Math 105A Summer 2017 Midterm Solutions Sketched

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Problem 1

(a) Define Machine Epsilon.
(b) List the following in order of increasing computational cost by number of operations.
i. Calculating Determinant ii. Gaussian Elimination with Partial Pivoting iii. Gaussian Elimination with Scaled Pivoting iv. Gaussian Elimination with Full Pivoting.

Solution. (a) Machine Epsilon, ϵ_m is the smallest value such that

$$fl(1+\epsilon_m) > 1.$$

So for any $\delta < \epsilon_m$, we would have $fl(1+\delta) = 1$.

(b) Determinant is an $O(n!)$ calculation. We see that from the recursive cofactor definition, we start with n-multiplications and additions of determinants of $(n-1)\times(n-1)$ matrices, and each of those has $(n-1)$ multiplications and additions, etc.

Gaussian Elimination is always $O(n^3)$ which is smaller than $n!$. We know Scaled Pivoting adds on more divisions from computing the rescalings; otherwise it does the same checks, so it must be more than Partial Pivoting. Full Pivoting involves a full search and also permuting the elements of the matrix so it's the most complex Gaussian Elimination.

The order is then

$$(i) > (iv) > (iii) > (ii).$$

Problem 2

(a) Define Quadratic Convergence of a sequence $\{p_n\}$.
(b) Suppose that p is a root of multiplicity 3 for $f(x)$. Show that this modified Newton's Method converges quadratically:

$$p_{n+1} = p_n - 3\frac{f(p_n)}{f'(p_n)}.$$

Solution. (a) For quadratic convergence of $p_n \to p$,

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \lambda, \quad s.t.\ \ 0 < \lambda < \infty.$$

(b) We know Newton's Method is really a fixed point iteration. Now we're equivalently solving

$$x = g(x), \quad g(x) = x - 3\frac{f(x)}{f'(x)}.$$

We know if we satisfy (Thm 2.9) $g'(p) = 0$ and $|g''(x)|$ bounded for points sufficiently close to p, we attain quadratic convergence.

$g'(p) = 0$: Here first note that we can write $f(x) = (x-p)^3 q(x)$ from the multiple root. So

$$f'(x) = 3(x-p)^2 q(x) + (x-p)^3 q'(x), \quad f''(x) = 6(x-p)q(x) + 6(x-p)^2 q'(x) + (x-p)^3 q''(x).$$

Then

$$g'(x) = 1 - 3\frac{f'(x)}{f'(x)} + 3\frac{f(x)f''(x)}{f'(x)^2}$$

simplifying a little and plugging in what we have for f, f', f'',

$$\begin{aligned} g'(x) &= -2 + 3\frac{(x-p)^3 q(x)(6(x-p)q(x) + 6(x-p)^2 q'(x) + (x-p)^3 q''(x))}{(3(x-p)^2 q(x) + (x-p)^3 q'(x))^2} \\ &= -2 + 3\frac{(x-p)^4 q(x)(6q(x) + 6(x-p)q'(x) + (x-p)^2 q''(x))}{(x-p)^4 (3q(x) + (x-p)q'(x))^2} \end{aligned}$$

so as we take $x \to p$,

$$g'(p) = -2 + 3\frac{6q(p)^2}{9q(p)^2} + \lim_{x\to p} O(x-p)$$

where the $O(x-p)$ terms go to zero. We got the middle term from picking out the smallest powers of $(x-p)$ in the numerator and denominator because they are the slowest to go to zero.
We simplify the derivative,

$$g'(p) = -2 + \frac{18}{9} + 0 = -2 + 2 = 0.$$

Good, $g'(p) = 0$.

