}^{d+1} \lambda_i^2\right] = \frac{1}{2}\left[1 - \sum_{i=1}^{d+1} \lambda_i^2\right].$$

□

To get the formula of the constant $C_{d,p}$, let us compute $E_p(\boldsymbol{t}^*)$ using (3.6)

$$\begin{aligned} E_p(\boldsymbol{t}^*)^{1/p} &= \frac{1}{N_d}\left\{\int_\tau \Big[\sum_{k=1}^{N_d} b_k(\boldsymbol{x})\Big]^p d\boldsymbol{x}\right\}^{1/p} \\ &= \frac{1}{d(d+1)}\left\{\int_\tau \Big[1 - \sum_{i=1}^{d+1} \lambda_i^2(\boldsymbol{x})\Big]^p d\boldsymbol{x}\right\}^{1/p} \end{aligned}$$

which leads to the formula of $C_{d,p}$ in (1.4). The formula (1.5) comes from the combination of the Taylor series

$$(1-x)^p = \sum_{j=0}^{\infty} \frac{p(p-1)\cdots(p-j+1)}{j!}(-1)^j x^j, \quad |x| < 1, \tag{3.7}$$

with the expression of a power of a sum using multi-index notation

$$\Big(\sum_{i=1}^{d+1} \lambda_i^2\Big)^j = \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}|=j} \frac{j!}{\boldsymbol{\alpha}!} \boldsymbol{\lambda}^{2\boldsymbol{\alpha}},$$

and the integral formula (2.4). Since $\lambda_i(\boldsymbol{x}) \in (0,1)$, for any $\boldsymbol{x} \in \tau$ and $\boldsymbol{x} \notin \partial\tau$,

$$\sum_{i=1}^{d+1} \lambda_i^2(\boldsymbol{x}) < \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x}) = 1,$$

Taylor series (3.7) for $x = \sum_i \lambda_i^2$ converges uniformly. We thus can exchange the summation and the integration in the derivation of $C_{d,p}$.

We shall derive another formula for integers $p \geq 1$. Given a $d(d+1)/2$-multi-index $\boldsymbol{\beta}$, we define a new $d+1$-multi-index $\boldsymbol{\gamma}$ such that

$$\boldsymbol{b}^{\boldsymbol{\beta}} = \boldsymbol{\lambda}^{\boldsymbol{\gamma}}. \tag{3.8}$$

With this notation, we can write out the error formulae for convex quadratic functions u

$$\|u - u_I\|_{L^p(\tau)}^p = \frac{p!d! \operatorname{vol}(\tau)}{(2p+d)!} \sum_{|\boldsymbol{\beta}|=p} \frac{\boldsymbol{\gamma}!}{\boldsymbol{\beta}!} \boldsymbol{t}^{\boldsymbol{\beta}}, \quad \text{for integers } p \geq 1. \tag{3.9}$$

and compute $E_p(\boldsymbol{t}^*)^{1/p}$ as

$$\begin{aligned} E_p(\boldsymbol{t}^*)^{1/p} &= \frac{1}{N_d} \left\{ \int_\tau \Big[\sum_{k=1}^{d(d+1)/2} b_k(\boldsymbol{x}) \Big]^p d\boldsymbol{x} \right\}^{1/p} = \frac{2}{d(d+1)} \left(\int_\tau \sum_{|\boldsymbol{\beta}|=p} \frac{p!}{\boldsymbol{\beta}!} \boldsymbol{b}^{\boldsymbol{\beta}} d\boldsymbol{x} \right)^{1/p} \\ &= \frac{2}{d(d+1)} \left(\int_\tau \sum_{|\boldsymbol{\beta}|=p} \frac{p!}{\boldsymbol{\beta}!} \boldsymbol{\lambda}^{\boldsymbol{\gamma}} d\boldsymbol{x} \right)^{1/p} = \frac{2 \operatorname{vol}(\tau)^{1/p}}{d(d+1)} \left[\frac{p!d!}{(2p+d)!} \sum_{|\boldsymbol{\beta}|=p} \frac{\boldsymbol{\gamma}!}{\boldsymbol{\beta}!} \right]^{1/p}, \end{aligned}$$

which leads to the following formula

$$C_{d,p} = \left[\frac{p!d!}{(2p+d)!} \sum_{|\boldsymbol{\beta}|=p} \frac{\boldsymbol{\gamma}!}{\boldsymbol{\beta}!} \right]^{1/p} \frac{2\, d!^{2/d}}{(d+1)^{1/d}}. \tag{3.10}$$

