Solutions

Math 105A Quiz 6 - July 21

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Show all of your work *There is a question on the back side.*

1. (a) [2pts] Find the l_1 and l_2 norms of $\vec{x} = (2, -1, -3, 4)^T$. Which one is larger?

- $\|\vec{x}\|_1 = |2| + |-1| + |-3| + |4| = 10.$ +1
- $\|\vec{x}\|_2 = (4+1+9+16)^{1/2} = \sqrt{30}.$ +1

$\|\vec{x}\|_1 > \|\vec{x}\|_2$ here, $\sqrt{100} > \sqrt{30}$.

(b) [8pts] Prove that for any $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\|_2 \leq \|\vec{x}\|_1$.

Pf: $\|\vec{x}\|_2 \leq \|\vec{x}\|_1 \iff \|\vec{x}\|_2^2 \leq \|\vec{x}\|_1^2$. +1

LHS: $\|\vec{x}\|_2^2 = |x_1|^2 + \dots + |x_n|^2 = \sum_{i=1}^{n} |x_i|^2$. +1

RHS: $\|\vec{x}\|_1^2 = (|x_1| + \dots + |x_n|)^2$

+4 expanding

$= |x_1|^2 + \dots + |x_n|^2 + \left(\text{Cross Terms of form } 2|x_i||x_j|\right)$ — $\sum_{i=1}^{n}\sum_{j \neq i}^{n} |x_i x_j|$, or just,

$= \sum_{i=1}^{n} |x_i|^2 + \left(\text{Non-negative Cross terms}\right)$

+2 $\|\vec{x}\|_2^2$ here.

$= \|\vec{x}\|_2^2 + \left(\text{non-negative cross terms}\right).$

Thus, $\|\vec{x}\|_1^2 \geq \|\vec{x}\|_2^2$ ✓ 𝍸 ▨

(which is equivalent to $\|\vec{x}\|_1 \geq \|\vec{x}\|_2$).

Solns Cont'd

2. [4pts] (a) Do ONE of parts (i) OR (ii). *Circle the one you are answering.*

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$. Either:

(i) Find its Matrix l_2 Norm, $||A||_2$ OR (ii) Find its Spectral Radius, $\rho(A)$.

(i) $||A||_2 = \rho(A^TA)^{1/2}$, +2

$A^TA = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$.

could solve $\det(A - \lambda I) = 0$,

But the eigenvalues of an upper or lower triangular matrix are its diagonal entries,

A^TA has $\lambda = 2, 8$.

larger!

Thus, $\boxed{||A||_2 = \sqrt{8}}$ +2

(ii) $\rho(A) = \max |\text{eigenvalues}|$.

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix}$

$= \lambda^2 - 3\lambda + 4$, +2

Roots: $\lambda = \dfrac{3 \pm \sqrt{9-16}}{2} = \dfrac{3 \pm i\sqrt{7}}{2}$.

For Both,

$|\lambda| = \left(\frac{9}{4} + \frac{7}{4}\right)^{1/2}$ +2

$= \left(\frac{16}{4}\right)^{1/2} \Rightarrow \boxed{\rho(A) = 2}$

(c) [6pts] Prove that for an $n \times n$ matrix B, $\det B = \prod_{i=1}^{n} \lambda_i$ where λ_i are the eigenvalues of B.

Pf: Recall $P_B(\lambda) = \det(B - \lambda I)$ +2 to get eigenvalues.

$= (\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$. +2

// It's $(\lambda_i - \lambda)$ because we do $\det(B - \lambda I)$. Think Δ-matrices. or diagonal.

Plugging in $\lambda = 0$,

$P_B(0) = \det(B)$ +2

$= \lambda_1 \cdot \ldots \cdot \lambda_n \Rightarrow \det B = \prod_{i=1}^{n} \lambda_i$ ✓

∎

#1b Alt Pf: Do by induction, on $\mathbb{R}^m$, and for $\|\vec{x}\|_2^2 \leq \|\vec{x}\|_1^2$.

$m=1$: $\vec{x} = x_1$ (one entry) and

$$\left[(|x_1|^2)^{1/2} \right]^2 = |x_1|^2, \quad \|\vec{x}\|_2^2 = \|\vec{x}\|_1^2 \text{ in 1-dim} \checkmark$$

Thus assume true for $m = n-1$.

$m=n$: • $\|\vec{x}\|_2 = \left[(|x_1|^2 + \dots + |x_n|^2)^{\cancel{1/2}} \right]^{\cancel{2}} = \sum_{i=1}^{n-1} |x_i|^2 + |x_n|^2$.

$$\bullet \ \|\vec{x}\|_1^2 = \left[|x_1| + \dots + |x_n| \right]^2$$

$$= \left[(|x_1| + \dots + |x_{n-1}|) + |x_n| \right]^2$$

$$= (|x_1| + \dots + |x_{n-1}|)^2 + 2|x_n|(|x_1| + \dots + |x_{n-1}|) + |x_n|^2.$$

Induction $\Rightarrow$ $(|x_1| + \dots + |x_{n-1}|)^2 \geq \sum_{i=1}^{n-1} |x_i|^2$ [the $\mathbb{R}^{n-1}$ case]

so $\|\vec{x}\|_1^2 - \|\vec{x}\|_2^2 \geq 2|x_n|(|x_1| + \dots + |x_{n-1}|) \geq 0$

$\Leftrightarrow \|\vec{x}\|_1^2 \geq \|\vec{x}\|_2^2$ for $\mathbb{R}^n$.

By induction for any $\mathbb{R}^n$, it holds that $\|\vec{x}\|_1^2 \geq \|\vec{x}\|_2^2$

$\Leftrightarrow \|\vec{x}\|_1 \geq \|\vec{x}\|_2$.

$\blacksquare$

* This proof is more complete and formal.

#1b Third Alt Pf: Because this was a "common" attempt.

$$\|\vec{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \leq \sqrt{|x_1|^2} + \cdots + \sqrt{|x_n|^2} = \|\vec{x}\|_1 .$$

Needs A LOT of Justification

Justification 1: Write $\vec{x} = x_1 \vec{e_1} + \cdots + x_n \vec{e_n}$ where $\vec{e_i}$ ~ standard basis.

Then $\|\vec{x}\|_2 \underset{\text{Inequality}}{\overset{\text{Triangle}}{\leq}} \sum_{i=1}^{n} \|x_i \vec{e_i}\|_2$

$= \sum_{i=1}^{n} |x_i| \cdot \|\vec{e_i}\|_2$ are all equal to one

$= \sum_{i=1}^{n} |x_i| = \|\vec{x}\|_1 .$

Justification 2: The function $f(t) = \sqrt{t}$ is a concave function for $t \geq 0$.

(I'd rather not go into this, but we could discuss in office or Piazza).

(you can see Wikipedia for example).