Math 2D MT Group Test - October 28th

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Show all of your work. Be neat, write clearly, and box your answers.
On the exam, don't spend too long on any one problem.

For this worksheet: Please work in a group or with friends!

1. (a) Eliminate the parameter to find a Cartesian equation of the curve

$$x = \frac{\cos t}{2}, \quad y = 2\sin t, \quad t \in [0, \pi].$$

Use this to sketch the curve on the given interval in t.
(b) Compute dy/dx and d^2y/dx^2. At which times is the tangent line vertical? Horizontal?

Solution. Using the trig id $\cos^2 t + \sin^2 t = 1$, we have $(2x)^2 + (y/2)^2 = 1$, or,

$$4x^2 + \frac{y^2}{4} = 1.$$

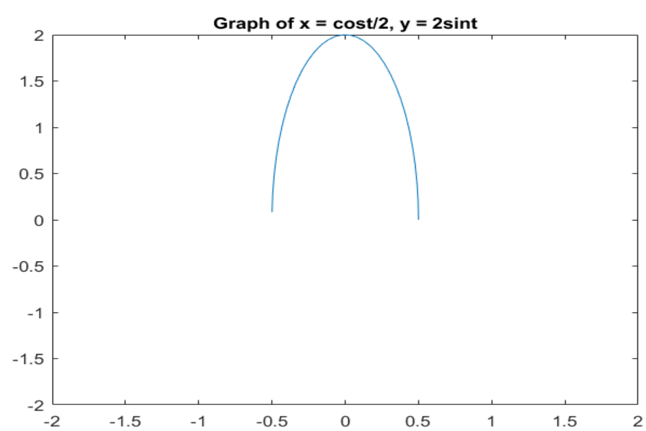
When we sketch it though, we only get the top half of the ellipse!

(b)

$$\frac{dy}{dx} = \frac{2 \cdot 2\cos t}{-\sin t} = -4\cot t.$$

$$\frac{d^2y}{dx^2} = \frac{-2 \cdot 4 \cdot (-\csc^2 t)}{-\sin t} = -8\csc^3 t.$$

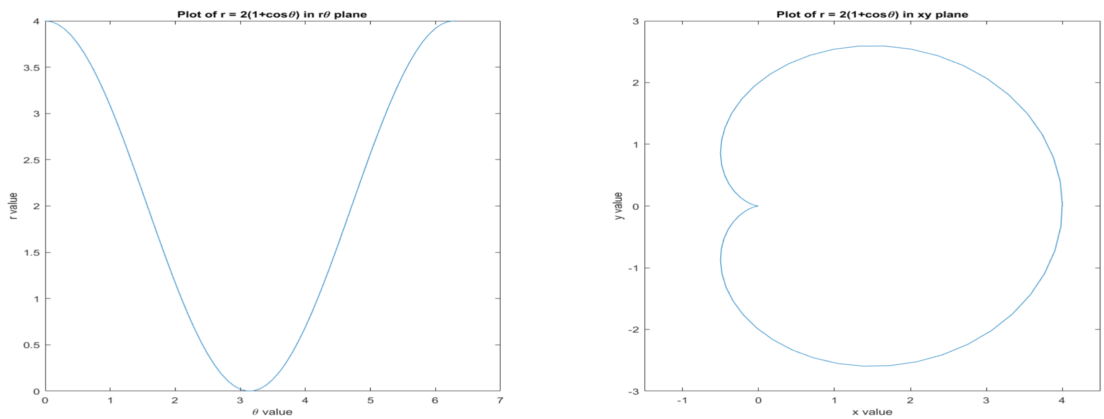
The tangent is horizontal when $\cot(t) = 0, \quad t = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, ...$
The tangent is vertical when $\cot(t) = \pm\infty, \quad t = 0, \pm\pi, \pm 2\pi, ...$
(Think about these times corresponding on the graph).



The graph of the curve in #1 on the given time range.

