Solutions

Math 105A Quiz 2 - July 3rd

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Show all of your work. *There is a question on the back side.*

1. Let $f(x) = x^2 - 2x - 4$ or in other words $f(x) = (x-1)^2 - 5$. Consider solving for $f(x) = 0$.

(a) [5pts] If $p_0 = 2$, find the next two approximations p_1, p_2 using Newton's Method.

+1 $f'(x) = 2x - 2$ or $f'(x) = 2(x-1)$.

+2 $p_1 = p_0 - \dfrac{f(p_0)}{f'(p_0)} = 2 - \dfrac{(-4)}{2} = 2+2$, $\boxed{p_1 = 4}$

+2 $p_2 = p_1 - \dfrac{f(p_1)}{f'(p_1)} = 4 - \dfrac{4}{6} = 3\tfrac{2}{6}$, $\boxed{p_2 = \dfrac{10}{3}}$

(or, $3\tfrac{1}{3}$).

(b) [2pts] If $p_0 = 2$ and $p_1 = 4$, find p_2 with the Secant Method.

$$p_2 = p_1 - \frac{f(p_1)\cdot(p_1 - p_0)}{f(p_1) - f(p_0)}$$

$$= 4 - \frac{4\cdot 2}{4-(-4)}$$

+2

$$= 4 - 1 = 3, \quad \boxed{p_2 = 3}$$

(c) [1pt] $p = 1+\sqrt{5} \approx 3.2361$ is the exact root. Which p_2 is closer to p, (a)'s or (b)'s?

Part (a) with Newton is, $|3.3333 - 3.2361| \sim 0.1$

+1 $|3 - 3.2361| \sim 0.2$

(solns cont'd)

2. (a) [6pts] Show that $p_n = 2 + \frac{1}{n^3}$ converges linearly to $p = 2$.

(+2) if needed.

Pf: Need to show $\frac{|p_{n+1} - p|}{|p_n - p|} \xrightarrow[n\to\infty]{\lim} \lambda$ with $\lambda \le 1$ for linear convergence.

$$\lim_{n\to\infty} \frac{\left|\cancel{2} + \frac{1}{(n+1)^3} - \cancel{2}\right|}{\left|\cancel{2} + \frac{1}{n^3} - \cancel{2}\right|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \longrightarrow 1^3 = 1$$

+3 +3

So, yes it converges linearly to $p = 2$. ▨

(b) [6pts] Let p be a zero of multiplicity m for an infinitely differentiable function $f(x)$, i.e. $f \in C^\infty(\mathbb{R})$. Compute the following limit:

$$\lim_{x\to p} \frac{f'(x)\cdot(x-p)}{f(x)}.$$

Hint: Recall we can rewrite our function as $f(x) = (x-p)^m \cdot q(x)$ with $q(p) \neq 0$.

From hint, $f'(x) = m(x-p)^{m-1} \cdot q(x) + (x-p)^m q'(x)$. ~~+1~~ (since it was given)

$$\text{So} \quad \lim_{x\to p} \frac{f'(x)\cdot(x-p)}{f(x)} = \lim_{x\to p} \left(\frac{m\,\cancel{(x-p)^m}\,\cancel{q(x)}}{\cancel{(x-p)^m}\,\cancel{q(x)}} + \frac{(x-p)^{m+1}\,q'(x)}{(x-p)^m\,q(x)} \right)$$

+2

$$= \lim_{x\to p} m + \underbrace{(x-p)\cdot\frac{q'(x)}{q(x)}}$$

+3

Goes to zero! (b/c $q(p) \neq 0$)

$$= m \checkmark$$

+1

▨