Solutions

Math 105A Quiz 7 - July 24th

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Show all of your work *There is a question on the back side.*

1. Let T be a square matrix with $||T|| < 1$. Consider the iteration $\vec{x}^{(k)} = T\vec{x}^{(k-1)} + \vec{c}$. We can start with any $\vec{x}^{(0)}$, and it converges to a solution $\vec{x}$ satisfying $\vec{x} = T\vec{x} + \vec{c}$.

(a) [7pts] Prove that the terms have error bounded by

$$||\vec{x}^{(k)} - \vec{x}|| \leq ||T||^k \cdot ||\vec{x}^{(0)} - \vec{x}||, \qquad k \geq 1.$$

Pf: $\|\vec{x}^{(k)} - \vec{x}\| = \| T\vec{x}^{(k-1)} + \cancel{\vec{c}} - T\vec{x} - \cancel{\vec{c}}\|$ +3

$= \| T(\vec{x}^{(k-1)} - \vec{x})\|$

$\leq \|T\| \cdot \|\vec{x}^{(k-1)} - \vec{x}\|$ +2

same

$\leq \|T\| \cdot \|T\| \cdot \|\vec{x}^{(k-2)} - \vec{x}\|$

$\vdots$

$\leq \|T\|^{k-1} \cdot \|\vec{x}^{(1)} - \vec{x}\|$

$\leq \|T\|^{k} \cdot \|\vec{x}^{(0)} - \vec{x}\|.$ +2 ✓

▨

(b) [3pts] Using part (a), prove that

$$||\vec{x}^{(k+1)} - \vec{x}^{(k)}|| \leq 2||T||^k \cdot ||\vec{x}^{(0)} - \vec{x}||.$$

Pf: $\|\vec{x}^{(k+1)} - \vec{x}^{(k)}\| = \|\vec{x}^{(k+1)} - \vec{x} + \vec{x} - \vec{x}^{(k)}\|$

Triangle Ineq

$\leq \|\vec{x}^{(k+1)} - \vec{x}\| + \|\vec{x} - \vec{x}^{(k)}\|$ +2

(a)

$\leq \|T\|^{k+1} \cdot \|\vec{x}^{(0)} - \vec{x}\| + \|T\|^{k} \cdot \|\vec{x}^{(0)} - \vec{x}\|$

Rewrote

$\|T\|^{k+1} = \|T\| \cdot \|T\|^k \longrightarrow$

$\|T\| < 1$

$\leq 2\|T\|^k \cdot \|\vec{x}^{(0)} - \vec{x}\|$ +1 ✓

▨

Solns contd

Recall: $A = D - L - U$ (careful with signs).

2. Let's reconsider our old tridiagonal friend, $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) [8pts] For its associated Gauss Seidel iterative $T_{GS} = (D-L)^{-1}U$, compute $\rho(T_{GS})$.

Here $D - L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

$(D-L)^{-1}$: $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{E_2 + \frac{E_1}{2}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$

$\overset{E_3 + \frac{E_2}{2}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 2 & 1/4 & 1/2 & 1 \end{array}\right]$ ⊕ divide by 2,

+3

$(D-L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$, so

+2

$(D-L)^{-1}U = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} = T_{GS}}$

$p_{T_{GS}}(\lambda) = \det \begin{bmatrix} -\lambda & 1/2 & 0 \\ 0 & 1/4-\lambda & 1/2 \\ 0 & 1/8 & 1/4-\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} 1/4-\lambda & 1/2 \\ 1/8 & 1/4-\lambda \end{bmatrix}$

+2   +1

$= -\lambda\left(\lambda^2 - \frac{\lambda}{2}\right)$, $\underline{\lambda = 0, 0, \frac{1}{2}} \Rightarrow \boxed{\rho(T_{GS}) = \frac{1}{2}.}$

2

(b) [2pts] The Gauss Seidel method $\vec{x}^{(k)} = (D-L)^{-1}\vec{b} + (D-L)^{-1}U\vec{x}^{k-1}$ converges to a unique solution $\vec{x}$ satisfying $\vec{x} = (D-L)^{-1}\vec{b} + (D-L)^{-1}U\vec{x}$. Show that $\vec{x}$ also satisfies $A\vec{x} = \vec{b}$.

Pf: $\vec{x} = (D-L)^{-1}(\vec{b} + U\vec{x}) \iff (D-L)\vec{x} = \vec{b} + U\vec{x}$

All or nothing.

$\iff (D-L-U)\vec{x} = \vec{b}$

$\iff A\vec{x} = \vec{b}$ ✓ ∎