Solns

☆ #1a has many solns actually. Here's the less obvious ones:

Math 2D Quiz 5 Evening, November 3rd

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Show all of your work. *There is a question on the back side.*

1. (a) [3pts] Determine if $\lim_{(x,y)\to(0,0)} \frac{5x+y^3}{x^3+y^3}$ exists or not. If it exists, find the limit.

Sol 1:

. It has a line of discontinuity, $y=-x$, so limit can't exist +3

Sol 2: Top ~ r^1 (from $5x$), Bot ~ r^3 ⇒ Expect DNE. (+1)

Try $\lim_{\substack{y=mx\\x\to 0}} \frac{5x+m^3x^3}{x^3+m^3x^3} = \lim_{x\to 0} \frac{5+m^3x^2}{(1+m^3)x^2}$

$= \lim_{x\to 0} \frac{5}{(1+m^3)x^2} \to \pm\infty$ (+1) $\begin{cases} +\infty & \text{if } m^3>-1 \\ -\infty & \text{if } m^3<-1 \end{cases}$.

so, regardless, the limit → ∞, DNE. (+1)

(b) [7pts] Show that $\lim_{(x,y)\to(0,0)} \frac{\tan(x)y^4}{x^4+y^4} = 0$. Recall: $\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}$.

Hint ϵ-δ: $\left|\frac{y^4}{x^4+y^4}\right| \le 1$. Hint Polar: $1 \ge \cos^4 t + \sin^4 t = 1 - \frac{1}{2}\sin^2(2t) \ge \frac{1}{2}$.

1st: $\lim_{(x,y)\to(0,0)} \frac{\tan x \, y^4}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{\left(\frac{\sin x}{x}\right)\cdot\left(\frac{x}{\cos x}\right)\cdot y^4}{x^4+y^4} \overset{\substack{\text{since}\\x\approx 0}}{=} \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$.

ϵ-δ: Pick any $\epsilon>0$.

If we take $\lim_{\substack{x=0\\y\to 0}} \frac{xy^4}{x^4+y^4} \to 0$ is our candidate limit value.

By Hint, $\left|\frac{xy^4}{x^4+y^4}\right| \overset{\text{Hint}}{\le} 1\cdot|x| \le \sqrt{x^2+y^2}$ +2

∴ For $\left|\frac{xy^4}{x^4+y^4} - 0\right| < \epsilon$, it suffices to have $\sqrt{x^2+y^2} \le \delta < \epsilon$ +3

Thus, for any $\epsilon>0$, we pick $\boxed{\delta = \epsilon/2}$ s.t. $\left|\frac{xy^4}{x^4+y^4} - 0\right| \le \sqrt{x^2+y^2} = \delta < \epsilon$, satisfying the defn. of limit existence that the limit equals 0 +2 ✓

Polar: Now, top ~ r^5 & bot ~ r^4 so we expect limit to exist (top heavy)

so $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = \lim_{r\to 0^{(+)}} \frac{r^5 \sin^4\theta\cos\theta}{r^4(\cos^4\theta+\sin^4\theta)}$

$= \lim_{r\to 0^+} r\cdot\left[\frac{\sin^4\theta\cos\theta}{\cos^4\theta+\sin^4\theta}\right]$. +2

By the Hint, and by bounding $\sin^4\theta\cos\theta$, +4

$-\frac{1}{1/2} r \le r\left[\frac{\sin^4\theta\cos\theta}{\cos^4\theta+\sin^4\theta}\right] \le \frac{1}{1/2} r$

independent of θ. Thus, in the limit,

$\underbrace{\lim_{r\to 0} -2r}_{\boxed{0}} \le \lim_{r\to 0} r\left[\frac{\sin^4\theta\cos\theta}{\cos^4\theta+\sin^4\theta}\right] \le \underbrace{\lim_{r\to 0} 2r}_{\boxed{0}}$

∴ By squeeze Thm, limit exists and is 0 +1 ✓

Soln's

2. (a) [5pts] Compute f_{xy} and f_{yx} with $f(x, y) = \cos(x^2y)$.

First, $f_x = -2xy\sin(x^2y)$ +1 , $f_y = -x^2\sin(x^2y)$. +1

So, $f_{xy} = -2x\sin(x^2y) - 2x^3y\cos(x^2y)$ +3

and $f_{yx} = -2x\sin(x^2y) - 2x^3y\cos(x^2y)$

And they match (as they should) confirming Clairaut. ✓

Details of f_{xy}: $\frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}\left[-2xy\sin(x^2y)\right]$

(similar for f_{yx})

rule: product

$= \frac{\partial}{\partial y}(-2xy)\cdot\sin(x^2y) - 2xy\frac{\partial}{\partial y}(\sin(x^2y))$

$= -2x\sin(x^2y) - 2x^3y\cos(x^2y)$ ✓

(b) [5pts] Show that $u(x,t) = \sin(kx - wt)$ satisfies the wave equation, $u_{tt} = \left(\frac{w}{k}\right)^2 u_{xx}$. In other words, show that $u_{tt} - (\frac{w}{k})^2 u_{xx} = 0$.

- $u_t = -w\cos(kx - wt)$ +1 so $u_{tt} = (-w)^2\cdot(-1)\sin(kx-wt)$
  $= -w^2\sin(kx-wt)$ +1
- $u_x = k\cos(kx-wt)$ +1 so $u_{xx} = -k^2\sin(kx-wt)$. +1

Thus, $u_{tt} - \frac{w^2}{k^2}u_{xx} = -w^2\sin(kx-wt) + \frac{w^2}{\cancel{k^2}}(\oplus\cancel{k^2}\sin(kx-wt))$ +1

$= \sin(kx-wt)\cdot(w^2 - w^2) = \boxed{0}$ ✓