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Postprocessing Mixed Finite Element Methods For Solving Cahn–Hilliard Equation: Methods and Error Analysis

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Abstract A postprocessing technique for mixed finite element methods for the Cahn–Hilliard equation is developed and analyzed. Once the mixed finite element approximations have been computed at a fixed time on a coarser space, the approximations are postprocessed by solving two decoupled Poisson equations in an enriched finite element space (either on a finer grid or a higher-order space) for which many fast Poisson solvers can be applied. The nonlinear iteration is only applied to a much smaller size problem and the computational cost using Newton and direct solvers is negligible compared with the cost of the linear problem. The analysis presented here shows that this technique remains the optimal rate of convergence for both the concentration and the chemical potential approximations. The corresponding error estimate obtained in our paper, especially the negative norm error estimates, are non-trivial and different with the existing results in the literatures.

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1 Introduction

The purpose of this paper is to study a postprocessing technique for mixed finite element (MFE) methods for the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta(\epsilon \Delta u - \phi(u)) = 0, \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

subject to the no flux boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(\phi(u) - \epsilon \Delta u) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^d (d = 2, 3)$ with a sufficiently smooth boundary $\partial\Omega$, ν is the outward unit normal vector along $\partial\Omega$, $\epsilon > 0$ is a phenomenological constant modeling the effect of interfacial energy, and ϕ is the derivative of a smooth double equal well potential. A typical example of ϕ is

$$\phi(u) = \Phi'(u); \qquad \Phi(u) = \frac{1}{4}(u^2 - 1)^2. \tag{1.4}$$

The differential equation (1.1) arises in continuum models of phase separation and spinodal decomposition, c.f. [4,8,33,35]. The field variable u is a scaled concentration of one species in a binary mixture.

Different numerical schemes have been proposed for solving Cahn–Hilliard equation. A main theme of the study is on the energy stable time discretization. Stabilization [37] or convex splitting scheme [11,30,42] is general technique to obtain an energy stable discretization. The energy stability for spectral methods is given in [26,37], for the local discontinuous Garlerkin method in [44], for a non-conforming finite method in [48], and for the finite difference scheme in [42]. We shall not explore more on the energy stability in this paper.

We are interested in efficient ways on enhancing the accuracy of numerical approximation to the Cahn–Hilliard equation. When the domain is rectangular, energy stable spectral methods developed in [6,27,37] can be used with high order accuracy. For unstructured grids of a general domain with possible complex geometry, finite element methods will be a better choice. The main difficulty in finite element approximations of the Cahn–Hilliard equation is that conforming finite element spaces for fourth order equations is not easy to construct especially in three dimensions. Possible remedy is non-conforming elements [9,40,48] or discontinuous Galerkin methods [1,7,14,31,41,44]. Here we consider the mixed finite element (MFE) approximation since it can give not only a numerical approximation to the concentration u but also a numerical approximation to the chemical potential $w = \phi(u) - \epsilon\Delta u$. The mixed finite element method for solving the Cahn–Hilliard equation is, to our best knowledge, first studied by Elliott et al. in [9]. The stability and the convergence of MFE for the Cahn–Hilliard equation are further investigated in [10,12,13].

We shall apply a postprocessing technique to improve the accuracy and computational efficiency of MFE methods for Cahn–Hilliard equations (1.1)–(1.3). The basic idea of this postprocessing technique is to solve two linear elliptic problems in an enriched finite element space (either on a finer grid or a higher-order space) once the time integration on the coarse space is completed. The linear problem in the enriched space can be solved efficiently by fast Poisson solvers, e.g., multigrid methods. The nonlinear iteration is only applied to a much smaller size problem and the computational cost using Newton method and direct solvers for inverting Jacobian matrix is negligible compared with the cost of the linear problem. We note that the nonlinear multigrid method, e.g., [1,32,43] can also solve the nonlinear system efficiently. Considering the fact that many fast Poisson solvers are available, the postprocessing method will thus be much easier to implement than nonlinear multigrid methods. The analysis presented in this paper shows that this technique remains the optimal rate of convergence for both the concentration and the chemical potential approximations.

The postprocessing technique we study here was originally developed for spectral methods for parabolic equations in [23,24]. Later on it was extended to methods based on Chebyshev and Legendre polynomial [15], spectral element methods [16,17], finite element methods [17,25], and mixed finite element methods for Navier–Stokes equations [2,3,19,22]. The analysis of fully discrete procedures can be developed along the same lines [20,21,47]. In these works, theoretical analysis and numerical experiments show that the postprocessed method is computationally more efficient than the method to which it is applied. On the other hand, we observe that the postprocessing technique is mainly applied to second order nonlinear evolutional equations like the Navier–Stokes equations and the reaction-diffusion equations. In the original paper [23] in which the postprocessing method is introduced, the postprocessing spectral method has been applied to fourth order problems like the Cahn–Hilliard equation and the Kuramoto–Sivashinsky equation. At that moment, either its analysis or its development seemed to depend heavily on the properties of the Fourier modes. The extension to finite element methods on unstructured grids is restricted to second order equations [17,25]. The analysis in [17,25] could be applied to fourth order equations provided a conforming finite element space is used. As we mentioned before, however, a conforming element for fourth order equations is not easy to construct.

In this paper, we shall extend the postprocessing technique to MFE methods for the Cahn–Hilliard equation. Unlike the postprocessing MFE applied to second order problems such as Navier–Stokes equations, the analysis of postprocessing MFE for fourth order problems is based on the coupled system of u and w which is much more complex than that for second order problems. Although the postprocessing method can be applied to fourth order problems and the mixed methods for fourth order problems are standard, the corresponding error estimates obtained in our paper, especially the negative norm error estimates, are non-trivial. We generalize the error estimate in the H^{-1} norm for the second order elliptic equations; see, e.g. [29] and [28], to both H^{-1} and H^{-2} norm for the fourth order equations.

The postprocessing method can also be regarded as a two-grid method, where the postprocessed (or fine-grid) approximation is an improvement of the previously computed (coarse-grid) approximation [45,46]. For the evolution problem, the difference between postprocessing methods and two-grid methods is that in the postprocessing method the fine grid computations can be done only at the final time T while in the two-grid methods, e.g. [34], the fine grid computation is applied at every time step.

The rest of the paper is as follows. In Sect. 2 we describe the postprocessing methods and present the main result. In Sect. 3 we briefly review the derivation of the Cahn–Hilliard equation and Sobolev spaces used in the analysis. In Sect. 4 we recall some properties of MFE methods and collect some inequalities to be used later. We prove the main result in Sect. 5 and present a numerical example which is only of academic value in Sect. 6. More realistic numerical tests have been presented in [49]. Finally, we give a summary.

2 Postprocessing Mixed Finite Element Methods

In this section we describe our methods and present the corresponding error estimate.

The Cahn–Hilliard equation (1.1) with boundary condition (1.3) can be written in the mixed formulation as

$$\frac{\partial u}{\partial t} - \Delta w = 0, \qquad x \in \Omega, \quad t > 0, \tag{2.1}$$

$$\phi(u) - \epsilon \Delta u - w = 0, \qquad x \in \Omega, \quad t > 0, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.3}$$

with the initial value conditions (1.2) and $w(x, 0) = \phi(u_0(x)) - \epsilon \Delta u_0(x)$ for all $x \in \Omega$. The Cahn–Hilliard equation (1.1) arises from a gradient flow of the Ginzburg-Landau free energy

$$\mathcal{E}(u) = \int_\Omega \left(\Phi(u) + \frac{\epsilon}{2}|\nabla u|^2\right) dx.$$

The Cahn–Hilliard equation is frequently referred to as the H^{-1} gradient flow:

$$u_t = -\mathrm{grad}_0 \mathcal{E}(u),$$

where the symbol "grad_0" denotes a constrained gradient in a Hilbert space, defined by $\int_\Omega u\, dx = $ constant. It is easy to verify the global mass conservation and energy dissipation of the model problem (1.1)–(1.3). Considering the mass conservation and $u - \bar{u} = u - \frac{1}{|\Omega|}\int_\Omega u\ dx$, we may consider functions with zero average. Therefore, we assume $\int_\Omega u_0(x)\ dx = 0$. For the chemical potential w, similarly, we replace $w - \bar{w}$ with w in the following analysis. Due to the boundary condition, the natural Sobolev space is $\dot{H}^1 := \{u \in H^1, \int_\Omega u\ dx = 0\}$. The weak formulation is: Find $u \in \dot{H}^1$, $w \in \dot{H}^1$, such that

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla w, \nabla v) = 0 \qquad \text{for all} \quad v \in \dot{H}^1, \tag{2.4}$$

$$\epsilon(\nabla u, \nabla \chi) + (\phi(u) - w, \chi) = 0 \qquad \text{for all} \quad \chi \in \dot{H}^1. \tag{2.5}$$

