IMA Journal of Numerical Analysis (2011) Page 1 of 18
doi:10.1093/imanum/drnxxx

Convergence Analysis of V-Cycle Multigrid Methods for Anisotropic Elliptic Equations

YONGKE WU[1]†, LONG CHEN[2]‡, XIAOPING XIE[1]§ AND JINCHAO XU[3]¶
1. School of Mathematics, Sichuan University, Chengdu 610064, China

2. Department of Mathematics, University of California at Irvine, Irvine, CA 92697, USA

3. School of Mathematical Sciences, Peking University, and Department of Mathematics, Pennsylvania State University, University Park, PA 16801, USA

[Received on xx; revised on xx]

Fast multigrid solvers are considered for the linear systems arising from the bilinear finite element discretizations of second order elliptic equations with anisotropic diffusion. Optimal convergence of V-cycle multigrid method in the semi-coarsening case and nearly optimal convergence of V-cycle multigrid method with line smoothing in the uniformly coarsening case are established using the XZ identity. Since the "regularity assumption" is not used in the analysis, the result can be extended to general domains consisting of rectangles.

Keywords: Bilinear element, Anisotropic Equations, Multigrid method.

1. Introduction

As one of the most efficient methods for obtaining approximations to solutions of partial differential equations, multigrid methods have been used extensively (see (1)-(15) and the references therein). This paper will present convergence analysis of multigrid methods for second order elliptic equations with anisotropic diffusion.

One of the standard multigrid analysis, proposed by Hackbusch (15) and later extended by Bramble and Pasciak (7), imposes the "regularity and approximation" assumption. This hypothesis can be verified by using the regularity theory of elliptic equations with isotropic diffusion and the approximation properties of multilevel discrete spaces. Along this approach, for anisotropic diffusion elliptic equations, Stevenson (19; 20) proved the uniform convergence of the W-cycle and V-cycle multigrid methods. Bramble and Zhang (4) extended the results to anisotropic problems with variable coefficients on a rectangular domain. Another framework of multigrid analysis was propose by Bramble, Pasciak, Wang and Xu (8; 5); see also Xu (24) and Yserentant (26). Using this framework, Neuss (17) gave an analysis for anisotropic elliptic equations. In all of these works, the regularity assumption is critical in the analysis.

We shall use a new framework of multigrid analysis developed recently by Xu and Zikatanov (22). By using the XZ identity (22), we are able to prove uniform or nearly uniform convergence of two variants of V-cycle multigrid methods for anisotropic diffusion elliptic equations *without any regularity*

†Email: wuyongkescu06@yahoo.com.cn
‡Email: chenlong@math.uci.edu
§Email: xpxiec@gmail.com
¶Email: xu@math.psu.edu

assumption. For simplicity of exposition, we mainly present our analysis in the unit square domain and briefly mention the generalization to domains which can be decomposed into rectangles. Note that for those domains the full regularity, in general, does not hold and thus cannot be covered by most existing work; see Section 5. It should be pointed out that, for the semi-coarsening mesh, Griebel and Oswald (14) obtained an optimal multilevel additive preconditioners without regularity assumption. But the result is restricted on tensor product-type grids. And Stevenson (19) proved the uniform convergence of W-cycle multigrid methods with sufficiently large number of smoothing on a L-shape domain for anisotropic elliptic problem using a refined regularity result.

The two variants of V-cycle multigrid methods considered in this paper include one with standard Gauss-Seidel smoothers on meshes obtained by semi-coarsening, and another with line Gauss-Seidel smoothers on meshes by the standard uniformly coarsening. For the semi-coarsening case, we define an interpolation operator in one direction only and prove its stability in the corresponding H^1-norm in that direction. For the uniformly coarsening with line smoothers case, we define a new quasi-interpolation and prove its stability in the energy norm. These two interpolations play an important role in the analysis.

The rest of this paper is organized as follows. In §2 we introduce the symmetric V-cycle multigrid algorithm following Bramble, Pasciak, Wang and Xu (5). In §3 we obtain convergence of the algorithm in the semi-coarsening case. In §4 we prove convergence of the algorithm in the uniformly coarsening case. In §5 we extend our results to more general domains.

Following Xu (23), we use notation $a \lesssim b$ to denote there exists a positive constant C independent of ε, h_k, h_k^x, h_y and J, such that $a \leqslant Cb$, and $a \eqsim b$ to denote $a \lesssim b \lesssim a$.

2. Multigrid Algorithms

In this section we present a model problem, describe multigrid algorithms and derive identities on the corresponding error operators.

2.1 *Problem*

Let $\Omega = (0,1)^2$ be the unit square. We consider the anisotropic diffusion equation

$$\left\{ \begin{array}{rcccc} -\partial_{xx}u - \varepsilon\partial_{yy}u & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{array} \right. \tag{2.1}$$

where $\varepsilon > 0$ is a constant and $f \in L^2(\Omega)$. We are interested in the case that $\varepsilon \ll 1$. The weak form of (2.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \text{for all } v \in H_0^1(\Omega),$$

where

$$a(u,v) = \int_\Omega (\partial_x u \partial_x v + \varepsilon \partial_y u \partial_y v)\,\mathrm{d}x\mathrm{d}y, \text{ and } (f,v) = \int_\Omega f v\,\mathrm{d}x\mathrm{d}y,$$

define two inner products on $H_0^1(\Omega)$. We define $\|\cdot\|_A^2 = a(\cdot,\cdot)$ the energy norm on $H_0^1(\Omega)$.

Assume that

$$\mathscr{T}_0 \subset \mathscr{T}_1 \subset \cdots \subset \mathscr{T}_J$$

is a sequence of nested rectangular partitions of Ω, and let h_k^x denote the mesh size in the x-direction and h_k^y the mesh size in the y-direction on the kth level mesh $\mathscr{T}_k$, $0 \leqslant k \leqslant J$. The finest mesh $\mathscr{T}_J$ is a uniform mesh by dividing Ω into $2^{J+1} \times 2^{J+1}$ small squares with equal size, and $\mathscr{T}_k$ for $0 \leqslant k \leqslant J-1$ is obtained by either

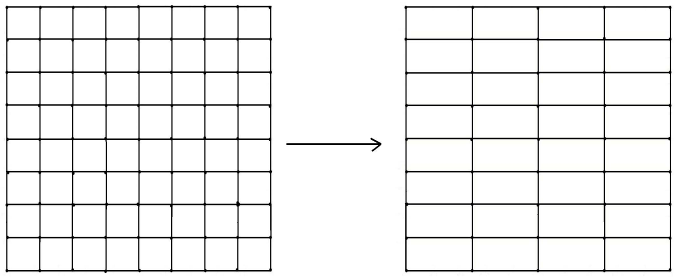


FIG. 1. Semi-Coarsening

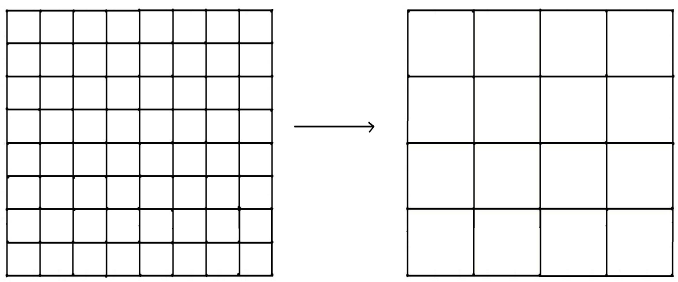


FIG. 2. uniformly coarsening

- semi-coarsening of $\mathscr{T}_{k+1}$ in the x-direction only (Fig. 1). Let $h_y = h_J^y$. The semi-coarsening should stop at the k_0th level when $k_0 = 0$ or $\frac{h_{k_0}^x}{h_y} \leqslant \frac{1}{\sqrt{\varepsilon}}$ and $\frac{2h_{k_0}^x}{h_y} > \frac{1}{\sqrt{\varepsilon}}$ and then continue with uniformly coarsening. By this semi-coarsening criterion, we always have

$$\frac{h_k^x}{h_k^y} \leqslant \frac{1}{\sqrt{\varepsilon}}, \quad \text{for } 0 \leqslant k \leqslant J, \text{ and } \qquad \frac{h_k^x}{h_k^y} \approx \frac{1}{\sqrt{\varepsilon}}, \text{ for } 0 \leqslant k \leqslant k_0. \tag{2.2}$$

- uniformly coarsening of $\mathscr{T}_{k+1}$ (Fig. 2). In this case, $h_k^x = h_k^y$ for $0 \leqslant k \leqslant J$. Therefore we use h_k to denote the mesh size on the kth level mesh.

