Solns

☆ There's many solns to #1a, but I chose the less-obvious ones.

Math 2D Quiz 5 Morning, November 3rd

Please put name on front & ID on back for redistribution!

Show all of your work. *There is a question on the back side.*

1. (a) [3pts] Determine if $\lim_{(x,y)\to(0,0)} \frac{5x+y^3}{x^3+y^3}$ exists or not. If it exists, find the limit.

Sol 1: Along $x=-y$, the function is undefined so as $(x,y)\to(0,0)$ along this line, the limit can't exist.

Sol 2: Top $\sim r^1$ Bot $\sim r^3$ so expect DNE.

If we try $\lim_{\substack{y=mx \\ x\to 0}} \frac{5x+m^3x^3}{x^3+m^3x^3} = \lim_{x\to 0} \frac{5+m^3x^2}{(1+m^3)x^2}$

$= \lim_{x\to 0} \frac{m^3}{1+m^3} + \frac{5}{x^2(1+m^3)} \to \pm\infty, \begin{cases} +\infty & m^3>-1 \\ -\infty & m^3<-1 \end{cases}$.

so, thus the limit does not exist.

(b) [7pts] Show that $\lim_{(x,y)\to(0,0)} \frac{x^4\sin(y)}{x^4+y^4} = 0$.

Hint ϵ-δ: $\left|\frac{x^4}{x^4+y^4}\right| \le 1$. Hint Polar: $1 \ge \cos^4 t + \sin^4 t = 1 - \frac{1}{2}\sin^2(2t) \ge \frac{1}{2}$.

1st: $\lim_{(x,y)\to(0,0)} \frac{x^4 \sin y}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^4\cdot\left(\frac{\sin y}{y}\right)\cdot y}{x^4+y^4} \overset{\substack{\sin y \approx y \\ \text{near } 0}}{=} \lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^4}$.

$\boxed{\epsilon-\delta}$: First, if we take $\lim_{\substack{x=0 \\ y\to 0}} \frac{x^4 \sin y}{x^4+y^4} = 0$, our limit candidate is 0.

Now, pick any $\epsilon > 0$.

By Hint, $\left|\frac{x^4 y}{x^4+y^4} - 0\right| \le \left|\frac{x^4}{x^4+y^4}\right| |y| \overset{\text{Hint}}{\le} |y| \le \sqrt{x^2+y^2}$.

so, for $\left|\frac{x^4y}{x^4+y^4} - 0\right| < \epsilon$, it suffices to have $\sqrt{x^2+y^2} \le \delta < \epsilon$.

Thus, for any $\epsilon > 0$, there exists $\boxed{\delta = \frac{\epsilon}{2}}$ s.t. $\left|\frac{x^4y}{x^4+y^4}\right| \le \sqrt{x^2+y^2} = \delta < \epsilon$

satisfying the defn that the limit exists & equals 0. ✓

$\boxed{\text{Polar:}}$ The top $\sim r^5$, bot $\sim r^4$ so should exist.

Using $x = r\cos\theta$, $y = r\sin\theta$, $\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4+y^4} = \lim_{r\to 0^{(+)}} \frac{r^5\cos^4\theta\sin\theta}{r^4(\cos^4\theta+\sin^4\theta)}$

$= \lim_{r\to 0} r \cdot \left[\frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right]$.

Using the hint to bound indep. of θ,

$-\frac{1}{1/2} r \le r\left[\frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right] \le \frac{1}{1/2} r$.

Thus in the limit,

$\underbrace{\lim_{r\to 0}(-2r)}_{=\boxed{0}} \le \lim_{r\to 0} r\left[\frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right] \le \underbrace{\lim_{r\to 0} 2r}_{=\boxed{0}}$

∴ By the squeeze thm,

$\lim_{r\to 0} r\left[\frac{\cos^4\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\right] = 0$, exists & equals 0 ✓

2. (a) [5pts] Verify that Clairaut's Theorem holds with f_{xy} and f_{yx} where $f(x,y) = \ln(x+2y)$.

$f_x = \dfrac{1}{x+2y}$ & $f_y = \dfrac{2}{x+2y}$.

$f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{x+2y}\right) = \boxed{\dfrac{(-1)\cdot 2}{(x+2y)^2}}$

$f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{2}{x+2y}\right) = \boxed{\dfrac{2\cdot(-1)}{(x+2y)^2}}$

$\longleftrightarrow$ equal ✓

(b) [5pts] Verify that the function $u(x,t) = e^{-k^2w^2t}\sin(wx)$ solves the heat equation $u_t = k^2u_{xx}$. In other words, show that $u_t - k^2u_{xx} = 0$.

- $u_t = -k^2\omega^2 e^{-k^2\omega^2 t}\sin(\omega x)$.
- $u_x = \omega e^{-k^2\omega^2 t}\cos(\omega x)$, so

$\hookrightarrow$ $u_{xx} = -\omega^2 e^{-k^2\omega^2 t}\sin(\omega x)$.

Then, $u_t - k^2 u_{xx} = -k^2\omega^2 e^{-k^2\omega^2 t}\sin(\omega x)$

$- k^2 \cdot (-\omega^2 e^{-k^2\omega^2 t}\sin \omega x)$

$= e^{-k^2\omega^2 t}\sin(\omega x)\cdot\left[\underbrace{-\cancel{k^2\omega^2}+\cancel{k^2\omega^2}}_{\text{zero}}\right]$

$= \underline{\underline{0}}$ ✓