Solutions

Math 105A Quiz 8 - July 28

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Show all of your work *There is a question on the back side.*

1. (a) Assume that A is similar to B. Prove the following two things.

(i) [4pts] They share the same characteristic polynomial, $p_A(\lambda) = p_B(\lambda)$. (Thus, this is another justification they share the same eigenvalues).

Pf: $p_A(\lambda) = \det(A - \lambda I)$ // since $A \sim B$, $A = SBS^{-1}$,

$+1$ $= \det(SBS^{-1} - \lambda SIS^{-1})$ // since $SIS^{-1} = SS^{-1} = I$,

$+1$ $= \det[S(B - \lambda I)S^{-1}]$

$+1$ $= \cancel{\det(S)} \cdot \det(B - \lambda I) \cdot \cancel{\det(S^{-1})}$

$+1$ $= \det(B - \lambda I) = p_B(\lambda)$, $\Rightarrow p_A(\lambda) = p_B(\lambda)$ ✓

∎

(ii) [4pts] If B is also similar to C, then A is similar to C.

Pf: Know $A = SBS^{-1}$ $(A \sim B)$,

$B = TCT^{-1}$ $(B \sim C)$. $+1$ defns

Plug B in, $A = STCT^{-1}S^{-1}$. $+1$

Let $R = ST$, $+1$ then $R^{-1} = (ST)^{-1} = T^{-1}S^{-1}$

So $A = RCR^{-1}$, $+1$ which defines $A \sim C$. ∎

(R invertible because S, T are).

Soln's cont'd

(b) [4pts] Give an example of two 2×2 matrices that have the same eigenvalues but are not similar. Provide a brief justification as well.

+1 Let $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (only one eigenvector) +2

D is already diagonal, But for $(B - \lambda I)\vec{v} = 0$, $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\hookrightarrow \vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

+1 B is not diagonalizable, so

$B \neq PDP^{-1}$ for any P $\Rightarrow B \not\sim D$.

2. [8pts] The rotation matrix $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ applied to any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ rotates the vector $\vec{x}$ through an angle θ [redacted]clockwise.

5 pts (a) Prove that Q preserves the l_2 norm for any $\vec{x} \in \mathbb{R}^2$.

3 pts (b) Give an example of a rotation Q changing the l_∞ norm of some $\vec{x} \in \mathbb{R}^2$.

Pf (a): Soln 1: Note $Q^T = Q^{-1}$ so Q is orthogonal. +2 (Justifying)

By Thm 9.10, then Q preserves l_2-norm ✓ +3

Soln 2: Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2}$.

And $Q\vec{x} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{bmatrix}$, so

$$\|Q\vec{x}\|_2 = \left[(x_1\cos\theta + x_2\sin\theta)^2 + (-x_1\sin\theta + x_2\cos\theta)^2 \right]^{1/2}$$

+1 $= \left[x_1^2\cos^2\theta + 2x_1x_2\cancel{\cos\theta\sin\theta} + x_2^2\sin^2\theta + x_1^2\sin^2\theta - 2x_1x_2\cancel{\cos\theta\sin\theta} + x_2^2\cos^2\theta \right]$

+2 $= \left[x_1^2(\cos^2\theta + \sin^2\theta) + x_2^2(\sin^2\theta + \cos^2\theta) \right]^{1/2}$

+2 $= \left[x_1^2 + x_2^2 \right]^{1/2} = \|\vec{x}\|_2$ ✓ +1 example.

(b): Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\|\vec{x}\|_\infty = 1$. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\theta = 45^\circ$.

$Q\vec{x} = \begin{bmatrix} 2/\sqrt{2} \\ 0 \end{bmatrix}$ so $\|Q\vec{x}\|_\infty = \frac{2}{\sqrt{2}} = \sqrt{2} \neq 1$ which is $\|\vec{x}\|_\infty$.

+2 justifying it.