Let M_k be the bilinear finite element space of $H_0^1(\Omega)$ associated to $\mathscr{T}_k$. We then obtain a sequence of nested spaces

$$M_0 \subset M_1 \subset \cdots \subset M_J.$$

We shall develop multigrid algorithms for solving the problem on the finest grid: Given $f \in M_J$, find $u \in M_J$ satisfying

$$a(u,v) = (f,v) \quad \text{for all } v \in M_J.$$

2.2 *Multigrid Algorithms*

To describe multigrid algorithms, we introduce the following auxiliary operators. For $k = 0, 1, 2, \cdots, J$, define the operators $A_k : M_k \mapsto M_k$ by

$$(A_k w, \phi) = a(w, \phi) \quad \text{ for all } \phi \in M_k.$$

The operator A_k is symmetric and positive definite with respect to the L^2-inner product. We define the projection operator $P_k : M_J \mapsto M_k$ in $a(\cdot,\cdot)$-inner product as

$$a(P_k w, \phi) = a(w, \phi) \quad \text{for all } \phi \in M_k$$

and the L^2 projection $Q^k_{k+1} : M_{k+1} \mapsto M_k$

$$(Q^k_{k+1} w, \phi) = (w, \phi) \quad \text{for all } \phi \in M_k.$$

To define the smoother, we introduce subspaces $M_{k,j}$ as follows:

- **The semi-coarsening case.** For $1 \leqslant k \leqslant J$, let N_k denote the number of interior nodes in the kth mesh $\mathscr{T}_k$. Let M_k be spanned by the basis $\{\varphi^k_j(x,y)\}_{j=1}^{N_k}$. We define the one dimensional subspace $M_{k,j} = \text{span}\{\varphi^k_j\}$ and the subdomain $\Omega_{k,j} = \text{supp}\{\varphi^k_j\}$, $1 \leqslant j \leqslant N_k$. For $k = 0$, to unify the notation of summation for all levels, we let $N_0 = 1,\ M_{0,1} = M_0$. Note that the dimension of M_0 may be bigger than one.

- **The uniformly coarsening case.** For $1 \leqslant k \leqslant J$, let N_k be the integer such that $\mathscr{T}_k$ partition Ω into $(N_k+1)\times(N_k+1)$ small squares. Define $\Omega_{k,j} = \{(x,y) \in \Omega :\ (j-1)h_k < y < (j+1)h_k\}$ for $2 \leqslant j \leqslant N_k$. Namely $\Omega_{k,j}$ is a horizontal strip with width $2h_k$. We define

  $$M_{k,j} = \{v \in M_k :\ v = 0 \ \text{ in } \ \Omega \backslash \Omega_{k,j}\} \quad \text{for } j = 1, \cdots, N_k.$$

  Namely the space $M_{k,j}$ is spanned by basis functions along the j-th horizontal line. The dimension of $M_{k,j}$ is N_k. Similarly for $k = 0$, let $N_0 = 1,\ M_{0,1} = M_0$.

Let $P_{k,j} : M_k \mapsto M_{k,j}$ and $Q_{k,j} : M_k \mapsto M_{k,j}$ be the projections with respect to the inner product $a(\cdot,\cdot)$ and to the L^2 inner product $(\cdot,\cdot)$, respectively. Let $A_{k,j} : M_{k,j} \mapsto M_{k,j}$ be the operator satisfying

$$a(w, v) = (A_{k,j} w, v) \qquad \text{for all } w,\ v \in M_{k,j},$$

which can be regarded as the restriction of A_k to $M_{k,j}$. It is easily to verify the following important relations

$$A_{k,i} P_{k,i} = Q_{k,i} A_k. \tag{2.3}$$

We define V-cycle multigrid algorithm as follows.

Algorithm 1. (Symmetric V-cycle multigrid)
Set $B_0^S = A_0^{-1}$.
For $k = 1, 2, \cdots, J$, define $B_k^S r$ for $r \in M_k$ as follows:
1) Pre-smoothing
   Set $u^0 = 0$.
   Define u^l for $l = 1, \cdots, m$ by
   $v \leftarrow u^{l-1}$;
   for $i = 1 : N_k$, $v \leftarrow v + A_{k,i}^{-1} Q_{k,i}(r - A_k v)$, endfor;
   $u^l \leftarrow v$.
2) Coarse grid correction
   Define $u^{m+1} = u^m + B_{k-1}^S Q_k^{k-1}(r - A_k u^m)$.
3) Post-smoothing
   Define u^l for $l = m+2, \cdots, 2m+1$ by
   $v \leftarrow u^{l-1}$;
   for $i = N_k : 1$, $v \leftarrow v + A_{k,i}^{-1} Q_{k,i}(r - A_k v)$, endfor;
   $u^l \leftarrow v$.
4) Let $B_k^S r = u^{2m+1}$.

2.3 *Convergence analysis*

We now derive formulae for error operators on the multigrid algorithm. Let

$$\begin{aligned} K_k &= (I - A_{k,N_k}^{-1} Q_{k,N_k} A_k) \cdots (I - A_{k,1}^{-1} Q_{k,1} A_k) = (I - P_{k,N_k}) \cdots (I - P_{k,1}), \\ T_k &= (I - K_k^m) P_k, \;\; k = 1, 2, \cdots, J, \; T_0 = P_0, \end{aligned}$$

and define B_J^N such that

$$I - B_J^N A_J = (I - T_0)(I - T_1) \cdots (I - T_J).$$

We then get

$$I - B_J^S A_J = (I - B_J^N A_J)^* (I - B_J^N A_J),$$

where the $(\cdot)^*$ is the adjoint in the inner product $a(\cdot, \cdot)$. Since $\|I - (I - K_k^m) P_k\|_A \leqslant \|I - (I - K_k) P_k\|_A$, we only need to consider the case $m = 1$. In this case, by using $P_{k,i} P_k = P_{k,i}$, we have

$$I - T_k = \prod_{i=1}^{N_k} (I - P_{k,i}), \tag{2.4}$$

and

$$\|I - B_J^S A_J\|_A = \|I - B_J^N A_J\|_A^2 = \| \prod_{k=0}^{J} \prod_{i=1}^{N_k} (I - P_{k,i}) \|_A^2. \tag{2.5}$$

Our analysis relies on the following fundamental identity developed by Xu and Zikatanov (22) for the multiplication of operators; see also (21; 12; 13) for alternative proofs.

THEOREM 2.1 (XZ Identity) Assume that V is a Hilbert space with A-inner product and $V_i \subset V$ $(i = 1,\dots,J)$ are closed subspaces satisfying $V = \sum_{i=1}^{J} V_i$. Let $P_i : V \mapsto V_i$ be the orthogonal projection in the A-inner product. Then the following identity holds:

$$\|(I-P_J)(I-P_{J-1})\cdots(I-P_1)\|_A^2 = 1 - \frac{1}{1+c_0},$$

$$\text{where} \quad c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{i=1}^{J} v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i+1}^{J} v_j\|_A^2.$$

In the following sections, we shall show the constant c_0 for Algorithm 1 is uniformly bounded with respect to ε and depends on h in a very weak way.