2.1 Standard Mixed Finite Element Methods

Let $\mathcal{T}_h,\ h \geq 0$, be a family of partitions of suitable domains Ω_h, where the parameter h is the maximum diameter of the elements in $\mathcal{T}_h$. For $r \geq 2$, we consider the finite element space

$$S_{h,r} = \left\{v_h \in \mathcal{C}(\overline{\Omega}_h) | v_{h|\tau} \in \mathcal{P}^{r-1}, \text{for all } \tau \in \mathcal{T}_h\right\},$$

where $\mathcal{P}^{r-1}$ denotes the space of polynomials of degree at most $r-1$. Note that for the linear finite element space, $r = 2$. The subspace $\dot{S}_{h,r}$ is defined by

$$\dot{S}_{h,r} = \{\chi_h \in S_{h,r} : (\chi_h, 1) = 0\}.$$

Then the mixed finite element method for Cahn–Hillard equation reads: Find $(u_h, w_h) : [0, T] \to \dot{S}_{h,r} \times \dot{S}_{h,r}$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + (\nabla w_h, \nabla v_h) = 0 \qquad \text{for all } v_h \in \dot{S}_{h,r}, \tag{2.6}$$

$$\epsilon(\nabla u_h, \nabla \chi_h) + (\phi(u_h) - w_h, \chi_h) = 0 \qquad \text{for all } \chi_h \in \dot{S}_{h,r}, \tag{2.7}$$

with a suitable starting approximation $u_h(0) \in \dot{S}_{h,r}$. For simplicity we will take $u_h(0)$ as the Galerkin projection of u_0 in the analysis, although other choices are possible. Let us introduce the standard L^2-orthogonal projection $P_h : L^2 \to \dot{S}_{h,r}$ and the operator $A_h : \dot{S}_{h,r} \to \dot{S}_{h,r}$

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h) \qquad \text{for all } u_h, v_h \in \dot{S}_{h,r}. \tag{2.8}$$

Then (2.6)–(2.7) can be equivalently written as

$$u_{h,t} + \epsilon A_h^2 u_h + A_h P_h \phi(u_h) = 0, \qquad t > 0. \tag{2.9}$$

By definition, A_h is self-adjoint positive definite (SPD) on $\dot{S}_{h,r}$. Its inverse will be denoted by G_h and extended to L^2 by $G_h f = G_h P_h f$ for $f \in L^2$. Then, $v_h = G_h f$ is equivalent to $A_h v_h = P_h f$. Note that G_h is also SPD on L^2 and $\dot{S}_{h,r}$. For the discrete chemical potential w_h in (2.6)-(2.7), we have $w_h = \epsilon A_h u_h + P_h \phi(u_h) = -G_h u_{h,t}$. In [9], it has been shown that $u_h(t)$ is bounded in H^1 independent of t, and the following error estimate holds:

$$\|u(t) - u_h(t)\| + \|w(t) - w_h(t)\| + \|u(t) - u_h(t)\|_1 \le Ch^r, \quad 0 \le t \le T. \tag{2.10}$$

2.2 Postprocessing Mixed Finite Element Methods

Given a finite element space $\dot{S}_{H,r} \subset \dot{H}^1$, the semi-discrete mixed finite element approximation of (2.1)–(2.2) is defined as: Find $(u_H, w_H) : [0, T] \to \dot{S}_{H,r} \times \dot{S}_{H,r}$ such that

$$\left(\frac{\partial u_H}{\partial t}, v_H\right) + (\nabla w_H, \nabla v_H) = 0 \qquad \text{for all} \quad v_H \in \dot{S}_{H,r}, \tag{2.11}$$

$$\epsilon(\nabla u_H, \nabla \chi_H) + (\phi(u_H) - w_H, \chi_H) = 0 \qquad \text{for all} \quad \chi_H \in \dot{S}_{H,r}, \tag{2.12}$$

with a suitable starting approximations $u_H(0) \in \dot{S}_{H,r}$ and $w_H(0) = P_H\phi(u_H(0)) + \epsilon A_H u_H(0) \in \dot{S}_{H,r}$.

We are interested in approximations at a certain time $T > 0$. The postprocessing method is as follows:

(i) First, integrate (2.11)–(2.12) up to T to obtain the MFE approximations $u_H(T)$ and $w_H(T)$.
(ii) Then solve the following two decoupled linear elliptic problems: Search for $(u^h, w^h) \equiv (u^h(T), w^h(T)) \in \dot{S}_{h,\tilde{r}} \times \dot{S}_{h,\tilde{r}}$ satisfying

$$\left(\nabla w^h, \nabla v_h\right) = -\left(\frac{\partial u_H}{\partial t}(T), v_h\right) \qquad \text{for all} \quad v_h \in \dot{S}_{h,\tilde{r}}, \tag{2.13}$$

$$\epsilon(\nabla u^h, \nabla \chi_h) = (w_H(T) - \phi(u_H(T)), \chi_h) \qquad \text{for all} \quad \chi_h \in \dot{S}_{h,\tilde{r}}. \tag{2.14}$$

The finite element space $\dot{S}_{h,\tilde{r}} \supset \dot{S}_{H,r}$ can be chosen as either

1. the same-order finite element over a finer grid: $\dot{S}_{h,\tilde{r}} = \dot{S}_{h,r}$ with $\tilde{r} = r,\ h < H$, or
2. a higher-order finite element over the same grid: $\dot{S}_{h,\tilde{r}} = \dot{S}_{H,\tilde{r}}$ with $h = H$, where $\tilde{r}$ is defined by, for L^2 norm error estimates

$$\tilde{r} = \begin{cases} r+1, & \text{if } r = 3, \\ r+2, & \text{if } r \ge 4, \end{cases}$$

and, for H^1 norm error estimates

$$\tilde{r} = \begin{cases} r+1, & \text{if } r = 2, \\ r+2, & \text{if } r \ge 3. \end{cases}$$

We present the error estimate below but defer the proof until Sect. 5.

Theorem 2.1 *Fix $T>0$, let (u,w) be the solution of (1.1)–(1.3), (u_H,w_H) be the MFE approximation obtained from (2.11)–(2.12), and $(u^h(T),w^h(T))\in \dot{S}_{h,\tilde r}\times \dot{S}_{h,\tilde r}$ be the post-processed MFE approximation defined in (2.13)–(2.14). Suppose $u\in C([0,T],\dot{H}^r)\cap C^2((0,T],L^2)$ and $w\in L^2([0,T],\dot{H}^r)\cap C^1((0,T],L^2)$. Then, for $r\ge 2$, there exists a positive constant $C>0$ such that*

$$\|w(T)-w^h(T)\|\le Ch^{\tilde r}+CH^{r+\delta(r,l)}\ell_H(l),\qquad l=3,4, \tag{2.15}$$

$$\|w(T)-w^h(T)\|_1\le Ch^{\tilde r-1}+CH^{r+\min\{r-2,1\}}, \tag{2.16}$$

$$\|u(T)-u^h(T)\|\le Ch^{\tilde r}+CH^{r+\delta(r,l)}\ell_H(l),\qquad l=3,4, \tag{2.17}$$

$$\|u(T)-u^h(T)\|_1\le Ch^{\tilde r-1}+CH^{r+\min\{r-2,1\}}. \tag{2.18}$$

Here and in the rest of the paper, $\delta(r,l)$ and $\ell_H(l)$ are defined, respectively, by

$$\delta(r,l)=\begin{cases}\min\{r-2,l\}, & \text{if } l=0,1,2,\\ \min\{r-2,l-2\}, & \text{if } l=3,4,\end{cases} \tag{2.19}$$

and

$$\ell_H(l)=\begin{cases}1 & \text{if } l=0,1,3,\\ |\log(H)|, & \text{if } l=2,4.\end{cases} \tag{2.20}$$

We observe that the postprocessing technique applied to the linear MFE method improves the rate of convergence of the error in the H^1 norm only but not in the L^2 norm which is standard for postprocessing methods (see, e.g., [18,22]). Especially, we can postprocess the MFE approximation under the same-order finite element over a finer grid to obtain

- for linear finite element

$$\|w(T)-w^h(T)\|_1+\|u(T)-u^h(T)\|_1\le C(h+H^2);$$

- for quadratic finite element

$$\|w(T)-w^h(T)\|_1+\|u(T)-u^h(T)\|_1\le C(h^2+H^4).$$

This theorem suggests that to achieve the same convergence rate, the postprocessing MFE methods can spend less computation time than the standard one by spending less nonlinear iterations. For example, for linear or quadratic elements, we can chose a coarse grid with $H=h^{1/2}$.