$|g''(x)|$ bounded: Here $g''(x) = 3\frac{d}{dx}\left[\frac{f(x)\cdot f''(x)}{f'(x)^2}\right]$. It suffices to show that, ignoring the 3, that derivative is bounded. The derivative is

$$\begin{aligned} \frac{d}{dx}\left[\frac{f(x)\cdot f''(x)}{f'(x)^2}\right] &= \frac{f''}{f'} + \frac{f \cdot f'''}{(f')^2} - \frac{2f\cdot(f'')^2}{(f')^3}, \qquad \text{get a common denominator,} \\ &= \frac{(f')^2 f'' + f\cdot f' \cdot f''' - 2f\cdot (f'')^2}{(f')^3}, \qquad \text{plug in forms of } f \text{ and let } x \to p, \\ &\overset{x\to p}{=} \frac{9\cdot 6(x-p)^5 q(p)^3 + 3\cdot 6(x-p)^5 q(p)^3 - 2\cdot 6^2 (x-p)^5 q(p)^3 + O(x-p)^6}{\lim_{x\to p} 27(x-p)^6 q(p)^3 + O(x-p)^7} \\ &= \frac{54 + 18 - 72}{27\lim_{x\to p}(x-p)} + O(1) \\ &= O(1), \end{aligned}$$

mainly that there's actually no division by 0, we can bound $g''(x)$ as long as we are sufficiently close to p so that the extra terms are all just $O(1)$.

Whew! So that means, Theorem 2.9 criteria are satisfied; the iteration converges quadratically.

Problem 3

(a) Using scaled partial pivoting, find the $PA = LU$ decomposition of $A = \begin{bmatrix} 1 & 1 & 20 \\ 2 & -1 & 10 \\ 1 & 2 & 0 \end{bmatrix}$.

(b) Use the $PA = LU$ decomposition to solve $\begin{bmatrix} 1 & 1 & 20 \\ 2 & -1 & 10 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. (a) First, we see the maxima in each row are 20,10, and 2. So we compare first,

$$\frac{|a_{11}|}{s_1} = \frac{1}{20}, \quad \frac{|a_{21}|}{s_2}\frac{2}{10}, \quad \frac{|a_{31}|}{s_3} = \frac{1}{2}.$$

so we need to swap $E_1 \leftrightarrow E_3$,

$$\begin{bmatrix} 1 & 1 & 20 \\ 2 & -1 & 10 \\ 1 & 2 & 0 \end{bmatrix} \overset{E_1 \leftrightarrow E_3}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 10 \\ 1 & 1 & 20 \end{bmatrix} \overset{\substack{E_2 = E_2 - 2E_1 \\ E_3 = E_3 - E_1}}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 10 \\ 0 & -1 & 20 \end{bmatrix}$$

now we compare again,

$$\frac{|a_{22}|}{s_2}\frac{5}{10}, \quad \frac{|a_{32}|}{s_3} = \frac{1}{20}$$

we don't need to pivot so we just reduce more,

$$\overset{E_3 = E_3 - \frac{1}{5}E_2}{\sim} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 10 \\ 0 & 0 & 18 \end{bmatrix} = U.$$

So now we can build P, L, U from what we did above.

We swapped rows 1 and 3. Since that was the very first swap, this fortunately doesn't swap any entries of multipliers. Thus,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 10 \\ 0 & 0 & 18 \end{bmatrix}.$$

(b) First $PA = LU \iff A = P^T LU$ so we solve $P^T LU = \vec{b} \iff LU = P\vec{b}$. Here, since $\vec{b} = [1,1,1]^T$, applying P does nothing. $P\vec{b} = \vec{b}$. Thus we're actually just solving $LU = [1,1,1]^T = \vec{b}$. To do this,

First solve $L\vec{y} = \vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1/5 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{c} y_1 = 1 \\ y_2 = 1 - 2y_1 = -1 \\ y_3 = 1 - y_1 - \frac{y_2}{5} = 1/5 \end{array}$$

Second solve $U\vec{x} = \vec{y}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -5 & 10 & -1 \\ 0 & 0 & 18 & 1/5 \end{array}\right] \longrightarrow \begin{array}{c} x_1 = 1 - 2x_2 = 5/9 \\ x_2 = \frac{1}{-5}(-1 - 10x_3) = 2/9 \\ x_3 = \frac{1}{5 \cdot 18} = 1/90 \end{array}$$

So the answer is $\vec{x} = \begin{bmatrix} 5/9 \\ 2/9 \\ 1/90 \end{bmatrix}$. (Checked with Matlab too).