3. Convergence of Algorithm with Semi-coarsening

In this section, we give convergence analysis for the semi-coarsening case. We first introduce a transformation, define a nodal interpolation in the x-direction and then use the XZ identity to estimate the rate of convergence.

3.1 *Transformation to the isotropic case*

For $0 \leqslant k \leqslant k_0$, let $\hat{\Omega} = (0,\sqrt{\varepsilon}) \times (0,1)$ and the mapping $F : \Omega \to \hat{\Omega}$ be defined as

$$F : \begin{cases} \hat{x} = \sqrt{\varepsilon} x \\ \hat{y} = y \end{cases} . \tag{3.1}$$

Then by using (2.2), we have $\hat{\mathscr{T}}_k = \{\hat{K} = F(K) : \text{ for all } K \in \mathscr{T}_k\}, 0 \leqslant k \leqslant k_0$, are quasi-uniform partitions of $\hat{\Omega}$ with mesh size h_k^y. The differential operator $-\partial_{xx} - \varepsilon\partial_{yy}$ is transformed to $-\varepsilon\partial_{\hat{x}\hat{x}} - \varepsilon\partial_{\hat{y}\hat{y}}$. We thus return to the territory of classic multigrid theories and shall outline the proof and present the main ingredients below.

First we denoted by

$$\langle \hat{u}, \hat{v} \rangle = \int_{\hat{\Omega}} \hat{u}\hat{v}\, d\hat{x} = \sqrt{\varepsilon}(u,v), \quad \hat{a}(\hat{u},\hat{v}) := a(u,v) = \sqrt{\varepsilon}\langle \hat{\nabla}\hat{u}, \hat{\nabla}\hat{v} \rangle. \tag{3.2}$$

Let $\hat{M}_k = \{\hat{v}(\hat{x},\hat{y}) = v(x,y) : \text{ for all } v \in M_k\}$ be the bilinear finite element space on $\hat{\mathscr{T}}_k, 0 \leqslant k \leqslant k_0$. We can define the L^2 projection $\hat{Q}_k : L^2(\hat{\Omega}) \mapsto \hat{M}_k,\ 0 \leqslant k \leqslant k_0$ as

$$\langle \hat{Q}_k \hat{u}, \hat{v} \rangle = \langle \hat{u}, \hat{v} \rangle, \qquad \text{for all } \hat{u} \in L^2(\hat{\Omega}), \text{ and } \hat{v} \in \hat{M}_k.$$

and the H^1 projection $\hat{P}_k : H^1(\hat{\Omega}) \mapsto \hat{M}_k,\ 0 \leqslant k \leqslant k_0$ as

$$\hat{a}(\hat{P}_k \hat{u}, \hat{v}) = \hat{a}(\hat{u}, \hat{v}) \qquad \text{for all } \hat{u} \in H^1(\hat{\Omega}), \text{ and } \hat{v} \in \hat{M}_k.$$

It is easy to verify that for any $u \in H^1(\Omega)$, we have

$$\widehat{Q_k u} = \hat{Q}_k \hat{u}, \quad \text{and } \widehat{P_k u} = \hat{P}_k \hat{u}.$$

We now recall the following important ingredients in the classic multigrid convergence theory; see, for example, (25).

LEMMA 3.1 1. (Stable decomposition). For any $\hat{v} \in \hat{M}_{k_0}$, it holds

$$\sum_{k=0}^{k_0} (h_k^y)^{-2} \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{v}\|_0^2 \lesssim |\hat{v}|_1^2, \tag{3.3}$$

where for convenience of notation $\hat{Q}_{-1} := 0$.

2. (Strengthened Cauchy-Schwarz inequality). For any $\hat{u}_l \in \hat{M}_l$ and $\hat{w}_k \in \hat{M}_k$, $0 \leqslant l \leqslant k \leqslant k_0$, it holds

$$\langle \hat{\nabla} \hat{u}_l, \hat{\nabla} \hat{w}_k \rangle \lesssim \left(\frac{1}{\sqrt{2}} \right)^{k-l} (h_k^y)^{-1} \|\hat{\nabla} \hat{u}_l\| \|\hat{w}_k\|. \tag{3.4}$$

With these two ingredients, we have the following estimate.

LEMMA 3.2 For any $v \in M_{k_0}$, it holds

$$\sum_{k=0}^{k_0} |(\hat{P}_k - \hat{Q}_k)\hat{v}|_1^2 \lesssim \sum_{k=1}^{k_0} (h_k^y)^{-2} \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{v}\|^2 \lesssim |\hat{v}|_1^2. \tag{3.5}$$

Proof. Note that for $v \in M_{k_0}$, $\hat{v} \in \hat{M}_{k_0}$. Then

$$\begin{aligned}
|(\hat{P}_k - \hat{Q}_k)\hat{v}|_1^2 = (\hat{\nabla}(\hat{P}_k - \hat{Q}_k)\hat{v}, \hat{\nabla}(I - \hat{Q}_k)\hat{v}) &= \sum_{l=k+1}^{k_0} (\hat{\nabla}(\hat{P}_k - \hat{Q}_k)\hat{v}, \hat{\nabla}(\hat{Q}_l - \hat{Q}_{l-1})\hat{v}) \\
&\lesssim \sum_{l=k+1}^{k_0} \left(\frac{1}{\sqrt{2}} \right)^{l-k} (h_l^y)^{-1} \|\hat{\nabla}(\hat{P}_k - \hat{Q}_k)\hat{v}\| \|(\hat{Q}_l - \hat{Q}_{l-1})\hat{v}\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{k_0} |(\hat{P}_k - \hat{Q}_k)\hat{v}|_1^2 &\lesssim \sum_{k=0}^{k_0} \sum_{l=k+1}^{k_0} \left(\frac{1}{\sqrt{2}} \right)^{l-k} (h_l^y)^{-1} |(\hat{P}_k - \hat{Q}_k)\hat{v}|_1 \|(\hat{Q}_l - \hat{Q}_{l-1})\hat{v}\| \\
&\lesssim \left(\sum_{k=0}^{k_0} |(\hat{P}_k - \hat{Q}_k)\hat{v}|_1^2 \right)^{1/2} \left(\sum_{k=1}^{k_0} (h_k^y)^{-2} \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{v}\|^2 \right)^{1/2}.
\end{aligned}$$

Hence it follows (3.5). □

By the relation of inner products (3.2), we can easily get the stable decomposition up to level k_0.

LEMMA 3.3 For any $v \in M_{k_0}$, it holds

$$\sum_{k=0}^{k_0} \|(P_k - Q_k)v\|_A^2 \lesssim \varepsilon \sum_{k=1}^{k_0} (h_k^y)^{-2} \|(Q_k - Q_{k-1})v\|^2 \lesssim \|v\|_A^2, \tag{3.6}$$

where again $Q_{-1} := 0$.

The main difficulty in the anisotropic case is to establish the stable decomposition and Strengthened Cauchy-Schwarz inequality.

3.2 *A One Dimensional Interpolation*

In this subsection, we use i and j as global indices of interior nodes and r and c as row and column indices. We use n_k^x (n_k^y) to denote the number of grid points in the $x-$direction ($y-$direction) on the kth level mesh. Notice that when $k_0 \leqslant k \leqslant J$ the coarsening is applied to the $x-$direction only, n_k^y is independent of k. We thus simplify the notation to n^y. Also the range for i and j is from 1 to N_k while r is from 1 to n_k^x, and c is from 1 to n^y. The indices i and j can be treated as functions of r and c, i.e., $i = i(r,c)$, $j = j(r,c)$.

