Solutions

Math 105A Quiz 3 - July 7th

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Show all of your work. *There is a question on the back side.*

1. Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Consider approximating $e^1 \approx 2.71828$ using this power series.
The first few approximations are given by the partial sums of the power series, i.e.

$$p_0 = 1, \quad p_1 = 2, \quad p_2 = \frac{5}{2}, \quad p_3 = \frac{8}{3}, \quad \ldots, \quad p_n = \sum_{k=0}^{n} \frac{1^k}{k!}, \quad \ldots$$

(a) [6pts] Find the first two terms $\hat{p}_0, \hat{p}_1$ from applying Aitken's Δ^2 Method to this sequence.

	p_n	Δp_n	$\Delta^2 p_n$
$n=0$	1	1	$-1/2$
$n=1$	2	$1/2$	$-1/3$
$n=2$	$5/2$	$1/6$	—
$n=3$	$8/3$	—	—

(+2)

And use either

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}, \quad \text{or}$$

$$= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

$$\hat{p}_0 = 1 - \frac{1^2}{-1/2} = 1 + \frac{2}{1} = \boxed{3} \quad (+2)$$

$$\hat{p}_1 = 2 - \frac{(1/2)^2}{-1/3} = 2 + \frac{3}{4} = \boxed{2.75} \quad (+2)$$

Note: $p_2 = 2.5$
$p_3 = 2.666\ldots$

(b) [2pts] Is $\hat{p}_0$ a better approximation than p_2? Is $\hat{p}_1$ a better approximation than p_3?

$\hat{p}_0$ vs p_2: $|\hat{p}_0 - 2.71828| \approx 0.28$; $|p_2 - 2.71828| \approx 0.21$.

so $\boxed{p_2 \text{ is better than } \hat{p}_0.}$ +1

$\hat{p}_1$ vs p_3: $|\hat{p}_1 - 2.71828| \approx 0.032$; $|p_3 - 2.71828| \approx 0.05$

So $\boxed{\hat{p}_1 \text{ is better than } p_3.}$ +1

(solus cont'd)

2. Consider solving the linear system by Guassian Eliminiation with Backsubstitution,

$$2x_1 - 4\alpha x_2 = 2$$
$$2\alpha x_1 - \quad x_2 = 1$$

(a) [5pts] What values of α have no solutions? (i.e. we get $0 = 1$ or $0 =$ something nonzero). Hint: Row reduce the matrix with Gaussian Elimination. It will be useful for parts (a)-(c).

$$\hookrightarrow \tilde{A}^{(1)} = \left[\begin{array}{cc|c} 2 & -4\alpha & 2 \\ 2\alpha & -1 & 1 \end{array}\right] \underset{\sim}{E_2^{(2)} = E_2^{(1)} - \alpha E_1^{(1)}} \tilde{A}^{(2)} = \left[\begin{array}{cc|c} 2 & -4\alpha & 2 \\ 0 & -1+4\alpha^2 & 1-2\alpha \end{array}\right].$$

+2

This has no solutions if $(-1+4\alpha^2) = 0$ when $(1-2\alpha) \neq 0$.

Roots of $(-1+4\alpha^2)$: Factor $(2\alpha+1)(2\alpha-1) = 0$,

$\alpha = \pm \frac{1}{2}$. +1

+2

But $(1-2\alpha)$ is zero at $\alpha = \frac{1}{2}$ so only $\boxed{\alpha = -\frac{1}{2} \text{ has no solution.}}$

(b) [3pts] What values of α yield infinitely many solutions? (i.e. we get $0 = 0$, free variable)

+1

Then we need both $(-1+4\alpha^2) = 0$; $(1-2\alpha) = 0$.

This corresponds to $\boxed{\alpha = \frac{1}{2}}$. +2

(c) [4pts] Suppose we are in the case that α gives a unique solution. Find the solution.

From $\tilde{A}^{(2)}$: $(-1+4\alpha^2)x_2 = 1-2\alpha \iff (1+2\alpha)(-1+2\alpha)x_2 = 1-2\alpha$.

so $\boxed{x_2 = -\frac{1}{1+2\alpha}}$. +2

Put Together $\Rightarrow$ $\boxed{\vec{x} = \begin{bmatrix} \frac{1}{1+2\alpha} \\ \frac{-1}{1+2\alpha} \end{bmatrix}}$ +1

Backsub: $2x_1 + \frac{4\alpha}{1+2\alpha} = 2$; $x_1 = 1 - \frac{2\alpha}{1+2\alpha}$, +1

or, $x_1 = \frac{(1+\cancel{2\alpha}) - \cancel{2\alpha}}{1+2\alpha} = \frac{1}{1+2\alpha}$, looks nicer.