Comparing (1.5) and (3.10), we obtain an interesting identity:

$$\frac{2^p}{(2p+d)!} \sum_{|\boldsymbol{\beta}|=p} \frac{\boldsymbol{\gamma}!}{\boldsymbol{\beta}!} = \sum_{j=0}^{p} \frac{(-1)^j}{j!(2j+d)!} \sum_{|\boldsymbol{\alpha}|=j} \frac{(2\boldsymbol{\alpha})!}{\boldsymbol{\alpha}!}, \quad \text{for integers } p \geq 1. \tag{3.11}$$

3.2. **Maximum norm.** Although the same conclusion holds for $p = \infty$, the proof and the determination of the best constant is different. The proof of the following results can be found at [9, 23, 24]. For the completeness, we include the proof here.

Lemma 3.5. *Let R be the minimal radius of all balls covering τ. Then for $u(\boldsymbol{x}) = \boldsymbol{x}^2$,*

$$\|u_I - u\|_{L^\infty(\tau)} = R^2.$$

Proof. By (2.7), $E(\boldsymbol{x}) := u_I(\boldsymbol{x}) - u(\boldsymbol{x})$ depends only on the quadratic part of the function. Let $\boldsymbol{x}_o$ be the circum-center of τ and R_τ the circum-radius. We consider another quadratic function $v(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{x}_o\|^2$. Then v_I is a constant function, i.e., $v_I = R_\tau^2$ since $v(\boldsymbol{x}_i) = R_\tau^2$. By looking at this way, we obtain

$$E(\boldsymbol{x}) = u_I(\boldsymbol{x}) - u(\boldsymbol{x}) = v_I(\boldsymbol{x}) - v(\boldsymbol{x}) = R_\tau^2 - \|\boldsymbol{x} - \boldsymbol{x}_o\|^2. \tag{3.12}$$

Obviously

$$\max_{\boldsymbol{x}\in\tau} E(\boldsymbol{x}) = R_\tau^2 - \min_{\boldsymbol{x}\in\tau} \|\boldsymbol{x} - \boldsymbol{x}_o\|^2.$$

If $\boldsymbol{x}_o \in \tau$, then $\max_{\boldsymbol{x}\in\tau} E(\boldsymbol{x}) = R_\tau^2$. Otherwise $E(\boldsymbol{x})$ attains its maximum at $\boldsymbol{x}^*$, the projection of $\boldsymbol{x}_o$ to τ. Namely

$$\max_{\boldsymbol{x}\in\tau} E(\boldsymbol{x}) = R_\tau^2 - \|\boldsymbol{x}_o - \boldsymbol{x}^*\|^2.$$

In this case $\boldsymbol{x}^*$ is on some facet σ of τ, which is a $(d-1)$-simplex. By the definition of the projection, for $\boldsymbol{x} \in \sigma$

$$\|\boldsymbol{x} - \boldsymbol{x}^*\|^2 + \|\boldsymbol{x}^* - \boldsymbol{x}_o\|^2 = \|\boldsymbol{x} - \boldsymbol{x}_o\|^2. \tag{3.13}$$

Without loss of generality, we may assume σ is opposite to the vertex $\boldsymbol{x}_{d+1}$, namely σ is generated by $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_d$. By (3.13), all the distances between $\boldsymbol{x}_i$ $(1 \le i \le d)$ and $\boldsymbol{x}^*$ are equal. Thus $\boldsymbol{x}^*$ is the circum-center of σ and $\max_{\boldsymbol{x}\in\tau} E(\boldsymbol{x}) = R_\sigma^2$, where R_σ is the radius of the circum-sphere of σ.

When $\boldsymbol{x}_o \in \tau$, obviously $B(\boldsymbol{x}_o, R_\tau)$ covers τ. When $\boldsymbol{x}_o \notin \tau$, we shall prove the ball centered at $\boldsymbol{x}^*$ with radius R_σ can also cover the simplex τ.