Let $\{\phi_r^k(x)\}_{r=1}^{n_k^x}$ be the linear nodal basis in the $x-$direction on the kth level mesh $\mathscr{T}_k$ ($k_0 \leqslant k \leqslant J$), and $\{\phi_c(y)\}_{c=1}^{n^y}$ be the linear nodal basis in the $y-$direction. We define $I_k : M_J \mapsto M_k,\ k_0 \leqslant k \leqslant J$ as the nodal interpolation operator in the x-direction only

$$(I_k v)(x,y) = \sum_{r=1}^{n_k^x} v(x_r^k, y)\phi_r^k(x).$$

LEMMA 3.4 For any $v \in M_J$, it holds

$$\sum_{k=k_0+1}^{J} (h_k^x)^{-2} \|(I_k - I_{k-1})v\|^2 \lesssim \|v\|_A^2. \tag{3.7}$$

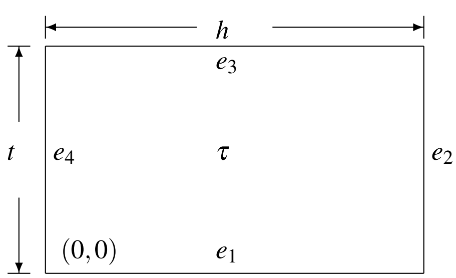
Proof. For a fixed $y \in (0,1)$, the stability

$$\sum_{k=k_0+1}^{J} (h_k^x)^{-2} \int_0^1 |(I_k - I_{k-1})v(x,y)|^2\, dx \lesssim \int_0^1 |\partial_x v(x,y)|^2\, dx,$$

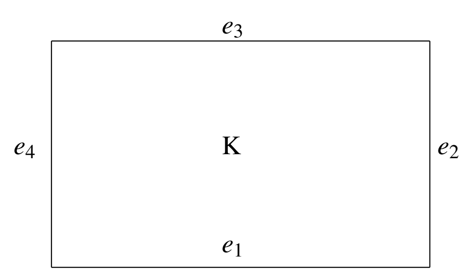
can be found in (25). Integrating over y and using Fubini's theorem for double integrals, we obtained the desirable inequality (3.7). □

3.3 *Strengthened Cauchy-Schwarz Inequality (SCS)*

To establish the SCS, we need a refined trace theorem.



(a) Rectangular τ and its four edges.



(b) An element $K \in \mathscr{T}_l$ and its four edges.

FIG. 3. A typical rectangle τ and element $K \in \mathscr{T}_l$

LEMMA 3.5 (Trace Theorem). Let τ (Fig. 3(a)) be a rectangle with width t, length h and with edges parallel to the $x-$ and $y-$axis. For any $v \in H^1(\tau)$, it holds

$$\|v\|_{0,e_i}^2 \lesssim t^{-1}\|v\|_{0,\tau}^2 + t\|\partial_y v\|_{0,\tau}^2, \qquad i = 1,3, \tag{3.8}$$

$$\|v\|_{0,e_i}^2 \lesssim h^{-1}\|v\|_{0,\tau}^2 + h\|\partial_x v\|_{0,\tau}^2, \qquad i = 2,4. \tag{3.9}$$

Proof. By density argument it suffices to prove the two estimates for $v \in C(\bar{\tau})$. Note that for $v \in C(\bar{\tau})$

$$\begin{aligned}|v(x,y)|^2 &= |v(x,0)|^2 + \int_0^y \frac{\partial(v(x,\eta))^2}{\partial_\eta}\mathrm{d}\eta \\ &= |v(x,0)|^2 + 2\int_0^y v(x,\eta)\frac{\partial v(x,\eta)}{\partial_\eta}\mathrm{d}\eta.\end{aligned}$$

Then

$$|v(x,0)|^2 \leqslant |v(x,y)|^2 + 2\int_0^y |v(x,\eta)|\left|\frac{\partial v(x,\eta)}{\partial_\eta}\right|\mathrm{d}\eta.$$

Integrating on τ and using Cauchy-Schwarz inequality, we get (3.8). Similarly, we can prove (3.9). □

LEMMA 3.6 (Strengthened Cauchy-Schwarz Inequality). For any $u_l \in M_l$ and $w_k \in M_k$, $k \geqslant l \geqslant k_0$, it holds

$$a(u_l, w_k) \lesssim \left(\frac{1}{\sqrt{2}}\right)^{k-l}(h_k^x)^{-1}\|u_l\|_A\|w_k\|. \tag{3.10}$$

Proof. For any $K \in \mathscr{T}_l$ (Fig. 3(b)), let $\mathbf{n} = (n_1, n_2)$ be the outward normal vector of ∂K. By the integration by part and Lemma 3.5, we have

$$\begin{aligned}\int_K \partial_x u_l \partial_x w_k \mathrm{d}x\mathrm{d}y &= \int_{\partial K}\partial_x u_l n_1 w_k \mathrm{d}s = \int_{e_2}\partial_x u_l w_k \mathrm{d}s - \int_{e_4}\partial_x u_l w_k \mathrm{d}s \\ &\leqslant \|\partial_x u_l\|_{0,e_2}\|w_k\|_{0,e_2} + \|\partial_x u_l\|_{0,e_4}\|w_k\|_{0,e_4} \\ &\lesssim (h_l^x)^{-1/2}\|\partial_x u_l\|_{0,K}(h_k^x)^{-1/2}\|w_k\|_{0,K} \\ &\lesssim \left(\frac{1}{\sqrt{2}}\right)^{k-l}(h_k^x)^{-1}\|\partial_x u_l\|_{0,K}\|w_k\|_{0,K}.\end{aligned}$$

Similarly

$$\begin{aligned}\int_K \partial_y u_l \partial_y w_k \mathrm{d}x\mathrm{d}y &= \int_{\partial K}\partial_y u_l n_2 w_k \mathrm{d}s = \int_{e_1}\partial_y u_l w_k \mathrm{d}s - \int_{e_3}\partial_y u_l w_k \mathrm{d}s \\ &\leqslant \|\partial_y u_l\|_{0,e_1}\|w_k\|_{0,e_1} + \|\partial_y u_l\|_{0,e_3}\|w_k\|_{0,e_3} \\ &\lesssim (h_y)^{-1/2}\|\partial_y u_l\|_{0,K}(h_y)^{-1/2}\|w_k\|_{0,K}.\end{aligned}$$

Combine the above inequalities and the relation that $\frac{\sqrt{\varepsilon}}{h_y} \leqslant \frac{1}{h_l^x} \leqslant \frac{1}{h_k^x}$, we then get our desired inequality on each element K. Summing over all K leads to (3.10). □

Using the above strengthened Cauchy-Schwarz inequality, we have

LEMMA 3.7 For any $v \in M_J$, it holds

$$\sum_{k=k_0}^{J} \|(P_k - I_k)v\|_A^2 \lesssim \sum_{k=k_0+1}^{J} (h_k^x)^{-2}\|(I_k - I_{k-1})v\|^2 \lesssim \|v\|_A^2. \tag{3.11}$$

Proof. For any $v \in M_J$, we have

$$\begin{aligned}\|(P_k - I_k)v\|_A^2 = a((P_k - I_k)v,(I - I_k)v) &= \sum_{l=k+1}^{J} a((P_k - I_k)v,(I_l - I_{l-1})v)\\ &\lesssim \sum_{l=k+1}^{J} \left(\frac{1}{\sqrt{2}}\right)^{l-k} (h_l^x)^{-1}\|(P_k - I_k)v\|_A\|(I_l - I_{l-1})v\|.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=k_0}^{J} \|(P_k - I_k)v\|_A^2 &\lesssim \sum_{k=k_0}^{J}\sum_{l=k+1}^{J} \left(\frac{1}{\sqrt{2}}\right)^{l-k} \|(P_k - I_k)v\|_A (h_l^x)^{-1}\|(I_l - I_{l-1})v\|\\ &\lesssim \left(\sum_{k=k_0}^{J} \|(P_k - I_k)v\|_A^2\right)^{1/2} \left(\sum_{k=k_0+1}^{J} (h_k^x)^{-2}\|(I_l - I_{l-1})v\|^2\right)^{1/2}.\end{aligned}$$

Hence it follows (3.11). $\square$

3.4 *Convergence Analysis*

THEOREM 3.1 Algorithm 1 with semi-coarsening meshes is convergent with a rate

$$\delta \leqslant 1 - \frac{1}{1+C},$$

where $C > 0$ is independent of ε and h.