We end this section by giving the essential idea of postprocessing procedure and a sketch of the proof of Theorem 2.1. Let us introduce the Ritz projection R_h which will play a prominent role in the understanding of the postprocessed method. For $v\in\dot{H}^1$, we define $R_hv\in\dot{S}_{h,\tilde r}$

$$(\nabla R_hv,\nabla\chi_h)=(\nabla v,\nabla\chi_h)\qquad\text{for all } \chi_h\in\dot{S}_{h,\tilde r}.$$

When applied to MFE approximation of Cahn–Hilliard equation, we have

$$A_h\left(w^h(T)-R_hw(T)\right)=-\left(u_H(T)-u(T)\right)_t, \tag{2.21}$$

$$A_h\left(u^h(T)-R_hu(T)\right)=\frac{1}{\epsilon}\left((w_H(T)-w(T))-(\phi(u_H(T))-\phi(u(T)))\right). \tag{2.22}$$

Therefore the norm of the error u^h-R_hu, w^h-R_hw will be bounded by a suitable negative norm of the right hand side in (2.21)-(2.22). Take (2.21) as an example. By the splitting

$(u-u_H)_t = (u-R_Hu)_t + (R_Hu-u_H)_t$ and the fact that the negative norm estimate for $(u-R_Hu)_t$ is well known, the key of the proof is then the negative norm estimate of $(R_Hu-u_H)_t$ which will be given in Sect. 5.1. For the estimate of $u^h - R_hu$, similarly, negative norm estimate of $w_H - R_Hw$ and $\phi(u_H)-\phi(u)$ is needed which is presented in Sect. 5.2.

We notice the undesirable factor $1/\epsilon$ in (2.22) which suggests that the parameter ϵ cannot be too small. Especially the postprocessing may fail if one is interested in the solution in the limit case, i.e., $\epsilon \to 0$.

3 The Cahn–Hilliard Equation

We define $L_0^2 = \{v \in L^2 : (v,1) = 0\}$, and let P be the L^2 orthogonal projection onto L_0^2, $Pf = f - \bar{f}$. We define the linear operator $A = -\Delta$ with domain of definition, i.e.,

$$\mathcal{D}(A) = \left\{ v \in H^2 \cap L_0^2 : \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

It is easy to verify A is a self-adjoint positive definite densely defined operator on L_0^2. We may write (1.1)–(1.3) as an abstract initial value problem

$$u_t + \epsilon A^2 u + AP\phi(u) = 0, \qquad t > 0, \tag{3.1}$$

$$u(0) = u_0. \tag{3.2}$$

A priori bound in the H^1 norm for the solution of (3.1)–(3.2) has been derived in [10].

Let $W^{s,p}(\Omega)$, $s \geq 0$, $p \geq 1$, be the standard Sobolev space with the norm $\|\cdot\|_{s,p}$. For convenience, we denote by $\|\cdot\|_s$ and $\|\cdot\|_\infty$ the norms of the space $H^s(\Omega) = W^{s,2}(\Omega)$ and $L^\infty(\Omega)$, respectively, and $\|\cdot\|$ for the usual norm in $L^2 = L^2(\Omega)$. Since A is self-adjoint positive semidefinite, for real s, we can define the spaces $\dot{H}^s = \mathcal{D}(A^{s/2})$ with norms $|v|_s = \|A^{s/2}v\|$. It is well known that, for integer $s \geq 0$, $\dot{H}^s$ is a subspace of $H^s \cap L_0^2$ and that the norms $|\cdot|_s$ and $\|\cdot\|_s$ are equivalent on $\dot{H}^s$. In particular, we have $\dot{H}^1 = H^1 \cap L_0^2$, see [10].

We define $G : L_0^2 \to \dot{H}^2$ as the inverse of A and extend to L^2 by $Gf = GPf$ for $f \in L^2$. Namely, $v = Gf$ if and only if $Av = Pf$. It can be easily verified that G is self-adjoint and positive definite on L_0^2 and positive semidefinite on L^2.

We recall the following embedding result: for $p, q \in [1,\infty)$, there exists a constant $C = C(\Omega, p)$ such that

$$\|v\|_{0,q} \leq C\|v\|_{s,p} \qquad \text{for } v \in W^{s,p}(\Omega) \text{ and } \frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{s}{d}; \tag{3.3}$$

For $q = \infty$ the above inequality holds for $\frac{1}{p} \geq \frac{1}{q} > \frac{1}{p} - \frac{s}{d}$, and, furthermore, in this case v is also a continuous function. From Hölder's inequality, we have the following inequality

$$|(f, v\chi)| \leq \|f\|\|v\|_{0,p}\|\chi\|_{0,q}, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad p, q > 0. \tag{3.4}$$

4 Error Analysis of Standard Mixed Finite Element Methods

4.1 Preliminaries

In this paper we consider quasi-uniform meshes $\mathcal{T}_h$ only. That is, we consider meshes $\mathcal{T}_h$ for which all elements are shape regular and of comparable size h. Then we have the following inverse estimate (see, e.g. [36]): for all $v_h \in S_{h,r}$,

$$\|v_h\|_{s,p} \le Ch^{l-s-d(1/q-1/p)}\|v_h\|_{l,q}, \quad 0 \le l \le s \le 2, \quad 1 \le q \le p \le \infty. \tag{4.1}$$

For simplicity, we will assume $\Omega_h = \Omega$, i.e., Ω is triangulated exactly. Then the following bounds hold (c.f. [25,39]): for $1 \le l \le r$ and $v \in H^l$

$$\|v - P_h v\| + \|v - R_h v\| + h\|v - R_h v\|_1 \le Ch^l\|v\|_l, \tag{4.2}$$

and, for $0 \le s \le 2$,

$$\|G^{s/2}(v - P_h v)\| + \|G^{s/2}(v - R_h v)\| \le Ch^{l+\delta(l,s)}\|v\|_l. \tag{4.3}$$

Here recall that $\delta(l, s)$ is defined in (2.19).

For $1 \le p \le \infty$, and for any $v \in \mathcal{D}(A) \cap W^{r,p}(\Omega)$, we consider the standard interpolant operator $I_h : \mathcal{D}(A) \cap H^r \to \dot{S}_{h,r}$. Under more specific assumption $v \in H^r$, the interpolant I_h satisfies

$$\|v - I_h v\| + h\|v - I_h v\|_1 \le Ch^r\|v\|_r. \tag{4.4}$$

Our error estimates obtained in this paper will depend on the following constants

$$K(u, t) = \|u(\cdot, t)\|_r + \|u_t(\cdot, t)\|_r + \|u_{tt}(\cdot, t)\|_r, \quad K(u) = \max_{0\le t\le T} K(u, t). \tag{4.5}$$

4.2 Error Analysis

We recall some error estimates obtained in [9] in this subsection. We first introduce the following error decomposition

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta_u + \rho_u, \tag{4.6}$$

$$w_h - w = (w_h - R_h w) + (R_h w - w) = \theta_w + \rho_w. \tag{4.7}$$

Lemma 4.1 *If (u, w) are sufficiently smooth then, for $t \in [0, T]$,*

$$\|D_t^j \rho_u\| + h\|D_t^j \rho_u\|_1 \le Ch^r, \qquad j = 0, 1, 2, \dots, \tag{4.8}$$

$$\|D_t^j \rho_w\| + h\|D_t^j \rho_w\|_1 \le Ch^r, \qquad j = 0, 1, 2, \dots, \tag{4.9}$$

where C is independent of h and t and $D_t^j = (\partial/\partial t)^j$.

In the following lemma we will need the bound $\|(R_h u)_t\|_\infty \le C$ for h sufficiently small. This can be obtained from

$$\begin{aligned}\|(R_h u)_t\|_\infty &\le \|(R_h u)_t - I_h u_t\|_\infty + \|u_t - I_h u_t\|_\infty + \|u_t\|_\infty \\ &\le h^{-d/2}\|(R_h u)_t - I_h u_t\| + Ch^r + \|u_t\|_r \\ &\le C(h^r + h^{r-d/2} + K(u)).\end{aligned} \tag{4.10}$$

Lemma 4.2 *Suppose (u, w) are sufficiently smooth. There exist constants C independent of h, u_h and w_h such that if $\|u_h(\cdot)\|_\infty$ is bounded independent of h then, $\forall t \in [0, T]$,*

$$\|\theta_u\|^2 + \int_0^t \|\theta_w\|^2 \le Ch^{2r} + \|\theta_u(0)\|^2, \tag{4.11}$$

$$\|\theta_u\|_1^2 + \|\theta_w\|^2 + \int_0^t \left(\|\theta_{u,t}\|^2 + \|\theta_w\|_1^2\right) \le Ch^{2r} + \|\theta_u(0)\|_1^2 + \|\theta_w(0)\|^2. \tag{4.12}$$

Then we have the following *a priori* error analysis:

Theorem 4.3 *If $\|u_h(0) - u_0\| \le Ch^r$, then*

$$\max_{0\le t\le T} \|u(t) - u_h(t)\| + \left(\int_0^T \|w(t) - w_h(t)\|dt\right)^{1/2} \le Ch^r. \tag{4.13}$$

If $\|u_h(0) - R_h u_0\|_1 \le Ch^r$ and $\|w_h(0) - R_h w_0\| \le Ch^r$,then

$$\max_{0\le t\le T} \|w(t) - w_h(t)\| + \left(\int_0^T \|(u - u_h)_t(t)\|dt\right)^{1/2} \le Ch^r, \tag{4.14}$$

$$\max_{0\le t\le T} \|u(t) - u_h(t)\|_1 + \left(\int_0^T \|(w - w_h)_t(t)\|_1 dt\right)^{1/2} \le Ch^{r-1}, \tag{4.15}$$

5 Proof of the Main Results

In our analysis we shall frequently use the following relation, for $v \in L^2$ and $\mu = 1, 2$:

$$\|G_H^{\mu/2} P_H v\| \le \|G^{\mu/2} v\| + CH^\mu \|v\|, \tag{5.1}$$

$$\|G^{\mu/2} v\| \le \|G_H^{\mu/2} P_H v\| + CH^\mu \|v\|. \tag{5.2}$$

These inequalities are readily deduced from the estimates $\|G^{\mu/2} - G_H^{\mu/2} P_H\| \le CH^\mu$ for $\mu = 1, 2$ (see, e.g., [25,38]).