2. (a) Sketch the curve $r = 2(1 + \cos\theta)$ in the θr plane, and then the xy plane.
(b) Find the equation of the tangent line at $\theta = \pi/2$.

Solution. (a) It is a cardioid going right to left.



The graph of the curve in #2 on $r\theta$ and xy planes.

(b) We first have that $x = 2(1+\cos\theta)\cos\theta,\quad y = 2(1+\cos\theta)\sin\theta$. Thus, (the 2's cancel)

$$\frac{dy}{dx} = \frac{2}{2}\cdot\frac{\cos\theta(1+\cos\theta)-\sin^2\theta}{-\sin\theta(1+\cos\theta)-\sin\theta\cos\theta}$$

so at $\pi/2$,

$$\left.\frac{dy}{dx}\right|_{\pi/2} = \frac{-1}{-1} = 1.$$

Also at $\pi/2$ we have that $(x, y) = (0, 2)$. This the tangent line is

$$y - 2 = x \iff y = x + 2.$$

3. Consider the vectors $\vec{a} = \langle 0, 2, -4\rangle$ and $\vec{b} = \langle -1, 3, 1\rangle$.
(a) Compute $\vec{c} = \vec{a}\times\vec{b}$. Verify that the angle between $\vec{c}$ and the vectors $\vec{a}, \vec{b}$ is 90 degrees. What does $|\vec{c}|$ represent geometrically?
(b) Find the scalar and vector projections of $\vec{a}$ onto $\vec{b}$. (Note which is projected onto which now!)

Solution. (a) $\vec{c} = \det\begin{bmatrix} i & j & k \\ 0 & 2 & -4 \\ -1 & 3 & 1\end{bmatrix} = \langle 14, 4, 2\rangle.$

The value of $|\vec{c}|$ is the area of the parallelogram with edges $\vec{a}, \vec{b}$.
We also see that $\vec{c}\cdot\vec{a} = 8 - 8 = 0$ and that $\vec{c}\cdot\vec{b} = -14 + 12 + 2 = 0$.

(b)

$$comp_{\ \vec{b}}\ \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2}{\sqrt{11}}$$

$$proj_{\ \vec{b}}\ \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{2}{11}\langle -1, 3, 1\rangle$$

4. (a) Find the equation of the plane through the points (0,1,1), (1,0,1), (1,1,0).
(b) Find the symmetric and parametric equations of the line of intersection of the planes

$$x + y + z = 1, \qquad x + 2y + 2z = 1.$$

(c) Find the angle between the two planes in part (b).

Solution. (a) Two vectors from these points are

$$\vec{u} = \langle 1, -1, 0 \rangle, \quad \vec{w} = \langle 0, 1, -1 \rangle$$

so then

$$\vec{n} = \vec{u} \times \vec{v} = \langle 1, 1, 1 \rangle.$$

Using the 1st point, our plane equation is hence

$$x + (y - 1) + (z - 1) = 0.$$

(b) The direction of the line should thus be the cross product of the normals,

$$\vec{d} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 2 \rangle = \langle 0, -1, 1 \rangle.$$

We also need to find a point on the intersection, so we have to solve

$$\begin{cases} x + y + z = 1 \\ x + 2y + 2z = 1 \end{cases} \quad \xrightarrow{\text{set } z=0} \quad \begin{cases} x + y = 1 \\ x + 2y = 1 \end{cases}$$

so subtracting, we get that $y = 0$ and plugging in, $x = 1$. So (1,0,0) is on the intersection. Now that we have a point and the direction, the equation of the line is

$$\vec{r}(t) = \langle 1, 0, 0 \rangle + t\langle 0, -1, 1 \rangle$$

which is the vector form, so in parametric form,

$$x = 1, \; y = -t, \; z = t.$$

In symmetric form, it's actually the following:

$$x = 1, \quad \text{and} \quad \frac{y}{-1} = z.$$

(c) The angle between the two planes is determined by the angle between their normal vectors, so