Proof. By the XZ identity, we only need to estimate the constant

$$c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{k=0}^{J}\sum_{i=1}^{N_k} v_{k,i}=v} \sum_{k=0}^{J}\sum_{i=1}^{N_k} \|P_{k,i} \sum_{(l,j)>(k,i)} v_{l,j}\|_A^2,$$

where the ordering $(l,j) > (k,i)$ is defined by

$$(l,j) > (k,i) \text{ if } \begin{cases} l = k \text{ but } j > i; \\ l > k. \end{cases}$$

For any $v \in M_J$, we define $v_k = (I_k - I_{k-1})(v - I_{k_0}v)$ for $k_0 + 1 \leqslant k \leqslant J$ and $v_k = (Q_k - Q_{k-1})I_{k_0}v$ for $0 \leqslant k \leqslant k_0$, where for convenience of notation we define $Q_{-1} = 0$. We further decompose as $v_k = \sum_{i=1}^{N_k} v_{k,i}$ with $v_{k,i} = v_k(x_r^k, y_c)\phi_r^k(x)\phi_c(y) \in M_{k,i}$ for $i = i(r,c)$ and $k_0 \leqslant k \leqslant J$ and $v_{k,i} = v_k(x_r^k, y_c^k)\varphi_i^k(x,y)$ for

$i = i(r,c)$ and $1 \leqslant k \leqslant k_0 - 1$. For convenience of notation, we use $\mathscr{I}_k$ to denote I_k for $k_0 \leqslant k \leqslant J$ or Q_k for $0 \leqslant k \leqslant k_0 - 1$. Then we have

$$\sum_{(l,j)>(k,i)} v_{l,j} = \sum_{l=k+1}^{J} \sum_{j=1}^{N_k} v_{l,j} + \sum_{j=i+1}^{N_k} v_{k,j} = \sum_{l=k+1}^{J} v_l + \sum_{j=i+1}^{N_k} v_{k,j} = v - \mathscr{I}_k v + \sum_{j=i+1}^{N_k} v_{k,j}.$$

Consequently we have

$$\begin{aligned}
\sum_{i=1}^{N_k} \|P_{k,i} \sum_{(l,j)>(k,i)} v_{l,j}\|_A^2 &\lesssim \sum_{i=1}^{N_k} \|P_{k,i}(P_k v - \mathscr{I}_k v)\|_A^2 + \|P_{k,i} \sum_{j=i+1}^{N_k} v_{k,j}\|_A^2 \\
&\leqslant \sum_{i=1}^{N_k} \left(\|P_k v - \mathscr{I}_k v\|_{A,\Omega_{k,i}}^2 + \sum_{(i+1)\leqslant j \leqslant N_k, \Omega_{k,j}\cap\Omega_{k,i}\neq\varnothing} \|v_{k,j}\|_{A,\Omega_{k,i}}^2 \right) \\
&\lesssim \|P_k v - \mathscr{I}_k v\|_A^2 + \sum_{i=1}^{N_k} \|v_{k,i}\|_A^2.
\end{aligned}$$

Here $\|v\|_{A,\Omega_{k,i}}^2 = \int_{\Omega_{k,i}} (\partial_x v)^2 + \varepsilon(\partial_y v)^2 \mathrm{d}x\mathrm{d}y$.

For $k_0 + 1 \leqslant k \leqslant J$, in the x-direction, by the inverse inequality and stability of L^2 decomposition, we get

$$\sum_{i=1}^{N_k} \|\partial_x v_{k,i}\|^2 \lesssim \sum_{i=1}^{N_k} (h_k^x)^{-2} \|v_{k,i}\|^2 \lesssim (h_k^x)^{-2} \|v_k\|^2.$$

For the y-direction, we have

$$\varepsilon \sum_{i=1}^{N_k} \|\partial_y v_{k,i}\|^2 \lesssim \varepsilon h_y^{-2} \sum_{i=1}^{N_k} \|v_{k,i}\|^2 \lesssim \varepsilon h_y^{-2} \|v_k\|^2 \lesssim (h_k^x)^{-2} \|v_k\|^2.$$

where in the last inequality, we have used the inequality (2.2) to bound εh_y^{-2} by $(h_k^x)^{-2}$. Therefore we obtain

$$\sum_{i=1}^{N_k} \|v_{k,i}\|_A^2 = \sum_{i=1}^{N_k} (\|\partial_x v_{k,i}\|^2 + \varepsilon \|\partial_y v_{k,i}\|^2) \lesssim (h_k^x)^{-2} \|v_k\|^2, \qquad \text{for } k_0 + 1 \leqslant k \leqslant J.$$

Then, by Lemma 3.4, we have

$$\sum_{k=k_0+1}^{J} \sum_{i=1}^{N_k} \|P_{k,i} \sum_{(l,j)>(k,i)} v_{l,j}\|_A^2 \lesssim \sum_{k=k_0+1}^{J} \|(P_k - I_k)v\|_A^2 + \sum_{k=k_0+1}^{J} (h_k^x)^{-2} \|v_k\|^2 \lesssim \|v\|_A^2.$$

For $0 \leqslant k \leqslant k_0$, using the mapping (3.1), the inequality (2.2) and (3.3), we easily obtain

$$\sum_{i=1}^{N_k} \|v_{k,i}\|_A^2 = \sqrt{\varepsilon} \sum_{i=1}^{N_k} |\hat{v}_{k,i}|_{1,\hat{\Omega}}^2 \lesssim \sqrt{\varepsilon} (h_k^y)^{-2} \sum_{i=1}^{N_k} \|\hat{v}_{k,i}\|_{0,\hat{\Omega}}^2 \lesssim \sqrt{\varepsilon} (h_k^y)^{-2} \|\hat{v}_k\|_{0,\hat{\Omega}}^2 = \varepsilon (h_k^y)^{-2} \|v_k\|^2.$$

From Lemma 3.2 and inequality (3.6), we have

$$\sum_{k=0}^{k_0}\sum_{i=1}^{N_k}\|P_{k,i}\sum_{(l,j)>(k,i)}v_{l,j}\|_A^2 \lesssim \sum_{k=0}^{k_0}\|(P_k-Q_k)v\|_A^2+\varepsilon\sum_{k=0}^{k_0}(h_k^y)^{-2}\|v_k\|^2 \lesssim \|I_{k_0}v\|_A^2.$$

Since

$$\varepsilon\|\partial_y(I-I_{k_0})v\|^2 \lesssim \varepsilon(h_k^y)^{-2}\|(I-I_k)v\|^2 \lesssim \varepsilon(h_k^y)^{-2}(h_k^x)^2\|\partial_x v\|^2 \lesssim \|\partial_x v\|^2,$$

and

$$\|\partial_x(I-I_{k_0})v\|^2 \lesssim \|\partial_x v\|^2,$$

we have

$$\|I_{k_0}v\|_A^2 \leqslant \|v\|_A^2+\|(I-I_{k_0})v\|_A^2 \lesssim \|v\|_A^2.$$

Combine the above estimates, we then get

$$\sum_{k=0}^{J}\sum_{i=1}^{N_k}\|P_{k,i}\sum_{(l,j)>(k,i)}v_{l,j}\|_A^2 \leqslant C\|v\|_A^2,$$

which implies $c_0 \leqslant C$, and the desired estimate of convergence rate follows from the XZ identity.