As stated in Sect. 2, in order to obtain the error estimate for the postprocessing method proposed here, we should give bounds for $\|G_H^{\mu/2} P_H(\phi(v) - \phi(\psi))\|$, $\|u_H - R_H u\|$ and $\|w_H - R_H w\|$. For $\|G_H^{\mu/2} P_H(\phi(v) - \phi(\psi))\|$, in view of (5.1), we need the following result.

Lemma 5.1 *(Lemma 3 in [25]) Let $v \in H^r \cap L_0^2$ and $\psi \in L_0^2 \cap L^\infty$. Assume that F is a smooth function. Then, there exists a constant $C = C(\|v\|_r, \|\psi\|_\infty)$ such that for $\mu = 0, 1, 2$, we have that*

$$\|G^{\mu/2}(F(v) - F(\psi))\| \le C\left(\|G^{\mu/2}(v - \psi)\| + \|v - \psi\|_{0,q}\|v - \psi\|\right). \tag{5.3}$$

Here $q = \max\{2, d/\mu'\}$, where $\mu' = \mu - 1/2$ if $d/\mu = 2$; otherwise $\mu' = \mu$.

Remark 5.1 In our application of Lemma 5.1, we will choose F to be either the function ϕ or its derivative ϕ', v to be the solution u of the Cahn–Hilliard equation, and ψ to be either the corresponding finite element approximation solution u_H or the Ritz projection $R_H u$. Obviously, these choices satisfy the assumptions of Lemma 5.1. For example, the bound of $\|u_H\|_\infty$ has been proved in [9] and $\|R_H u\|_\infty \le C$ has been shown in [36]; see also (4.10).

In the following we will focus on the superconvergence estimate of $\|u_H - R_H u\|$ and $\|w_H - R_H w\|$.

5.1 Superconvergence for the Concentration

In order to estimate $\theta_u = u_H - R_H u$, we first derive the corresponding error equation. From (2.11) we have for each $\chi \in \dot{S}_{H,r}$

$$(\theta_{u,t}, \chi) + (\nabla\theta_w, \nabla\chi) = -((R_H u)_t, \chi) - (\nabla R_H w, \nabla\chi)$$
$$= -(\rho_{u,t}, \chi) - (u_t, \chi) + (\nabla w, \nabla\chi), \tag{5.4}$$

where now $\rho_u = R_H u - u$ and $\theta_w = w_H - R_H w$, and since $u_t = \Delta w$, $\partial_\nu w = 0$ we have

$$(\theta_{u,t}, \chi) + (\nabla\theta_w, \nabla\chi) = -(\rho_{u,t}, \chi) \quad \text{for all } \chi \in \dot{S}_{H,r}. \tag{5.5}$$

From (2.2) and (2.12) we obtain

$$(\theta_w, \chi) - \epsilon(\nabla\theta_u, \nabla\chi) = (\phi(u_H) - \phi(u), \chi) - (\rho_w, \chi) \quad \text{for all } \chi \in \dot{S}_{H,r}, \tag{5.6}$$

where $\rho_w = R_H w - w$. It follows from (5.5) and (5.6) that

$$\theta_{u,t} + \epsilon A_H^2 \theta_u + A_H P_H(\phi(u_H) - \phi(R_H u)) = T_H, \tag{5.7}$$

where T_H is defined by

$$T_H = A_H P_H \rho_w - A_H P_H(\phi(R_H u) - \phi(u)) - P_H \rho_{u,t}. \tag{5.8}$$

As a consequence of the error equation (5.7), the following stability inequality can be obtained.

Proposition 5.2 *Fix $T > 0$. Let u be the solution of (1.1)–(1.3), let u_H be its discrete approximation via (2.11)–(2.12), and let $R_H u$ be the elliptic projection of u onto $\dot{S}_{H,r}$. Then, there exists a positive constant $K_s > 0$ such that $\forall t_1 \leq T$, the following estimate holds:*

$$\max_{0\leq t\leq t_1} \|u_H - R_H u\| \leq K_s \max_{0\leq t\leq t_1} \left\| \int_0^t e^{-\epsilon(t-s)A_H^2} T_H(s)ds \right\|, \tag{5.9}$$

where $T_H(s)$ is given in (5.8).

Proof It follows from (5.7) that

$$\theta_u(t) = e^{-\epsilon t A_H^2} P_H \theta_u(0) + \int_0^t e^{-\epsilon(t-s)A_H^2} A_H P_H(\phi(R_H u) - \phi(u_H))ds$$
$$+ \int_0^t e^{-\epsilon(t-s)A_H^2} T_H(s)ds. \tag{5.10}$$

We write the integrand of the second term on the right-hand side of (5.10) as

$$e^{-\epsilon(t-s)A_H^2} A_H P_H(\phi(R_H u) - \phi(u_H)) = A_H e^{-\epsilon(t-s)A_H^2} P_H(\phi(R_H u) - \phi(u_H)),$$

and use the mean-value theorem to estimate

$$\left\| \int_0^t e^{-\epsilon(t-s)A_H^2} A_H P_H(\phi(R_H u) - \phi(u_H))ds \right\| \leq \frac{C_{1/2}}{\sqrt{\epsilon}} \int_0^t \frac{\|P_H(\phi(R_H u) - \phi(u_H))\|}{\sqrt{t-s}} ds$$
$$\leq \frac{K C_{1/2}}{\sqrt{\epsilon}} \int_0^t \frac{\|\theta_u(s)\|}{\sqrt{t-s}} ds. \tag{5.11}$$

Then, taking $\theta_u(0) = 0$ into consideration, we get

$$\|\theta_u(t)\| \le \frac{KC_{1/2}}{\sqrt{\epsilon}} \int_0^t \frac{\|\theta_u(s)\|}{\sqrt{t-s}} ds + \int_0^t e^{-\epsilon(t-s)A_H^2} T_H(s) ds. \tag{5.12}$$

Application of the generalized Gronwall lemma allows us to conclude the proof. □

We then estimate the right-hand side of (5.9).

Lemma 5.3 *(Lemma 5 in [25]) For any $f \in C([0,T]; L^2)$, the following estimate holds: $\forall t \in [0,T]$*

$$\int_0^t \|A_H^{\mu/2} e^{-\epsilon(t-s)A_H^2} P_H f(s)\| ds \le \frac{C}{\epsilon^{\mu/4}} \ell_H(\mu) \max_{0\le t\le T} \|f(t)\|, \quad \mu = 3, 4. \tag{5.13}$$

Consequently

$$\max_{0\le t\le t_1} \|u_H - R_H u\| \le \frac{C}{\epsilon^{\mu/4}} \ell_H(\mu) \max_{0\le t\le T} \|G_H^{\mu/2} T_H\|, \quad \mu = 3, 4.$$

We split the truncation error in negative norm as

$$\|G_H^{\mu/2} T_H(t)\| \le \|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]\| + \|G_H^{\mu/2} P_H \rho_{u,t}\| + \|G_H^{(\mu-2)/2} P_H \rho_w\|. \tag{5.14}$$

and estimate the three terms on the right-hand side of (5.14). For the second and third terms, we have the following lemma.

Lemma 5.4 *There exists a constant C, that depends on $K(u)$ given in (4.5), such that for $t \in [0,T]$ the following bounds hold:*

$$\|G_H^{\mu/2} P_H D_t^j \rho_w(t)\| \le C H^{r+\delta(r,\mu)}, \qquad \mu = 1, 2, \quad j = 0, 1, \tag{5.15}$$

$$\|G_H^{\mu/2} P_H D_t^j \rho_u(t)\| \le C H^{r+\delta(r,\mu)}, \qquad \mu = 1, 2, 3, 4, \quad j = 0, 1, 2. \tag{5.16}$$

Proof By (5.1) and (4.3), we have

$$\begin{aligned} \|G_H^{\mu/2} P_H D_t^j \rho_w(t)\| &\le \|G^{\mu/2} D_t^j \rho_w(t)\| + C H^{\mu} \|D_t^j \rho_w(t)\| \\ &\le C H^{\delta(r,\mu)} \|D_t^j \rho_w(t)\|. \end{aligned} \tag{5.17}$$

Then (5.15) follows from the estimate (4.9).