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

so computing the dot product and solving for theta with $\vec{n_1} = \langle 1, 1, 1 \rangle$, $\vec{n_2} = \langle 1, 2, 2 \rangle$,

$$\theta = \arccos\left(\frac{5}{\sqrt{3}}\sqrt{9}\right) = \arccos\left(\frac{5}{3\sqrt{3}}\right).$$

5. (a) What is the domain of validity for $f(x, y) = \sqrt{4 - x^2 - 4y^2}$?
(b) Draw the contour plots and sketch the graph of $f(x, y) = \sqrt{4 - x^2 - 4y^2}$.
(c) Classify and sketch $x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$.

Solution. (a) We need the square root to be non-negative, so $4 - x^2 - 4y^2 \geq 0$. But in the xy plane, this is just the graph of

$$x^2 + 4y^2 \leq 4, \quad \text{a filled in ellipse.}$$

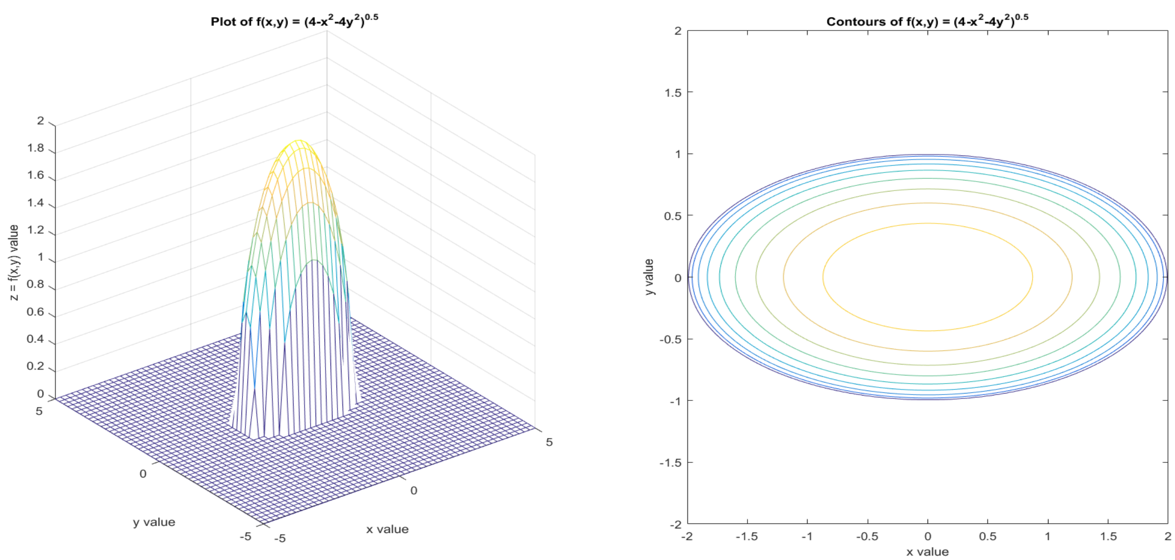
So, your domain should be written like

$$D = \{(x, y) \in \mathbb{R}^2 \text{ such that } x^2 + 4y^2 \leq 4.\}$$

(b) The level sets are graphs of the curves $f(x, y) \equiv k$, so here

$$\sqrt{4 - x^2 - 4y^2} = k \iff x^2 + 4y^2 = 4 - k^2.$$

We see that then, $0 \leq k \leq 2$ because we need $4 - k^2 \geq 0$ to make sense, and also the value of f is never negative so $k \geq 0$. These are just ellipses that get larger as $k \to 0$.



The graph of $f(x, y)$ in #5b, along with the contours.