□

4. Convergence of Algorithm with Line Smoother

In this section, we give convergence analysis for the multigrid algorithm using the uniformly coarsening with the line smoothers. We first give a property of the line smoothers, then define a stable quasi-interpolation operator, and finally use the XZ identity to estimate the rate of convergence.

4.1 *The Line Gauss-Seidel Smoother*

Since uniformly coarsening is applied, we use one mesh parameter h_k. We first prove the following property for the line Gauss-Seidel smoother.

LEMMA 4.1 For $v_k \in M_k$, let $v_k = \sum_{i=1}^{N_k} v_{k,i},\ v_{k,i}\in M_{k,i}, i=1,\cdots,N_k$. Then we have

$$\sum_{i=1}^{N_k}\|v_{k,i}\|_A^2 \lesssim \|v_k\|_A^2+\frac{\varepsilon}{h_k^2}\|v_k\|^2. \tag{4.1}$$

Proof. Note that we do not decompose in the x-direction and thus the summation is for the y-direction only. Therefore by the stability of the decomposition in L^2-norm, we have

$$\sum_{i=1}^{N_k}\|\partial_x v_{k,i}\|^2 \lesssim \|\sum_{i=1}^{N_k}\partial_x v_{k,i}\|^2 = \|\partial_x v_k\|^2. \tag{4.2}$$

For the y-direction, we use inverse inequality to get

$$\sum_{i=1}^{N_k}\|\partial_y v_{k,i}\|^2 \lesssim h_k^{-2}\sum_{i=1}^{N_k}\|v_{k,i}\|^2 \lesssim h_k^{-2}\|v_k\|^2. \tag{4.3}$$

Linear combination of (4.2) and (4.3) leads to the desirable estimate (4.1). □

4.2 *A Stable Quasi-Interpolation Operator*

On the edge (x_i^k, x_{i+1}^k), we choose $\theta_i^k = h_k^{-1}(4\phi_i^k - 2\phi_{i+1}^k) \in \mathscr{P}_1(x_i^k, x_{i+1}^k)$. Direct computation shows

$$\int_{x_i^k}^{x_{i+1}^k} \theta_i^k \phi_i^k = 1, \quad \int_{x_i^k}^{x_{i+1}^k} \theta_i^k \phi_{i+1}^k = 0,$$

where $\mathscr{P}_1(x_i^k, x_{i+1}^k)$ is the space of linear polynomial on the edge (x_i^k, x_{i+1}^k). Similar definition applies to $\theta_j^k(y)$ for the edge (y_j^k, y_{j+1}^k).

For a function $v \in H^1(\Omega)$, we define $\mathscr{I}_k^x$ and $\mathscr{I}_k^y$ as follows:

$$(\mathscr{I}_k^x v)(x,y) = \sum_{i=1}^{N_k} v_i^x(y)\phi_i^k(x), \quad (\mathscr{I}_k^y v)(x,y) = \sum_{j=1}^{N_k} v_j^y(x)\phi_j^k(y),$$

where

$$v_i^x(y) = \int_{x_i^k}^{x_{i+1}^k} \theta_i^k(x) v(x,y)\mathrm{d}x, \quad v_j^y(x) = \int_{y_j^k}^{y_{j+1}^k} \theta_j^k(y) v(x,y)\mathrm{d}y.$$

We then introduce a quasi-interpolation $\mathscr{I}_k : H_0^1(\Omega) \to M_k$ by

$$\mathscr{I}_k v = \sum_{i=1}^{N_k}\sum_{j=1}^{N_k} v_{i,j}\phi_i^k(x)\phi_j^k(y), \tag{4.4}$$

where

$$v_{i,j} = \int_{x_i^k}^{x_{i+1}^k}\int_{y_j^k}^{y_{j+1}^k} \theta_i^k(x)\theta_j^k(y) v(x,y)\,\mathrm{d}x\mathrm{d}y.$$

In this definition, since the boundary nodes are not included, we can easily to see that $\mathscr{I}_k v|_{\partial\Omega} = 0$.

LEMMA 4.2 The following properties hold for the interpolation $\mathscr{I}_k, \mathscr{I}_k^x$, and $\mathscr{I}_k^y$:

(1) Preservation of bilinear finite element functions: $\mathscr{I}_k v_k = v_k$ for $v_k \in M_k$.

(2) Approximation property: $\|v - \mathscr{I}_k v\| \lesssim h_k |v|_1$ for $v \in H_0^1(\Omega)$.

(3) Operators $\mathscr{I}_k^x$ and $\mathscr{I}_k^y$ are interchangeable and $\mathscr{I}_k = \mathscr{I}_k^x \mathscr{I}_k^y = \mathscr{I}_k^y \mathscr{I}_k^x$.

(4) $\mathscr{I}_k^x$ and $\mathscr{I}_k^y$ are stable in both L^2-norm and corresponding one dimensional H^1-norm. Namely, for all $v \in H^1$, we have

$$\|\mathscr{I}_k^x v\| \lesssim \|v\|,\ \|\mathscr{I}_k^y v\| \lesssim \|v\|, \text{ and } \|\partial_x \mathscr{I}_k^x v\| \lesssim \|\partial_x v\|,\ \|\partial_y \mathscr{I}_k^y v\| \lesssim \|\partial_y v\|.$$

(5) For $v \in M_J$,

$$\|\partial_x \mathscr{I}_k^y v\| \lesssim \|\partial_x v\|, \text{ and } \|\partial_y \mathscr{I}_k^x v\| \lesssim \|\partial_y v\|. \tag{4.5}$$

Proof. The properties (1) and (3) can be easily verified by the definition and (2) and (4) can be found in (18).

We now prove (5). First by the definition, we have, for $x \neq x_j^k, j = 1, \cdots, N_k$,

$$(\partial_x \mathscr{I}_k^y v)(x,y) = \sum_{j=1}^{N_k} \partial_x v_j(x) \phi_j^k(y) = \sum_{j=1}^{N_k} \left(\int_{y_j^k}^{y_{j+1}^k} \theta_j^k(y) \partial_x v(x,y) \mathrm{d}y \right) \phi_j^k(y) = (\mathscr{I}_k^y \partial_x v)(x,y).$$

Therefore

$$\begin{aligned} \|\partial_x \mathscr{I}_k^y v\|^2 &= \sum_{j=1}^{N_k} \|\partial_x \mathscr{I}_k^y v\|^2_{(x_j^k, x_{j+1}^k)\times(0,1)} = \sum_{j=1}^{N_k} \|\mathscr{I}_k^y \partial_x v\|^2_{(x_j^k, x_{j+1}^k)\times(0,1)} \\ &\lesssim \sum_{j=1}^{N_k} \|\partial_x v\|^2_{(x_j^k, x_{j+1}^k)\times(0,1)} = \|\partial_x v\|^2. \end{aligned}$$

The second inequality in (4.5) can be proved similarly.

□

Using these properties, we prove $\mathscr{I}_k$ is stable in the energy norm.

LEMMA 4.3 There is a positive constant C independent of ε, h_k and J such that

$$\|\mathscr{I}_k v\|_A \leqslant C\|v\|_A \quad \text{for all } v \in M_J. \tag{4.6}$$

Proof. For any $v \in M_J$, using the stability results (4) and (5) in Lemma 4.2, we get

$$\|\partial_x(\mathscr{I}_k v)\|^2 = \|\partial_x \mathscr{I}_k^x(\mathscr{I}_k^y v)\|^2 \lesssim \|\partial_x \mathscr{I}_k^y v\|^2 \lesssim \|\partial_x v\|^2.$$

The estimate $\|\partial_y(\mathscr{I}_k v)\| \lesssim \|\partial_y v\|$ is proved similarly. The stability (4.6) then follows.