Concerning (5.16), using the same arguments as for (5.15), we can prove the cases $\mu = 1, 2$. For the cases $\mu = 3, 4$, using

$$\begin{aligned} G_H^{\mu/2} &= G_H G_H^{(\mu-2)/2} = (G_H - G) G_H^{(\mu-2)/2} + G G_H^{(\mu-2)/2} \\ &= (R_H - I) G G_H^{(\mu-2)/2} + G G_H^{(\mu-2)/2} \end{aligned}$$

and taking into account that G is bounded, we obtain (5.16). □

We are now ready to give the estimate of $\|G_H^{\mu/2} T_H(t)\|$.

Lemma 5.5 *There exists a constant C, that depends on $K(u)$ given in (4.5), such that for $t \in [0,T]$ the following bounds hold for $\mu = 3, 4$:*

$$\|G_H^{\mu/2} T_H(t)\| \le C H^{r+\delta(r,\mu)}. \tag{5.18}$$

Proof Due to (5.15) and (5.16), we only need to estimate the first term on the right-hand side of (5.14). Using (5.1), we have

$$\begin{aligned}&\|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]\|\\ &\quad\le \|G^{(\mu-2)/2}[\phi(R_H u) - \phi(u)]\| + CH^{\mu-2}\|\phi(R_H u) - \phi(u)\|. \end{aligned} \tag{5.19}$$

Then, applying Lemma 5.1, (4.2) and (4.3), we obtain

$$\begin{aligned}&\|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]\|\\ &\quad\le \|G^{(\mu-2)/2}(R_H u - u)\| + \|R_H u - u\|_{0,q}\|R_H u - u\| + CH^{\mu-2}\|R_H u - u\|\\ &\quad\le CH^{r+\delta(r,\mu)} + CH^r\|R_H u - u\|_{0,q}. \end{aligned} \tag{5.20}$$

Here $q = \max\{2, d/\mu'\}$, where $\mu' = \mu - 2 - 1/2$ if $d/(\mu-2) = 2$; otherwise $\mu' = \mu - 2$. To finish the proof we use the error estimate [25]

$$\|R_H u - u\|_{0,q} = O(H^{\mu-2}). \tag{5.21}$$

□

From Lemma 5.3 and 5.5, we are in the position to formulate our main result in this subsection.

Theorem 5.6 *(Superconvergence for the concentration u) For $\mu = 3, 4$, there exists positive constant $C = C(K(u))$ such that*

$$\max_{0\le t\le T} \|R_H u(t) - u_H(t)\| \le CH^{r+\delta(r,\mu)}\ell_H(\mu). \tag{5.22}$$

5.2 Superconvergence for the Chemical Potential

In this subsection we give the superconvergence estimate of $\|\theta_w\| = \|w_H - R_H w\|$, which is more technical. Let us start by noticing that $P_H\rho_{u,t} = (R_H u)_t - P_H u_t$ satisfies

$$\frac{d}{dt}R_H u + A_H R_H w = P_H\rho_{u,t}, \tag{5.23}$$

and

$$A_H^{(2-\mu)/2}\theta_w = -G_H^{\mu/2}\theta_{u,t} - G_H^{\mu/2}\rho_{u,t}, \qquad \mu = 1, 2. \tag{5.24}$$

The second term on the right-hand side of (5.24) can be bounded by (5.16). Thus we only need to bound $\|G_H^{\mu/2} P_H\theta_{u,t}\|$.

For this purpose, similar to the case $\|\theta_u\|$, we derive an error equation satisfied by $\theta_{u,t}$. Differentiating equation (5.7) we get

$$\theta_{u,tt} + \epsilon A_H^2\theta_{u,t} + A_H P_H[\phi(u_H) - \phi(R_H u)]_t = T_{H,t}. \tag{5.25}$$

Hence, by Duhamel's principle

$$\begin{aligned}G_H^{\mu/2}\theta_{u,t}(t) &= e^{-\epsilon t A_H^2} G_H^{\mu/2}\theta_{u,t}(0) + \int_0^t e^{-\epsilon(t-s)A_H^2} A_H^{(2-\mu)/2} P_H[\phi(R_H u) - \phi(u_H)]_t ds\\ &\quad + \int_0^t e^{-\epsilon(t-s)A_H^2} G_H^{\mu/2} T_{H,t}(s) ds, \qquad \mu = 1, 2. \end{aligned} \tag{5.26}$$

To obtain the bound of the second term on the right-hand side of (5.26), we write $[\phi(R_H u) - \phi(u_H)]_t$ as

$$[\phi(R_H u) - \phi(u_H)]_t = [\phi'(R_H u) - \phi'(u_H)](R_H u)_t + \phi'(u_H)[R_H u - u_H]_t$$

and estimate the first term by the following two lemmas.

Lemma 5.7 *(Lemma 6.2 in [10]) Let $\|v\|_1$ and $\|\psi\|_1$ be bounded. Then*

$$\|G_H^{1/2}[(\phi'(v) - \phi'(\psi))z]\| \le C\|v - \psi\|\|z\|_1. \tag{5.27}$$

Based on this lemma, we have the following result.

Lemma 5.8 *There exists a constant C such that for $\mu = 1, 2$,*

$$\|G_H^{\mu/2}[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t]\| \le CH^{r+\delta(r,l)}\ell_H(l), \quad l = 3, 4. \tag{5.28}$$

Proof Taking $\|(R_H u)_t\|_1 \le \|(R_H u - u)_t\|_1 + \|u_t\|_1 \le C(u)$ and the boundness of $\|u_H\|_1$, which has been proved in [10], using Lemma 5.7, we have

$$\|G_H^{1/2}[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t]\| \le C\|R_H u - u_H\|\|(R_H u)_t\|_1, \tag{5.29}$$

which implies (5.28) holds for $\mu = 1$.

We now turn to the case $\mu = 2$. From (5.1), using the fact $\|(R_H u)_t\|_\infty$ and G are bounded, we have

$$\begin{aligned}
&\|G_H[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t]\| \\
&\quad \le \|G[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t]\| + CH^2\|(\phi'(R_H u) - \phi'(u_H))(R_H u)_t\| \\
&\quad \le C\|R_H u - u_H\| + C(K)H^2\|R_H u - u_H\| \\
&\quad \le CH^{r+\delta(r,l)}\ell_H(l).
\end{aligned} \tag{5.30}$$

This completes the proof. □

To give the bound of the third term on the right-hand side of (5.26), we present the following lemma.

Lemma 5.9 *Assume $\|v\|_r$, $\|v_t\|_\infty$ and $\|v_t\|_r$ are bounded. Then for $\mu = 1, 2$,*

$$\|G^{\mu/2}[\phi'(v)z]\| \le C\|G^{\mu/2}z\|, \tag{5.31}$$

$$\|G^{\mu/2}[(\phi'(v) - \phi'(\psi))v_t]\| \le C\|G^{\mu/2}(v - \psi)\| + C\|v - \psi\|_{0,q}\|v - \psi\|, \tag{5.32}$$

where $q = \max\{2, d/\mu'\}$, with $\mu' = \mu - 1/2$ if $d/\mu = 2$; otherwise $\mu' = \mu$.

Proof Let us first take p such that

$$\frac{1}{p} = \begin{cases} 1/2 - \mu/d & \text{if } \ d/2 > \mu, \\ 1/2 - (\mu - 1/2)/d & \text{if } \ d/2 = \mu, \\ 0 & \text{if } \ d/2 < \mu. \end{cases} \tag{5.33}$$

Then it is easy to verify $1/p + 1/q = 1/2$.

