Solutions

Math 105A Quiz 1 - June 30th

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Show all of your work. *There is a question on the back side.*

1. Consider solving for the solution to $x^3 + x - 4 = 0$ on [0,4]. In other words $[a_1, b_1] = [0, 4]$

(a) [4pts] Using the Bisection Method, $p_1 = 2$ is the first approximation. What are the next two intervals $[a_2, b_2]$ and $[a_3, b_3]$, and the next two approximations p_2, p_3 to the solution?

First endpts satisfy $f(0) = -4 < 0$, $f(4) = 64 > 0$.

$[f(x) = x^3 + x - 4]$

$p_1 = 2 \Rightarrow f(2) = 8 + 2 - 4 = 6 > 0$. So

$\boxed{[a_2, b_2] = [0, 2] \Rightarrow p_2 = 1.}$ +2

$p_2 = 1 \Rightarrow f(1) = 1 + 1 - 4 = -2 < 0$, so

$\boxed{[a_3, b_3] = [1, 2] \Rightarrow p_3 = 1.5}$ +2

(b) [6pts] If we wanted to find a solution with 10^{-3} accuracy, estimate a lower bound for the number of iterations required for Bisection Method. (You can leave your answer with a logarithm with any base).

+2

Error is $|p_n - p| \leq \frac{b-a}{2^n} \Rightarrow$ want $< 10^{-3}$.

Here, $\frac{4-0}{2^n} < 10^{-3} \iff 2^n > 4 \times 10^3$,

+2

$\boxed{n > \log_2 (4 \times 10^3)}$ +2

$(n \geq 12$ with calculator$)$

(Solns contd)

2. Consider solving for a fixed point of $\boxed{g(x) = \pi + \frac{1}{2}\sin(x/2)}$ on $[0, 2\pi]$.

Is a continuous fn ✓

(a) [8pts] Prove that $g(x)$ has a unique fixed point on this interval.
Hint: Remember that sine and cosine are bounded between [-1,1].

Pf: (i) Need g to map $[0, 2\pi]$ back within itself.

Since $-1 \le \sin\left(\frac{x}{2}\right) \le 1$, $g(x)$ is between $\left[\pi - \frac{1}{2}, \pi + \frac{1}{2}\right]$

which is contained in $[0, 2\pi]$ ✓ +4

(ii) Need $|g'(x)| \le k$ for $0 < k < 1$.

Here $g'(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right)$ and since $-1 \le \cos\left(\frac{x}{2}\right) \le 1$,

$|g'(x)| \le \frac{1}{4}$ so $\boxed{k = 1/4}$ works. +4

By the Fixed Pt Thm (Thm 2.3) $\Rightarrow$ g has a unique fixed pt in $[0, 2\pi]$

[It's ok if you don't know Thm #]
[Need to use its parts correctly]

(b) [2pts] If we start with $p_0 = 2\pi$, find the next two approximations p_1, p_2 of the Fixed-Point Iteration $p_{n+1} = g(p_n)$.

zero

$P_1 = g(P_0) = \pi + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) = \pi + \frac{1}{2}\sin\pi$,

$\boxed{P_1 = \pi}$ +1

$P_2 = g(P_1) = \pi + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = \pi + \frac{1}{2}$,

$\boxed{P_2 = \pi + \frac{1}{2}}$ +1