By the characterization of the projection $(\boldsymbol{x}_o - \boldsymbol{x}^*) \cdot (\boldsymbol{x}_{d+1} - \boldsymbol{x}^*) \le 0$, we obtain

$$\begin{aligned}
\|\boldsymbol{x}_{d+1} - \boldsymbol{x}^*\|^2 &= \|\boldsymbol{x}_{d+1} - \boldsymbol{x}_o\|^2 + \|\boldsymbol{x}_o - \boldsymbol{x}^*\|^2 + 2(\boldsymbol{x}_{d+1} - \boldsymbol{x}_o) \cdot (\boldsymbol{x}_o - \boldsymbol{x}^*) \\
&= R_\tau^2 - \|\boldsymbol{x}_o - \boldsymbol{x}^*\|^2 + 2(\boldsymbol{x}_{d+1} - \boldsymbol{x}^*) \cdot (\boldsymbol{x}_o - \boldsymbol{x}^*) \\
&\le R_\tau^2 - \|\boldsymbol{x}_o - \boldsymbol{x}^*\|^2 = R_\sigma^2.
\end{aligned}$$

This proves the vertex $\boldsymbol{x}_{d+1}$ is covered by the ball $B(\boldsymbol{x}^*, R_\sigma)$. As a circum-sphere of σ, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$, vertices of σ are covered by this ball. Therefore $\tau \subset B(\boldsymbol{x}^*, R_\sigma)$.

Now we prove R is the minimal radius of all balls covering τ, where $R = R_\tau$ or R_σ. The intersection of all balls centered at vertices $\boldsymbol{x}_i$ with radius R is only $\boldsymbol{x}^*$, i.e., $\cap_{i=1}^{d+1} B(\boldsymbol{x}_i, R) = \boldsymbol{x}^*$. Thus for any other sphere $B(\boldsymbol{y}, R_y)$ covers τ. If $\boldsymbol{y} \neq \boldsymbol{x}^*$, then there exists i such that $\boldsymbol{y} \notin B(\boldsymbol{x}_i, R)$. Therefore $R_y \ge \|\boldsymbol{x}_i - \boldsymbol{y}\| > R$. □

Proof of Theorem 1.1 for $p = \infty$. When $\|u_I - u\|_{L^\infty(\tau)} = R_\tau^2$, we use the geometric inequality [20] (p.515) between the circum-radius and the volume of a simplex:

$$R_\tau^2 \ge \frac{d}{d+1} \frac{d!^{2/d}}{(d+1)^{1/d}} \operatorname{vol}(\tau)^{2/d}. \tag{3.14}$$

The equality holds if and only if τ is regular.

When $\|u_I - u\|_{L^\infty(\tau)} = R_\sigma^2$, we shall construct a simplex τ' with $\operatorname{vol}(\tau') \ge \operatorname{vol}(\tau)$ and τ' is inscribed to $B(\boldsymbol{x}^*, R_\sigma)$. Let us choose a coordinate such that $\boldsymbol{x}^*$ is the origin and σ is on the hyperplane $\boldsymbol{x}_{d+1} = 0$. We project the vertex $\boldsymbol{x}_{d+1}$ to the boundary of the ball and denote the projection as $\boldsymbol{x}'_{d+1}$. Then $\boldsymbol{x}'_{d+1}$ and σ generate an inscribed simplex τ' with $\operatorname{vol}(\tau') \ge \operatorname{vol}(\tau)$.

Applying (3.14) to τ', we obtain

$$R_\sigma^2 \ge \frac{d}{d+1} \frac{d!^{2/d}}{(d+1)^{1/d}} \operatorname{vol}(\tau')^{2/d} \ge \frac{d}{d+1} \frac{d!^{2/d}}{(d+1)^{1/d}} \operatorname{vol}(\tau)^{2/d}. \tag{3.15}$$