Let us denote $E = Gz$ and take $\chi \in C_0^\infty(\Omega)$. When $\mu = 1$, we then get

$$\begin{aligned}
|(G^{\mu/2}[\phi'(v)z], \chi)| &= |(\phi'(v)z, G^{\mu/2}\chi)| \\
&= |(\nabla E, \nabla((G^{\mu/2}\chi)\phi'(v)))| \\
&= |(\nabla E, \nabla(G^{\mu/2}\chi)\phi'(v) + (G^{\mu/2}\chi)\nabla(\phi'(v)))| \\
&\le C\|\nabla E\|\|\chi\| + C\|\nabla E\|\|G^{\mu/2}\chi\|_{0,p}\|v\|_{1,q}.
\end{aligned}$$

Let us recall that with the choice of p, due to Sobolev's inequality, we always have

$$\|G^{\mu/2}\chi\|_{0,p} \le C\|\chi\|.$$

Then we have

$$\begin{aligned}|(G^{\mu/2}[\phi'(v)z], \chi)| &\le C\|\nabla E\|\|\chi\| + C\|\nabla E\|\|G^{\mu/2}\chi\|_{0,p}\|v\|_{1,q} \\ &\le C\|\nabla E\|\|\chi\| + C(\|v\|_r)\|\nabla E\|\|\chi\|.\end{aligned}$$

This implies

$$\|G^{1/2}[\phi'(v)z]\| \le \|G^{1/2}z\|. \tag{5.34}$$

Now consider $\mu = 2$. In view of

$$\Delta(\phi'(v)(G\chi)) = (G\chi)\Delta\phi'(v) + 2\nabla(\phi'(v))\cdot\nabla(G\chi) + \phi'(v)\chi,$$

we have

$$\begin{aligned}&|(G[\phi'(v)z], \chi)| = |(\phi'(v)z, G\chi)| = |(E, \Delta((G\chi)\phi'(v)))| \\ &\quad\le \|E\|\|G\chi\|_{0,p}\|\Delta\phi'(v)\|_{0,q} + 2|(E, \nabla(\phi'(v))\cdot\nabla(G\chi))| + C\|E\|\|\phi'(v)\|_\infty\|G\chi\|.\end{aligned} \tag{5.35}$$

Since $\Delta\phi'(v) = \phi''(v)\Delta v + \phi'''(v)\nabla v\cdot\nabla v$, $\|\cdot\|_{\mu,q} \le C\|\cdot\|_r$ and $\|\cdot\|_{0,2q} \le C\|\cdot\|_{r-1}$, we have that

$$\|\Delta\phi'(v)\|_{0,q} \le C\left(\|v\|_{2,q} + \|\nabla v\|^{1/2}_{L^{2q}(\Omega)^2}\right) \le C\|v\|_r.$$

We then can bound the first term on the right-hand side of (5.35) as follows:

$$\|E\|\|G\chi\|_{0,p}\|\Delta\phi'(v)\|_{0,q} \le C\|E\|\|\chi\|\|v\|_r. \tag{5.36}$$

Using similar arguments, we obtain the following estimate for the second term on the right-hand side of (5.35):

$$|(E, \nabla(\phi'(v))\cdot\nabla(G\chi))| \le C\|E\|\|v\|_r\|\chi\|. \tag{5.37}$$

Thus, in view of (5.36) and (5.37), it follows that

$$\|G[\phi'(v)z]\| \le C\|Gz\|.$$

Using this result together with (5.34), we get (5.31).

Now we turn to the inequality (5.32). Observe first that ϕ is smooth and therefore $\phi^{(3)}$ is bounded. Then using the mean-value theorem and the fact that $\|v_t\|_\infty \le C$, it follows that

$$\|G^{\mu/2}[(\phi'(v) - \phi'(\psi))v_t]\| \le \|G^{\mu/2}[\phi''(v)(v-\psi)v_t]\| + C\|v-\psi\|_{0,q}\|v-\psi\|. \tag{5.38}$$

For the first term on the right-hand side of (5.38), the arguments for (5.31) can be applied to it, yielding

$$\|G^{\mu/2}[\phi''(v)(v-\psi)v_t]\| \le C\|G^{\mu/2}(v-\psi)\|,$$

for $\mu = 1, 2$. Substituting this into (5.38), we arrive at (5.32). □

We then estimate $\|G_H^{\mu/2}T_{H,t}(t)\|$.

Lemma 5.10 *There exists a constant C, that depends on $K(u)$ which is given in (4.5), such that for $t \in [0, T]$ the following bounds hold for $\mu = 3, 4$:*

$$\|G_H^{\mu/2} T_{H,t}(t)\| \le CH^{r+\delta(r,\mu)}. \tag{5.39}$$

Proof Similar to (5.14), we have

$$\|G_H^{\mu/2} T_{H,t}(t)\| \le \|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]_t\| + \|G_H^{\mu/2} \rho_{u,tt}\| + \|G_H^{(\mu-2)/2} \rho_{w,t}\|. \tag{5.40}$$

Then thanks to (5.15) and (5.16), we only need to bound $\|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]_t\|$. For this purpose we write

$$\begin{aligned}\|G_H^{(\mu-2)/2} P_H[\phi(R_H u) - \phi(u)]_t\| \le\; & \|G_H^{(\mu-2)/2} P_H[(\phi'(R_H u) - \phi'(u))(R_H u - u)_t]\| \\ & +\|G_H^{(\mu-2)/2} P_H[(\phi'(R_H u) - \phi'(u))u_t]\| \\ & +\|G_H^{(\mu-2)/2} P_H[\phi'(u)(R_H u - u)_t]\| \\ =\; & I_1 + I_2 + I_3.\end{aligned} \tag{5.41}$$

For I_1, when $\mu = 3$, from (5.27) we have

$$\begin{aligned}\|G_H^{1/2}[(\phi'(R_H u) - \phi'(u))(R_H u - u)_t]\| &\le \|R_H u - u\| \|(R_H u - u)_t\|_1 \\ &\le CH^{2r-1}.\end{aligned} \tag{5.42}$$

When $\mu = 4$, we have

$$\begin{aligned}&\|G_H[(\phi'(R_H u) - \phi'(u))(R_H u - u)_t]\| \\ &\quad\le \|G[(\phi'(R_H u) - \phi'(u))(R_H u - u)_t]\| + CH^2\|(\phi'(R_H u) - \phi'(u))(R_H u - u)_t\| \\ &\quad\le C\|R_H u - u\|_{L^4}\|(R_H u - u)_t\|_{L^4} + CH^2\|R_H u - u\|_{L^4}\|(R_H u - u)_t\|_{L^4} \\ &\quad\le C\|R_H u - u\|_1\|(R_H u - u)_t\|_1 + CH^2\|R_H u - u\|_1\|(R_H u - u)_t\|_1 \\ &\quad\le CH^{2r-2} + CH^{2r}.\end{aligned} \tag{5.43}$$

For I_2 and I_3, taking $\|(R_H u)_t\|_\infty \le C$ and (4.8) into consideration, we have

$$\begin{aligned}&I_2 + I_3 \\ &\le \|G^{(\mu-2)/2}[(\phi'(R_H u) - \phi'(u))u_t]\| + CH^{\mu-2}\|[\phi'(R_H u) - \phi'(u)]u_t\| \\ &\quad+\|G^{(\mu-2)/2}[\phi'(u)(R_H u - u)_t]\| + CH^{\mu-2}\|\phi'(u)[R_H u - u]_t\| \\ &\le \|G^{(\mu-2)/2}[(\phi'(R_H u) - \phi'(u))u_t]\| + CH^{r+\mu-2} \quad + \|G^{(\mu-2)/2}[\phi'(u)(R_H u - u)_t]\|.\end{aligned} \tag{5.44}$$

Since $\|u\|_r$, $\|u_t\|_\infty$ and $\|u_t\|_r$ are bounded, from (5.31) and (5.32), we have

$$\begin{aligned}\|G^{(\mu-2)/2}[(\phi'(R_H u) - \phi'(u))u_t]\| &\le C\|G^{(\mu-2)/2}(R_H u - u)\| \\ &\quad+C\|R_H u - u\|_{0,q}\|R_H u - u\| \\ &\le CH^{r+\delta(r,\mu)},\end{aligned} \tag{5.45}$$

and

$$\|G^{(\mu-2)/2}[\phi'(u)(R_H u - u)_t]\| \le C\|G^{(\mu-2)/2}(R_H u - u)_t\| \le CH^{r+\delta(r,\mu)}. \tag{5.46}$$

Combine (5.40), (5.15), (5.16), (5.44), (5.45) and (5.46) to obtain

$$\|G_H^{\mu/2} T_{H,t}(t)\| \leq C(K) H^{r+\delta(r,\mu)}. \tag{5.47}$$

$\square$

Theorem 5.11 *Fix $T > 0$. Let u be the solution of (1.1)-(1.3), let u_H be its discrete approximation via (2.11)–(2.12), and let $R_H u$ be the elliptic projection of u onto $\dot{S}_{H,r}$. Then, there exists a positive constant $C > 0$ such that $\forall t_1 \leq T$, $\mu = 0, 1, 2$, the following estimate holds:*

$$\max_{0 \leq t \leq t_1} \|G_H^{\mu/2} \theta_{u,t}\| \leq C H^{r+\delta(r,l)} \ell_H(l), \qquad l = 3, 4. \tag{5.48}$$

Proof For the second term on the right-hand side of (5.26), we write $[\phi(R_H u) - \phi(u_H)]_t = [\phi'(R_H u) - \phi'(u_H)](R_H u)_t + \phi'(u_H)[R_H u - u_H]_t$, so that

$$\begin{aligned}
&\int_0^t e^{-\epsilon(t-s)A_H^2} A_H^{(2-\mu)/2} P_H[\phi(R_H u) - \phi(u_H)]_t ds \\
&= \int_0^t e^{-\epsilon(t-s)A_H^2} A_H^{(2-\mu)/2} P_H[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t] ds \\
&\quad + \int_0^t e^{-\epsilon(t-s)A_H^2} A_H^{(2-\mu)/2} P_H[\phi'(u_H)(R_H u - u_H)_t] ds \\
&= I_1 + I_2.
\end{aligned} \tag{5.49}$$