□

4.3 *Convergence Analysis*

THEOREM 4.1 Algorithm 1 applied to uniformly coarsening meshes with line Gauss-Seidel smoothers is convergent with rate

$$\delta \leqslant 1 - \frac{1}{1 + C|\log h|},$$

where C is a positive constant independent of h and ε.

Proof. For any $v \in M_J$, let $v_k = (\mathscr{I}_k - \mathscr{I}_{k-1})v,\ k = 0, 1, \cdots, J$, where $\mathscr{I}_{-1} = 0$. It is obvious that $v = \sum_{k=0}^{J} v_k$. Let $v_k = \sum_{i=1}^{N_k} v_{k,i}$ with $v_{k,i} \in M_{k,i},\ i = 1, \cdots, N_k$. Following the same line as in the proof of Theorem 3.1 , we have

$$\sum_{(l,j)>(k,i)} v_{l,j} = v - \mathscr{I}_k v + \sum_{j=i+1}^{N_k} v_{k,j},$$

and

$$\sum_{i=1}^{N_k} \|P_{k,i} \sum_{(l,j)>(k,i)} v_{l,j}\|_A^2 \lesssim \|v - \mathscr{I}_k v\|_A^2 + \sum_{i=1}^{N_k} \|v_{k,i}\|_A^2.$$

By the stability of $\mathscr{I}_k$, the first term is bounded by $\|v\|_A^2$. For the second term, when $1 \leqslant k \leqslant J$, by using Lemma 4.1, we have

$$
\begin{aligned}
\sum_{i=1}^{N_k} \|v_{k,i}\|_A^2 &\lesssim \|v_k\|_A^2 + \frac{\varepsilon}{h_k^2}\|v_k\|^2 \\
&\lesssim \|v\|_A^2 + \frac{\varepsilon}{h_k^2}(\|v-\mathscr{I}_k v\|^2 + \|v-\mathscr{I}_{k-1}v\|^2) \\
&\lesssim \|v\|_A^2 + \varepsilon|v|_1^2 \lesssim \|v\|_A^2.
\end{aligned}
$$

For $k=0$, we have

$$\sum_{i=1}^{N_0} \|v_{0,i}\|_A^2 = \|v_0\|_A^2 = \|\mathscr{I}_0 v\|_A^2 \lesssim \|v\|_A^2.$$

Using the XZ identity, we can estimate c_0 as

$$c_0 \leqslant \sup_{v\in M_J, \|v\|_A=1} \sum_{k=0}^{J}\sum_{i=1}^{N_k} \|P_{k,i} \sum_{(l,j)>(k,i)} v_{l,j}\|_A^2 \leqslant C(J+1),$$

which implies the desirable rate by noting that $J \eqsim |\log h|$. $\square$

5. Convergence analysis for more general domains

In the previous sections, we give the convergence analysis of Algorithm 1 in both the semi-coarsening and the uniformly coarsening with line smoothers on the unite square. In this section, we will extend our results to more general domains consisting of rectangles.

Assume the boundaries of Ω in equation (2.1) are parallel to $x-$ or $y-$axis and Ω can be partitioned into rectangles with $\mathscr{O}(1)$ size and this partition is denoted by $\mathscr{T}_\Omega$. We then uniformly refine these rectangles J times to get the finest grid $\mathscr{T}_J$.

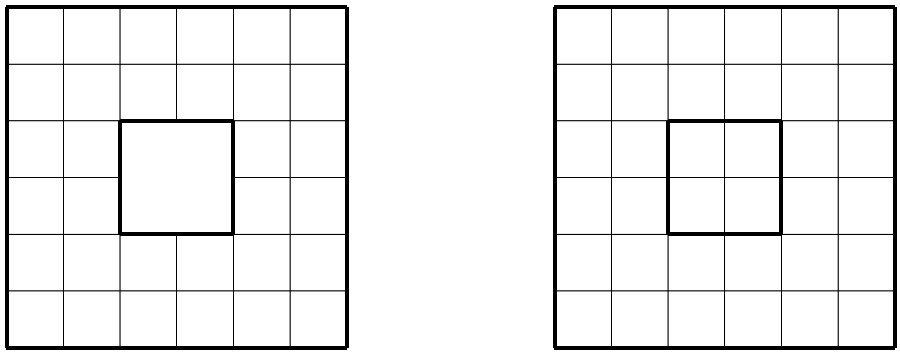
To simplify the notation, we find a rectangle R with the smallest diameter such that $\Omega \subset R$ and extend $\mathscr{T}_J$ to $\tilde{\mathscr{T}}_J$ by uniformly refined rectangles in $R\backslash\Omega$ J times. For the rectangle R and corresponding grids $\tilde{\mathscr{T}}_J$, we can apply the coarsening process in Section 2.2 to get a sequence of meshes $\tilde{\mathscr{T}}_k, 0 \leqslant k \leqslant J-1$ and obtain $\mathscr{T}_k$ by the restriction of $\tilde{\mathscr{T}}_k$ to Ω. Note that the coarsening should stop at a level such that the geometry of Ω can be seen in $\mathscr{T}_0$, i.e., $\mathscr{T}_\Omega \subset \mathscr{T}_0$. To illustrate the idea, in Fig. 4, we give the rectangle R and meshes $\mathscr{T}_k$, $\tilde{\mathscr{T}}_k$ of a square domain with a small square hole.

For semi-coarsening with point-wise smoother, once the sequence of meshes $\mathscr{T}_0 \subset \mathscr{T}_1 \subset \ldots \subset \mathscr{T}_J$ is given, we can define M_k, the bilinear finite element space of $H_0^1(\Omega)$ and $M_{k,i}$ the one dimensional subspace spanned by one basis of M_k as before. Consequently the multigrid algorithm Algorithm 1 is well defined.

For uniformly coarsening, we need to specify the line smoother. For $1 \leqslant k \leqslant J$, let N_k be the integer such that $\mathscr{T}_k$ partition R into $(N_k+1)\times(N_k+1)$ small squares. Define $R_{k,j} = \{(x,y) \in R : (j-1)h_k < y < (j+1)h_k\}$ for $2 \leqslant j \leqslant N_k$ as a horizontal strip. We define

$$M_{k,j} = \{v \in M_k : v = 0 \text{ in } \Omega\backslash R_{k,j}\} \quad \text{for } j = 1,\cdots,N_k.$$

Note that the line corresponding to $M_{k,j}$ could be broken; see, for example, the middle horizontal line in Fig 4 (a). Based on M_k and $M_{k,j}$, the Algorithm 1 is then well defined.



(a) Ω and a grid $\mathscr{T}_k$ (b) Rectangle R and the grid $\tilde{\mathscr{T}}_k$

FIG. 4. An example of a general domain and its extension.

THEOREM 5.1 For general domains Ω with boundaries parallel to $x-$ or $y-$axis and a fine grid $\mathscr{T}_J$ defined as above, Algorithm 1 is convergent with rate

$$\delta \leqslant 1 - \frac{1}{1+C}$$

when applied to semi-coarsening meshes, and with rate

$$\delta \leqslant 1 - \frac{1}{1+C|\log h|}$$

when applied to uniformly coarsening meshes with line Gauss-Seidel smoothers, where C is a positive constant independent of h and ε.

Proof. For functions $v \in H_0^1(\Omega)$, we use zero extension to get $\tilde{v} \in H_0^1(R)$, i.e., $\tilde{v}|_{\Omega} = v$ and $\tilde{v}|_{R\setminus\Omega} = 0$. Denoted this zero extension operator by E. Note that for $v \in H_0^1(\Omega)$, this extension will preserve both the L^2 and energy norm.