It is only the top half of an ellipsoid! You might notice it's an ellipsoid because if we plug in z for f, we get $z = \sqrt{4 - x^2 - 4y^2}$. Then

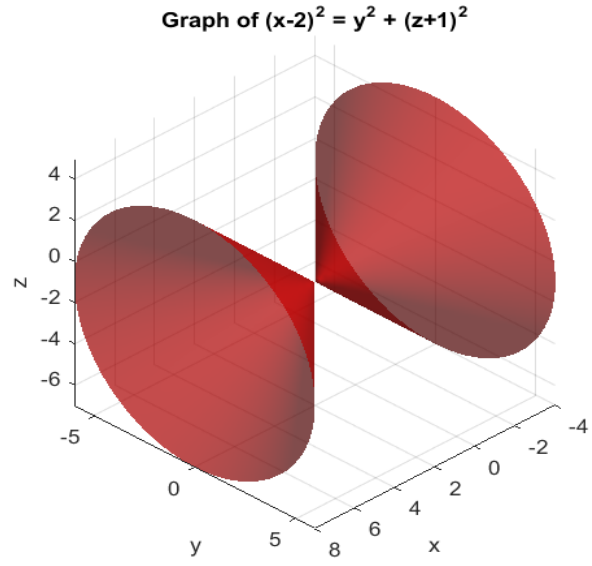
$$z^2 = 4 - x^2 4y^2 \iff x^2 + 4y^2 + z^2 = 4.$$

But, from our function $f(x, y)$, we only take the top half.

(c) First completing the square,

$$(x - 2)^2 - y^2 - (z + 1)^2 + 3 = 4 - 1 \iff (x - 2)^2 = y^2 + (z + 1)^2.$$

This is a cone!



The graph of the cone in 5c.

6. (a) Find the velocity and position vectors of a particle with:

$$\vec{a}(t) = \langle 2, 0, 2t \rangle, \quad \vec{v}(0) = \langle 3, -1, 0 \rangle, \quad \vec{r}(0) = \langle 0, 1, 1 \rangle.$$

(b) Compute the unit tangent vector $\vec{T}(t)$ for the particle's position, $\vec{r}(t)$.
(c) *Set up* the computation for the curvature of the position function $\vec{r}(t)$.
(d) An application of arclength, sort of.
Set up an integral that computes the total distance traveled from $t = 10$ to $t = 2016$.
(The total distance traveled is the integral of its speed).

Solution. (a) Let us use the definite integral approach,

$$\vec{v}(t) = \langle 3, -1, 0 \rangle + \int_0^t \langle 2, 0, 2t \rangle dt = \langle 3 + 2t, -1, t^2 \rangle.$$

Then,

$$\vec{r}(t) = \langle 0, 1, 1 \rangle + \int_0^t \langle 3 + 2t, -1, t^2 \rangle = \langle t^2 + 3t, -t + 1, \frac{t^3}{3} + 1 \rangle.$$

(b) We actually just need to look at $\vec{v}(t)$ and unitize it!

$$\vec{T}(t) = \frac{\langle 3 + 2t, -1, t^2 \rangle}{\sqrt{t^4 + 4t^2 + 12t + 10}}$$

(c) We can now just use $\vec{v}(t), \vec{a}(t)$, and we get

$$\kappa(t) = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|^3} \text{ so now, plug in from part (a)}$$

$$= \frac{|\langle 3 + 2t, -1, t^2 \rangle \times \langle 2, 0, 2t \rangle|}{[t^4 + 1 + (2t + 3)^2]^{3/2}}$$

The computation to do this isn't really that illustrative, but anyways, I think we get the following?

$$= \frac{|\langle -2t, -2t^2 - 6t, 2\rangle|}{[t^4 + 1 + (2t+3)^2]^{3/2}}$$

so then

$$\kappa(t) = \frac{\sqrt{4t^2 + (2t^2 + 6t)^2 + 4}}{[t^4 + 1 + (2t+3)^2]^{3/2}}.$$

(d) Lastly, (also plug in from (a)'s answers)

$$L = \int_{10}^{2016} |\vec{v}(t)| dt = \int_{10}^{2016} \sqrt{t^4 + 4t^2 + 12t + 10}\ dt.$$