Solutions

Math 105A Quiz 5 - July 14th

Please put name on front & ID on back for redistribution!

Show all of your work. *There is a question on the back side.*

Consider the *tridiagonal* matrix $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ for both problems.

1. (a) [2pts] Is A diagonally dominant? What about strictly diagonally dominant?

$a_{ii} = 2$ for all i.

Rows 1, 4: $|2| > |-1|$ ✓

Rows 2, 3: $|2| \geq |-1| + |-1|$ $\Rightarrow$

Not strictly ($>$)

A is only diagonally dominant. +1

Not strictly. +1

(b) [6pts] Is A (symmetric) positive definite? - Let $A_k = A(1:k, 1:k)$.

Note: A is symmetric ✓ +1

- A_1: $\det[2] = \boxed{2 > 0}$ ✓ +1
- A_2: $\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = \boxed{3 > 0}$ ✓ +1
- A_3: $\det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$

$= 2 \cdot 3 - 2 = \boxed{4 > 0}$ ✓ +1

- A_4: $\det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = 2 \underbrace{\det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}}_{\text{is } A_3} - (-1)\det\begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$= 2 \cdot 4 + (-1)\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$= 8 - 3 = \boxed{5 > 0}$ +1

Since all $\det(A_k) > 0 \Leftrightarrow$ A is positive definite. ✓
(and A is symmetric). +1

(solns)

2. (a) [6pts] Find the LU Factorization of the matrix A.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \underset{E_2+\frac{1}{2}E_1}{\sim} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \underset{E_3+\frac{2}{3}E_2}{\sim} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\underset{E_4+\frac{3}{4}E_3}{\sim} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

+3

+3

so $\boxed{U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}}$

(b) [6pts] Use $A = LU$ to solve $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. You must use L and U in some way.

$\boxed{1^{st}:}$ Solve $L\vec{y} = \vec{b}$, $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -1/2 & 1 & 0 & 0 & 0 \\ 0 & -2/3 & 1 & 0 & 0 \\ 0 & 0 & -3/4 & 1 & 1 \end{array}\right] \Rightarrow \boxed{\vec{y} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 5/4 \end{bmatrix}}$

so $y_1 = 1$; $y_2 = \frac{1}{2}y_1 = \frac{1}{2}$; $y_3 = \frac{2}{3}y_2 = \frac{1}{3}$; $y_4 = 1 + \frac{3}{4}y_3 = 5/4$. +3

+3

$\boxed{2^{nd}:}$ Solve $U\vec{x} = \vec{y}$, $\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 1 \\ 0 & 3/2 & -1 & 0 & 1/2 \\ 0 & 0 & 4/3 & -1 & 1/3 \\ 0 & 0 & 0 & 5/4 & 5/4 \end{array}\right] \Rightarrow \boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$

$x_4 = 1$, $x_3 = \frac{3}{4}\left(\frac{1}{3} + x_4\right) = \frac{3}{4}\left(\frac{4}{3}\right) = 1$, $x_2 = \frac{2}{3}\left(\frac{1}{2} + x_3\right) = \frac{2}{3}\left(\frac{3}{2}\right) = 1$

and $x_1 = \frac{1}{2}(1 + x_2) = \frac{1}{2}(2) = 1$.