Here, by Lemma 5.3 and Lemma 5.8, we have

$$\begin{aligned}
\|I_1\| &\leq \int_0^t \left\| A_H^{(3-\mu)/2} e^{-\epsilon(t-s)A_H^2} G_H^{1/2}[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t] \right\| ds \\
&\leq \frac{C}{\epsilon^{(3-\mu)/4}} \max_{0 \leq t \leq T} \left\| G_H^{1/2}[(\phi'(R_H u) - \phi'(u_H))(R_H u)_t] \right\| \\
&\leq C H^{r+\delta(r,l)} \ell_H(l), \qquad l = 3 \text{ or } 4.
\end{aligned} \tag{5.50}$$

Then, taking into account that $\|\theta_{u,t}(0)\| = 0$, we have

$$\begin{aligned}
\|G_H^{\mu/2} \theta_{u,t}(t)\| &\leq C H^{r+\delta(r,l)} \ell_H(l) + \|I_2\| + \left\| \int_0^t e^{-\epsilon(t-s)A_H^2} G_H^{\mu/2} T_{H,t}(s) ds \right\| \\
&= C H^{r+\delta(r,l)} \ell_H(l) + \|I_2\| + \left\| \int_0^t A_H^{(l-\mu)/2} e^{-\epsilon(t-s)A_H^2} G_H^{l/2} T_{H,t}(s) ds \right\| \\
&\leq C H^{r+\delta(r,l)} \ell_H(l) + \|I_2\| + \int_0^t \left\| A_H^{(l-\mu)/2} e^{-\epsilon(t-s)A_H^2} G_H^{l/2} T_{H,t}(s) \right\| ds.
\end{aligned}$$

In view of Lemma 5.3 and Lemma 5.10, the above inequality becomes

$$\begin{aligned}
\|G_H^{\mu/2} \theta_{u,t}(t)\| &\leq C H^{r+\delta(r,l)} \ell_H(l) + \|I_2\| + \frac{C}{\epsilon^{m/4}} \ell_H(m) \max_{0 \leq s \leq t} \left\| G_H^{l/2} T_{H,t}(s) \right\| \\
&\leq C H^{r+\delta(r,l)} \ell_H(l) + \|I_2\| + C H^{r+\delta(r,l)} \ell_H(m),
\end{aligned} \tag{5.51}$$

where $m = l - \mu$.

We consider the cases $\mu=0$ and $\mu=1,2$ separately. First let us consider the case $\mu=0$. Then for I_2 we argue as follows. Since $\|\phi'(u_H)[R_H u-u_H]_t\|\le C\|\theta_{u,t}\|$, we obtain that

$$\begin{aligned}\|I_2\| &\le \int_0^t \left\|A_H e^{-\epsilon(t-s)A_H^2} G_H^{\mu/2}[\phi'(u_H)\theta_{u,t}(s)]\right\| ds\\ &\le \frac{C}{\sqrt{\epsilon}}\int_0^t \frac{\|\theta_{u,t}(s)\|}{\sqrt{t-s}}ds. \end{aligned}\tag{5.52}$$

A standard application of the generalized Gronwall lemma leads to the inequality (5.48) for the case $\mu=0$.

We now turn to the cases $\mu=1,2$. From (5.1) and the fact that G and ϕ' are bounded, we have

$$\begin{aligned}\|G_H[\phi'(u_H)\theta_{u,t}(s)]\| &\le \|G[\phi'(u_H)\theta_{u,t}(s)]\| + CH^2\|\phi'(u_H)\theta_{u,t}(s)\|\\ &\le C\|\theta_{u,t}(s)\| \le CH^{r+\delta(r,l)}\ell_H(l). \end{aligned}\tag{5.53}$$

Hence, Lemma 5.3 may be applied to yield

$$\begin{aligned}\|I_2\| &\le \int_0^t \left\|A_H^{(4-\mu)/2} e^{-\epsilon(t-s)A_H^2} G_H[\phi'(u_H)\theta_{u,t}(s)]\right\| ds\\ &\le \frac{C}{\epsilon^{(4-\mu)/2}}\max_{0\le s\le t}\|G_H[\phi'(u_H)\theta_{u,t}(s)]\|\\ &\le CH^{r+\delta(r,l)}\ell_H(l). \end{aligned}\tag{5.54}$$

Substituting this into (5.51), we obtain the desired result for the cases $\mu=1,2$. □

We are ready to formulate our main result in this subsection.

Theorem 5.12 *(Superconvergence for the chemical potential w) There exists positive constant $C=C(K(u))$ such that for $\mu=1,2$, $l=3,4$*

$$\max_{0\le t\le T}\|R_H w(t)-w_H(t)\|_{2-\mu}\le C\left(H^{r+\delta(r,l)}\ell_H(l)+H^{r+\delta(r,\mu)}\right). \tag{5.55}$$

Proof From (2.11) and (5.23), we have $A_H^{(2-\mu)/2}\theta_w=-G_H^{\mu/2}\theta_{u,t}-G_H^{\mu/2}\rho_{u,t}$. Then (5.55) is a direct consequence of (5.48) and (5.16). □

As a consequence of the above analysis, we have the following corollary, which gives an H^1 error estimate of the chemical potential w and an L^2 error estimate of u_t.

Corollary 5.13 *There exists positive constant $C=C(K(u))$ such that*

$$\|u_t-u_{H,t}\|+\|w-w_H\|+H\|w-w_H\|_1\le CH^r. \tag{5.56}$$

Proof From Lemma 4.1 and Theorem 5.12, we have the desired estimate for the chemical potential w. The L^2 error estimate of u_t is a direct consequence of (4.8) and (5.48). □

It is convenient to give results providing negative norm estimate.

Theorem 5.14 *There exists positive constant $C=C(K(u))$ such that for $\mu=1,2$*

$$\|G^{\mu/2}(w-w_H)\|\le C\left(H^{r+\delta(r,l)}\ell_H(l)+H^{r+\delta(r,\mu)}\right),\quad l=3,4; \tag{5.57}$$

$$\|G^{\mu/2}D_t^j(u-u_H)\|\le C\left(H^{r+\delta(r,l)}\ell_H(l)+H^{r+\delta(r,\mu)}\right),\ l=3,4,\ j=0,1. \tag{5.58}$$

Proof It follows from the splitting $w_H - w = w_H - R_H w + R_H w - w$ and (4.3) that

$$\begin{aligned}\|G^{\mu/2}(w_H - w)\| &\le \|G^{\mu/2}(R_H w - w)\| + \|G^{\mu/2}(w_H - R_H w)\| \\ &\le C H^{r+\delta(r,\mu)} + \|G^{\mu/2}(w_H - R_H w)\|. \end{aligned} \tag{5.59}$$

Since G is bounded, we have $\|G^{\mu/2}(w_H - R_H w)\| = \|G A^{(2-\mu)/2}(w_H - R_H w)\|$ and use (5.55) to get (5.57).

To prove (5.58), we also consider the splitting

$$u_H - u = u_H - R_H u + R_H u - u. \tag{5.60}$$

Then from (4.3) and (4.8), we get

$$\begin{aligned}\|G^{\mu/2} D_t^j (u_H - u)\| &\le \|G^{\mu/2} D_t^j (u_H - R_H u)\| + \|G^{\mu/2} D_t^j (R_H u - u)\| \\ &\le \|G^{\mu/2} D_t^j (u_H - R_H u)\| + C H^{r+\delta(r,\mu)}. \end{aligned} \tag{5.61}$$

Now in the case $j = 0$, using the fact that $G^{\mu/2}$ is bounded and the superconvergence estimate (5.22), we have

$$\begin{aligned}\|G^{\mu/2}(u_H - u)\| &\le C\|u_H - R_H u\| + C H^{r+\delta(r,\mu)} \\ &\le C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right), \qquad l = 3, 4. \end{aligned} \tag{5.62}$$

For the case $j = 1$, it follows from (5.61), (5.2) and (5.48) that

$$\begin{aligned}\|G^{\mu/2}(u_H - u)_t\| &\le \|G_H^{\mu/2}(u_H - R_H u)_t\| + C H^{\mu}\|(u_H - R_H u)_t\| + C H^{r+\delta(r,\mu)} \\ &\le C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right), \qquad l = 3, 4. \end{aligned} \tag{5.63}$$

This completes the proof. □

In [29] and [28], the H^{-1}, i.e., $\mu = 1$, negative norm error estimates of the finite element methods were obtained for the nonstationary Navier–Stokes equations and the incompressible MHD equations, respectively. In this paper, we generalize their negative norm error estimates to the fourth order problem and the case $\mu = 2$.

5.3 Proof of Theorem 2.1

In this subsection, we give a detailed proof of our main result Theorem 2.1.