For the semi-coarsening case, by following the same line in subsection 3.2, we can prove the desirable results. When $0 \leqslant k \leqslant k_0$, we use L^2 projection in the decomposition. By mapping the anisotropic problem to an isotropic problem, we can establish Lemmas 3.2-3.3 on the domain Ω since the proof holds for quasi-uniform grids on general domains. For $k_0 \leqslant k \leqslant J$, we can define nodal interpolation operators $I_k : M_J \mapsto M_k,$ in the $x-$direction through the extension operator E: for $v \in \mathscr{M}_J$,

$$I_k v = I_k(E(v)).$$

Note that $I_k(E(v)) \in M_k$ since the zero extension and point-wise interpolation. Lemmas 3.4-3.7 can be obtained by simple changing of notation. With these ingredients, the desired result follows from the the same line as in the proof of Theorem 3.1.

For the case of uniformly coarsening with line Gauss-Seidel smoothers, let us examine the ingredients in the proof. First Lemma 4.1 holds for $\tilde{v}_k \in \tilde{M}_k$ which is the rectangular case. Note that for $v_k \in M_k$, if $v_k = \sum_{i=1}^{N_k} v_{k,i}$, then $E(v_k) = \sum_{i=1}^{N_k} E(v_{k,i})$. We apply Lemma 4.1 to $E(v_k) \in \tilde{M}_k$ and use the fact the extension will preserve the norm to get

$$\sum_{i=1}^{N_k} \|v_{k,i}\|_{A,\Omega}^2 = \sum_{i=1}^{N_k} \|E(v_{k,i})\|_{A,R}^2 \lesssim \|E(v_k)\|_{A,R}^2 + \frac{\varepsilon}{h_k^2}\|E(v_k)\|_R^2 = \|v_k\|_{A,\Omega}^2 + \frac{\varepsilon}{h_k^2}\|v_k\|_\Omega^2.$$

To define the quasi-interpolation operator $\mathscr{I}_k$, for a vertex on the boundary of Ω, we modify the choice of the edges such that they are on the boundary Ω. This will ensure $\mathscr{I}_k E(v)|_{\partial\Omega} = 0$ for $v \in H_0^1(\Omega)$ and thus $\mathscr{I}_k v = \mathscr{I}_k E(v) \in M_k$. Now Lemma 4.2 and 4.3 holds for $E(v)$. The stability follows from the preservation of norms

$$\|\mathscr{I}_k v\|_{A,\Omega} = \|\mathscr{I}_k E(v)\|_{A,R} \lesssim \|E(v)\|_{A,R} = \|v\|_{A,\Omega}, \qquad \text{for all } v \in M_J.$$

Then the desired result follows from the same line as in the proof of Theorem 4.1.

□

Acknowledgements

We would like to thank the reviewers for their helpful comments and suggestions, which led to the removal of $|\log h|$ factor in the semi-coarsening case.

The work of the first and third authors was supported by the National Natural Science Foundation of China (10771150), the National Basic Research Program of China (2005CB321701), and the Program for New Century Excellent Talents in University (NCET-07- 0584). The second author was supported by NSF Grant DMS-0811272, DMS-1115961, and by UC Irvine Academic Senate Council on Research, Computing and Libraries. The fourth author was supported in part by NSF DMS-0915153, NSFC-10528102 and Alexander von Humboldt Research Award for Senior US Scientists.

REFERENCES

1. R. E. BANK, AND C. C. DOUGLAS (1985) Sharp estimates for multigrid rates of convergence with general smoothing and acceleration. *SIAM J, Numer. Anal.*, **22**, 617–633.
2. R. E. BANK, T. DUPONT (1981) An optimal order process for solving finite element equations. *Math. Comp.*, **36**, 35–51.
3. D. BRAESS AND W. HACKBUSCH (1983) A new convergence proof for the multigrid method including the V-cycle. *SIAM J. Numer. Anal.*, **20**, 967–975.
4. J. H. BRAMBLE, X. ZHANG (1998) Uniform convergence of the multigrid V-cycle for an anisotropic problem. *Math. Comp.*, **70**, 453–470.
5. J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU (1991) Convergence estimates for multigrid algorithms without regularity assumptions. *Math. Comp.*, **57**, 23–45.
6. J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU (1991) Convergence estimates for product iterative methods with applications to domain decomposition. *Math. Comp.*, **195**, 1–21.
7. J. H. BRAMBLE AND J. E. PASCIAK (1987) New convergence estimates for multigrid algorithms. *Math. Comp.*, **49**, 311–329.
8. J. H. BRAMBLE AND J. E. PASCIAK (1993) New estimates for multilevel algorithms including the V-cycle. *Math. Comp.*, **60**, 447–471.
9. J. H. BRAMBLE, J. E. PASCIAK, AND J. XU (1991) The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms. *Math. Comp.*, **56**, 1–34.
10. J. H. BRAMBLE, AND J. XU (1991) Some estimates for a weighted L^2 projection. *Math. Comp.*, **56**, 463–476.
11. A. BRANDT (1977) Multi-level adaptive solutions to boundary-value problems. *Math. Comp.*, **31**, 333–390.

12. L. CHEN. (2010) Deriving the X-Z Identity from Auxiliary Space Method. In Y. Huang, R. Kornhuber, O. Widlund, and J. Xu, editors, *Domain Decomposition Methods in Science and Engineering XIX*, pages 309–316.
13. D. CHO, J. XU, AND L. ZIKATANOV. (2008) New estimates for the rate of convergence of the method of subspace corrections. *Numer. Math. Theor. Meth. Appl.*, 1:44–56.
14. M.GRIEBEL AND P. OSWALD (1995) Tensor-product-type subspace splittings and multilevel methods for anisotropic problems. *Adv. Comput. Math.*, **4**, 171–206.
15. W. HACKBUSCH (1985) *Multi-grid methods and applications*. Springer series in computational mathematics, vol. 4, Spring-Verlag, BErlin, New York.
16. P. W. HEMKER (1984) Multigrid methods for problems with a small parameter in the highest derivative. *Numerical Analysis*. Proceedings Dundee 1983. Lecture Notes in Mathematics, 1066 (Springer, Berlin Heidelberg New York)(D.F. Griffiths, ed.), 106–121.
17. N. NEUSS (1998) V-cycle convergence with unsymmetric smoothers and application to an anisotropic model problem. *SIAM. Numer. Anal.*, **35**, 1201–1212.
18. L. R. SCOTT AND S. ZHANG (1990) Finite element interpolation of nonsmoothing functions satisfying boundary conditions. *Math. Comp.*, **54**, 483–493.
19. R. STEVENSON (1993) Robustness of multi-grid applied to anisotropic equations on convex domains and on domains with re-entrant corners. *Numer. Math.*, **66**, 373–398.
20. R. STEVENSON (1993) New estimates of the contraction number of V-cycle multi-grid with applications to anisotropic equations, Incomplete Decompositions, Proceedings of the Eiglish GAMM Seminar. *Notes on Numerical Fluid Mechanics*, **41**, 159–167.
21. P. S. VASSILEVSKI. (2008) *Multilevel Block Factorization Preconditioners: Matrix-based Analysis and Algorithms for Solving Finite Element Equations*. Springer Verlag.
22. J. XU, AND L. ZIKATANOV (2002) The method of alternating projections and the method of subspace corrections in Hilbert space. *Journal of the American mathematical society*, **15**, 573–597.
23. J. XU (1989) *Theory of multilevel methods*. PH. D. Thesis, Cornell University and Penn State University, Dept. Math. Rep AM-48.
24. J. XU (1992) Iterative methods by space decomposition and subspace correction. *SIAM Review*, **34**, 581–613.
25. J. XU. (1997) *An introduction to multigrid convergence theory.* In R. Chan, T. Chan, and G. Golub, editors, *Iterative Methods in Scientific Computing*. Springer-Verlag.
26. H. YSERENTANT (1993) Old and new convergence proofs for multigrid methods. *Acta Numerica*, **2**, 285–326.