Combining (3.14) and (3.15), we finish the proof. □

REFERENCES

[1] V. Babenko, Y. Babenko, A. Ligun, and A. Shumeiko. On asymptotical behavior of the optimal linear spline interpolation error of C^2 functions. *East J. Approx.*, 12(1):71–101, 2006.
[2] V. F. Babenko. Interpolation of continuous mappings by piecewise linear ones. *Mat. Zametki*, 24(1):43–52, 141, 1978.
[3] V. F. Babenko and A. A. Ligun. The order of the best unilateral approximations by polynomials and splines in the L_p metric. *Mat. Zametki*, 19(3):323–329, 1976.
[4] M. Berzins. A solution-based triangular and tetrahedral mesh quality indicator. *SIAM Journal on Scientific Computing*, 19:2051–2060, 1998.
[5] K. Böröczky. Approximation of general smooth convex bodies. *Advances in Mathematics*, 153(2):325–341, 2000.
[6] K. Böröczky. About the error term for best approximation with respect to the Hausdorff related metrics. *Discrete and Computational Geometry*, 25(2):293–309, 2001.
[7] W. Cao. On the error of linear interpolation and the orientation, aspect ratio, and internal angles of a triangle. *SIAM Journal on Numerical Analysis*, 43(1):19–40, 2005.
[8] L. Chen. Mesh smoothing schemes based on optimal Delaunay triangulations. In *13th International Meshing Roundtable*, pages 109–120, Williamsburg, VA, 2004. Sandia National Laboratories.
[9] L. Chen. New analysis of the sphere covering problems and optimal polytope approximation of convex bodies. *Journal of Approximation Theory*, 133(1):134–145, 2005.
[10] L. Chen. *Robust and Accurate Algorithms for Solving Anisotropic Singularities*. PhD thesis, Department of Mathematics, The Pennsylvania State University, 2005.
[11] L. Chen, P. Sun, and J. Xu. Optimal anisotropic simplicial meshes for minimizing interpolation errors in L^p-norm. *Mathematics of Computation*, 76(257):179–204, 2007.
[12] E. F. D'Azevedo and R. B. Simpson. On optimal interpolation triangle incidences. *SIAM Journal on Scientific and Statistical Computing*, 6:1063–1075, 1989.
[13] E. F. D'Azevedo and R. B. Simpson. On optimal regular meshes for minimizing the gradient error. *Numerische Mathematik*, 59:321–348, 1991.
[14] C. de Boor. Good approximation by splines with variable knots. *Int. Seines Numer. Math, Birkhauser Verlag, Basel*, 21:57–72, 1973.
[15] S. Glasauer and P. M. Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies III. *Forum Math.*, 9:383–404, 1997.
[16] P. M. Gruber. Volume approximation of convex bodies by inscribed polytopes. *Math. Ann.*, 281:229–245, 1988.
[17] P. M. Gruber. Aspects of approximation of convex bodies. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry.*, volume A, pages 319–345. Amsterdam: North-Holland, 1993.
[18] K. Lipnikov and Y. Vassilevski. Optimal triangulations: Existence, approximation and double differentiation of P_1 finite element functions. *Comput. Math. Math. Phys.*, 43(6):827–835, 2003.
[19] M. Ludwig. Asymptotic approximation of smooth convex bodies by general polytopes. *Mathematika*, 46(1):103–125, 1999.
[20] D. S. Mitrinovic, J. E. Pecaric, and V. Volenec. *Recent Advances in Geometric Inequalities*. Mathematics and its applications: East European Series 28, 1989.
[21] E. Nadler. *Piecewise Linear Approximation on Triangulations of a Planar Region*. PhD thesis, Brown University, 1985.
[22] E. Nadler. Piecewise linear best L_2 approximation on triangulations. In C. K. Chui, L. L. Schumaker, and J. D. Ward, editors, *Approximation Theory*, volume V, pages 499–502. Academic Press, 1986.
[23] H. Pottmann, R. Krasauskas, B. Hamann, K. Joy, and W. Seibold. On piecewise linear approximation of quadratic functions. *J. Geometry and Graphics*, 4(1):31–53, 2000.
[24] V. T. Rajan. Optimality of the Delaunay triangulation in R^d. *Proc. of the Seventh Annual Symp. on Comp. Geom*, pages 357–363, 1991.
[25] R. Schneider. Zur optimalen approximation konvexer hyperflachen durch polyeder. *Math. Ann.*, 256:289–301, 1981.
[26] R. Schneider. Smooth approximation of convex bodies. *Rend. Circ. Mat. Palermo (2)*, 33(3):436–440, 1984.
[27] L. F. Tóth. *Lagerungen in der Ebene, auf der Kugel und im Raum*. Springer, Berlin, reprinted 1972.