Proof of Theorem 2.1. We consider the splitting $u - u^h = u - R_h u + R_h u - u^h$ and $w - w^h = w - R_h w + R_h w - w^h$. The term $u - R_h u$ and $w - R_h w$ can be readily estimated by using (4.8)–(4.9), so that,

$$\|w - R_h w\| + h\|w - R_h w\|_1 + \|u - R_h u\| + h\|u - R_h u\|_1 \le C h^{\tilde{r}}. \tag{5.64}$$

We will concentrate on the estimates of $u^h - R_h u$ and $w^h - R_h w$. For $\mu = 1, 2$,

$$A_h^{(2-\mu)/2}(w^h - R_h w) = -G_h^{\mu/2}\left[(u_H - u)_t\right], \tag{5.65}$$

$$A_h^{(2-\mu)/2}(u^h - R_h u) = \frac{1}{\epsilon} G_h^{\mu/2}\left[w_H - w + \phi(u) - \phi(u_H)\right]. \tag{5.66}$$

Then from (5.65), by (5.1), (5.58) and the fact that $H \leq h$ we have

$$\begin{aligned} \|w^h - R_h w\|_{2-\mu} &= \|G_h^{\mu/2}(u_H - u)_t\| \\ &\leq \|G^{\mu/2}(u_H - u)_t\| + Ch^{\mu}\|(u_H - u)_t\| \\ &\leq C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right) + Ch^{\mu}H^r \\ &\leq C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right), \qquad l = 3, 4. \end{aligned} \tag{5.67}$$

Using this result together with (5.64), we get (2.15) and (2.16). □

We now direct our attention to the proof of (2.17) and (2.18). From (5.66), we have

$$\|u^h - R_h u\|_{2-\mu} \leq C\left(\|G_h^{\mu/2}(w_H - w)\| + \|G_h^{\mu/2}(\phi(u_H) - \phi(u))\|\right). \tag{5.68}$$

For the first term on the right-hand side of (5.68), in a similar way, using (5.1), (5.56) and (5.57), we find

$$\begin{aligned} \|G_h^{\mu/2}(w_H - w)\| &\leq \|G^{\mu/2}(w_H - w)\| + Ch^{\mu}\|w_H - w\| \\ &\leq C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right), \quad l = 3, 4. \end{aligned} \tag{5.69}$$

For the second term on the right-hand side of (5.68), we have

$$\begin{aligned} &\|G_h^{\mu/2}(\phi(u_H) - \phi(u))\| \\ &\quad \leq \|G^{\mu/2}(\phi(u_H) - \phi(u))\| + Ch^{\mu}\|\phi(u_H) - \phi(u)\| \\ &\quad \leq C\|G^{\mu/2}(u_H - u)\| + C\|u_H - u\|_{0,q}\|u_H - u\| + Ch^{\mu}\|u_H - u\| \\ &\quad \leq C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right) + C\|u_H - u\|_{0,q}\|u_H - u\|, \quad l = 3, 4, \end{aligned} \tag{5.70}$$

where q is the value in Lemma 5.1. To estimate $\|u_H - u\|_{0,q}$, we use the inverse estimate (4.1), the fact that $\delta(r,l) = \delta(r,l-2)$ when $l = 3, 4$, and Theorem 5.6 to get

$$\|u_H - R_H u\|_{0,q} \leq CH^{d/q-d/2}\|u_H - R_H u\| \leq CK(u)H^{r+\delta(r,l-2)-d/p}\ell_H(l),$$

where $d/q - d/2 = -d/p$ has been used. In view of (5.33), we have $r + \delta(r,l-2) - d/p \geq \delta(r,l-2)$. This and (5.21) lead to

$$\|u - u_H\|_{0,q} \leq CK(u)H^{\min\{r+\delta(r,l-2)-d/p,\, l-2\}}\ell_H(l) \leq CH^{\delta(r,l-2)}\ell_H(l).$$

Hence, by (4.13), we have

$$\|u_H - u\|_{0,q}\|u_H - u\| \leq CH^{r+\delta(r,l)}\ell_H(l). \tag{5.71}$$

Substituting this inequality into (5.70) yields

$$\|G_h^{\mu/2}(\phi(u_H) - \phi(u))\| \leq C\left(H^{r+\delta(r,l)}\ell_H(l) + H^{r+\delta(r,\mu)}\right). \tag{5.72}$$

Combine (5.64), (5.68), (5.69) and (5.72) to get (2.17) and (2.18). This completes the proof.

6 Numerical Experiments

In this section, we present one numerical example in order to support the analysis developed in this paper. Due to the limitation of space, here we only give an example for which the exact solution is constructed. More examples involving practical applications have been presented in the paper [49]. We implemented the schemes by using the MATLAB© software package iFEM [5].

We consider the Cahn–Hilliard equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \Delta(\epsilon \Delta u - \phi(u)) &= f \qquad && x \in \Omega, \quad t > 0, \\ u(x,0) &= u_0(x) && x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(\phi(u) - \epsilon \Delta u) &= 0 && x \in \partial\Omega. \end{aligned} \tag{6.1}$$

over the domain $\Omega = (0, 1)^2$, where ϕ takes the typical type (1.4). Let $\epsilon = 0.1$, and the exact solution be

$$u = e^{-t} \sin^2(\pi x) \sin^2(\pi y).$$

Then functions f and $u_0(x)$ can be chosen to satisfy (6.1). Observe that our analysis can be easily generalized to the case where f is a known function. After rewriting the above equation as a mixed formulation, we apply MFE methods with linear element (P_1) and quadratic element (P_2), respectively. For time discretization, we use the backward Euler scheme

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + \left(\nabla w_h^{n+1}, \nabla v_h\right) = f \quad \text{for all } v_h \in \dot{S}_{h,r}, \tag{6.2}$$

$$\epsilon \left(\nabla u_h^{n+1}, \nabla \chi_h\right) + \left(\phi(u_h^{n+1}) - w_h^{n+1}, \chi_h\right) = 0 \quad \text{for all } \chi_h \in \dot{S}_{h,r}, \tag{6.3}$$

with $\Delta t = 1.0e\text{-}5$, where u_h^n and w_h^n denote the approximations at $t_n = n\Delta t$, respectively. This fully implicit scheme is energy-stable and uniquely solvable when $\Delta t \le 4\epsilon$ (see [10]). The errors at $T = 0.01$ for P_1 and P_2 mixed finite element approximation are presented in Tables 1 and 2.

We then apply our postprocessed MFE methods. The numerical results are presented in Tables 3 and 4. A comparison of Tables 1, 2, 3 and 4 confirms that the convergence order matches our theoretical estimate for linear and quadratic elements.

Table 1 $\Delta t = 1.0e{-}5$, $T = 0.01$, P_1 mixed finite element approximation

h	$\|\|u - u_h\|\|_1$	Ratio	$\|\|w - w_h\|\|_1$	Ratio
1/16	2.805653e−01		1.798280	
1/32	1.396404e−01	2.01	9.063605e−01	1.98
1/64	6.972192e−02	2.00	4.541101e−01	2.00

Table 2 $\Delta t = 1.0e{-}5$, $T = 0.01$, P_2 mixed finite element approximation

h	$\|\|u - u_h\|\|_1$	Ratio	$\|\|w - w_h\|\|_1$	Ratio
1/16	1.149667e−02		8.782236e−02	
1/32	2.900157e−03	3.96	2.225154e−02	3.95
1/64	7.273173e−04	3.99	5.588638e−03	3.98

Table 3 $\Delta t = 1.0e{-}5$, $T = 0.01$, P_1 postprocessing mixed finite element approximation

H	h	$\|\|u - u^h\|\|_1$	Ratio	$\|\|w - w^h\|\|_1$	Ratio
1/4	1/16	6.388851e−01		1.916974	
1/8	1/64	1.921115e−01	3.32	5.069748e−01	3.78
1/16	1/256	5.060618e−02	3.79	1.284365e−01	3.96
1/32	1/1024	1.282033e−02	3.80	3.221215e−02	3.99

Table 4 $\Delta t = 1.0e{-}5$, $T = 0.01$, P_2 postprocessing mixed finite element approximation

H	h	$\|\|u - u^h\|\|_1$	Ratio	$\|\|w - w^h\|\|_1$	Ratio
1/4	1/16	5.986256e−02		9.686167e−02	
1/8	1/64	4.980803e−03	12.02	6.285356e−03	15.01
1/16	1/256	3.343822e−04	14.90	3.975132e−04	15.81

7 Summary

In this paper, the postprocessing mixed finite element methods have been applied to solving the Cahn–Hilliard equation, a fourth order nonlinear differential equation. The methods can be described as: on the coarser mesh we first compute the mixed finite element approximations until a fixed time at which we need a higher accuracy solution, then at the fixed time the approximations are postprocessed by solving two decoupled linear elliptic problems on a finer grid (or higher-order space). The analysis presented here shows that this technique remains the optimal rate of convergence for both the concentration and the chemical potential approximations. The negative norm error estimates, which are new and non-trivial, are also obtained. A numerical example confirms the theoretical results and illustrates the effectiveness of the postprocessing MFE methods.

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