Problem 4

(a) Let A be $n \times n$. What does it mean for A to be strictly diagonally dominant?

(b) Find all $a, b > 0$ such that $A = \begin{bmatrix} 3 & 1 & a \\ 1 & b & 4 \\ a & 6 & b \end{bmatrix}$ is strictly diagonally dominant.

(c) For the choices of a, b from part (b), what would be the permutation matrix P in the $PA = LU$ decomposition of that matrix? Justify.

Solution. (a) A is strictly diagonally dominant if the diagonal elements outweigh the absolute row sums of the off diagonal elements, $|a_{ii}| > \sum_{j \neq i}^{n} |a_{ij}|$ for all $i = 1, ..., n$.

(b) Going row by row, (note we assume $a, b > 0$ so we can drop absolute value signs), we need

$$\begin{cases} E_1 & : 3 > 1 + a \iff 0 < a < 2 \\ E_2 & : b > 4 + 1 \iff b > 5 \\ E_3 & : b > 6 + a \end{cases}$$

The condition in E_3 overrides the one in E_2. It also implies b can be dependent on a.
The answer is

$$0 < a < 2, \quad b > 6 + a.$$

Common error: Note that $b > 8$ does not give all the possibilities. We could have $a = 0.0001$ and $b = 6.1$ for instance. There is truly a dependence of b on a.

(c) We picked a, b such that our matrix was strictly diagonally dominant. We know that strictly diagonally dominant matrices do not need row interchanges when performing Gaussian Elimination. (The same goes for symmetric positive definite matrices - see Quiz 5). That means we should never have to swap so $P = I$, the 3 by 3 identity in particular.

Problem 5

(a) Define the determinant of an $n \times n$ matrix A.

(b) Is it true that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d + a & b + e & f + c \\ g - a & h - b & i - c \end{bmatrix}$? Justify why or why not.

(c) Compute $\det \begin{bmatrix} 1 & 7 & 9 & 3 \\ 0 & 2 & 0 & 0 \\ 4 & 6 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. (a) The determinant is computed either as

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{for any fixed } j = 1, .., n$$

or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij}) \quad \text{for any fixed } i = 1, .., n$$

where in both, M_{ij} is the matrix left from deleting column j and row i.

(b) We notice that the 2nd matrix is just the result of row replacement operations from the 1st matrix. In particular, from the 1st matrix, it did $E_2 = E_2 + E_1$ and $E_3 = E_3 - E_1$. Replacement does not change determinant.

Warning: The other row operations can change it. So you should specifically write “it used row replacement.” Row swaps change the sign, and row scalings scale the determinant.

Also, though not advised, you could have literally computed the two determinants and showed that the 2nd one reduces to the 1st. More on this approach: you should show when terms cancel. **Always show your work!!**

(c) For $\det \begin{bmatrix} 1 & 7 & 9 & 3 \\ 0 & 2 & 0 & 0 \\ 4 & 6 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix}$, go along the 2nd row,

$$\det \begin{bmatrix} 1 & 7 & 9 & 3 \\ 0 & 2 & 0 & 0 \\ 4 & 6 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} = (-1)^{2+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 9 & 3 \\ 4 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{go along 3rd row,}$$

$$= 2 \left[(-1)^{1+3} \cdot 1 \cdot \begin{bmatrix} 9 & 3 \\ 1 & 3 \end{bmatrix} + (-1)^{3+3} \cdot 1 \cdot \det \begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix} \right]$$

$$= 2\left[(27-3) + (1-36)\right]$$

$$= 2[24 - 35]$$

$$= -22.$$