Real Analysis Qualifying Exam
September 2011

Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. Let a measurable bounded set $X \subset \mathbb{R}^n$ have the property that every continuous map $f : X \to \mathbb{R}$ is uniformly continuous. Show that X is compact.

2. (a) For any $\epsilon > 0$ construct an open set $U_\epsilon \subset (0,1)$ so that $U_\epsilon \supset \mathbb{Q} \cap (0,1)$ and $m(U_\epsilon) < \epsilon$ .Here and later m stands for the Lebesgue measure.

   (b) Let $A = \cup_{n=1}^{\infty} U_{1/n}^c$. Find $m(A)$. Show that $A^c \cap (1/2, 5/16) \neq \emptyset$.

3. Show that either a σ-algebra of subsets of a set X is finite, or else it has uncountably many elements.
   Hint: if the σ-algebra is not finite, how many pairwise disjoint elements does it necessarily contain?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous and assume that $f' \in \mathrm{L}^2([0,1])$ and that $f(0) = 0$. Show that the following limit exists and compute its value.
$$\lim_{x \searrow 0} x^{-1/2} f(x).$$

5. Let $S \subset \mathbb{R}$ be closed and let $f \in \mathrm{L}^1([0,1])$. Assume that for all measurable $E \subset [0,1]$ with $m(E) > 0$ we have $\frac{1}{m(E)} \int_E f \in S$. Prove that $f(x) \in S$ for a.e. $x \in [0,1]$.

6. If g is a Lebesgue measurable real-valued function on $[0,1]$ such that the function $f(x,y) = 5g(x) - 7g(y)$ is Lebesgue integrable over the square $[0,1] \times [0,1]$, show that g is Lebesgue integrable over $[0,1]$.