$\boxed{\text{Solutions}}$

Math 2D Quiz 7 Evening, November 17th

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Show all of your work. *There is a question on the back side.*

1. Consider the function $f(x,y) = xy^2$ on the domain $\mathcal{D} = \{(x,y) | x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$.

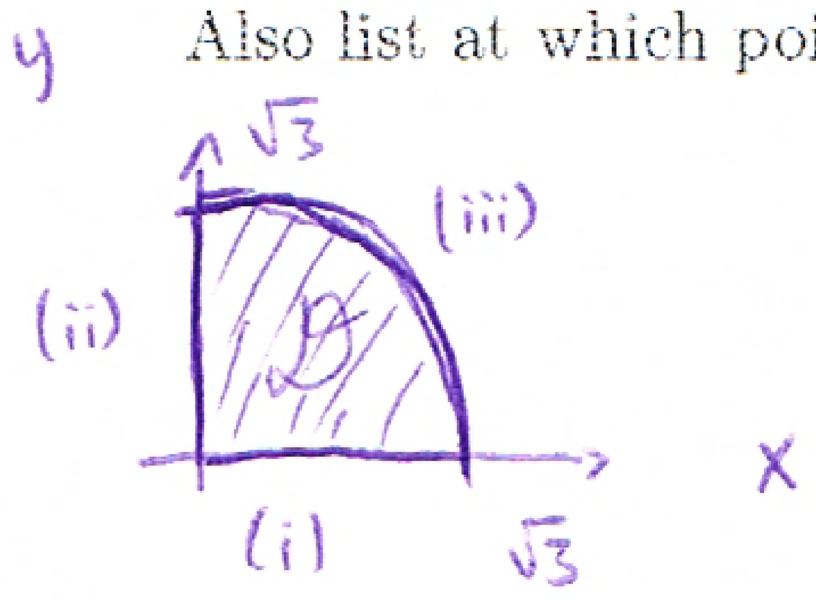
(a) [3pts] Find and classify all the critical points of f that are in $\mathcal{D}$. Evaluate f at these points. Hint: There's infinitely many, and they're all the same type and have the same function value.

$\nabla f(x,y) = \langle y^2, 2xy \rangle$ +1 is $(0,0)$ when $y=0$, $x \sim$ anything.

$\rightsquigarrow$ $\boxed{(x,0) \text{ for } 0 \leq x \leq \sqrt{3}}$ +1 are all crit. pts, and $f(x,0) = 0$.

$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = 0 \cdot 2x - (2y)^2 = -4y^2.$ At $(x,0)$, $D = 0$ +1 $\hookrightarrow$ Inconclusive!

(b) [7pts] Find the absolute max and min of $f(x,y)$ on $\mathcal{D}$. It may help to draw $\mathcal{D}$. Also list at which points the absolute max and min occur at.



- Crit. pts taken care of in (a) ✓

$\boxed{(i)}$ $y=0$, $x \sim \text{var}$, $f(x,0) = 0$ +1 (crit pts) ✓

$\boxed{(ii)}$ $x=0$, $y \sim \text{var}$, $f(0,y) = 0$ +1 too ✓

$\boxed{(iii)}$ $x^2+y^2 = 3 \rightsquigarrow y = \pm\sqrt{3-x^2}$ // Pick $(+)$ b/c top $y \geq 0$ of circle.

$\rightsquigarrow f(x, \sqrt{3-x^2}) \overset{+1}{=} 3x - x^3 \rightsquigarrow \frac{df}{dx} = 3 - 3x^2$, +2 $x = \pm 1$ are local extrema

Only $x = 1$ is in $\mathcal{D}$ $\rightsquigarrow$ $\boxed{x=1,\ y=\pm\sqrt{2}}$ +1 (only $+\sqrt{2}$ here).

Compare: $\boxed{f(1,\sqrt{2}) = 2 \quad \text{max}}$ +1

$\boxed{f(x,0) = f(0,y) = 0 \quad \text{Min}}$ at all pts $(x,0)$, $(0,y)$

$\uparrow$ $0 \leq x \leq \sqrt{3}$ $\quad$ $\uparrow$ $0 \leq y \leq \sqrt{3}$

(solu)

2. [10pts total] You may or may not know that a cube maximizes the volume while minimizing surface area for a rectangular shape, but why? We'll see. Assume that the dimensions $x, y, z \geq 0$.
→ Using Lagrange Multipliers, Prove that the dimensions of a rectangular box constrained to have surface area 150 cm^2 which maximizes volume are $x = y = z = 5$ cm. Follow this outline:

(a) [2pts] What is the system of equations we have to solve for Lagrange Multipliers?
Hint: $V = xyz$, $A = 2xy + 2yz + 2xz$ are the volume and surface area equations, respectively.

Maximizing V, so

$$\nabla V = \lambda \nabla A, \quad A = 150\ (\text{cm}^2) \Rightarrow \begin{array}{ll} (i) & yz = 2\lambda(y+z) \\ (ii) & xz = 2\lambda(x+z) \\ (iii) & xy = 2\lambda(x+y) \\ (iv) & 2(xy+yz+xz) = 150\ (\text{cm}^2) \end{array}$$

+2

(b) [3pts] Trivial Cases. How can we have $0 = 0$ for any equation above? What is the volume in these cases? Is it maximizing or minimizing the volume? (Recall: We are assuming $x, y, z \geq 0$).

+1

In any case, if any eqn reads $0 = 0$ $\Rightarrow$ $\boxed{\text{One of } x, y, z \text{ is } 0}$ (at least)

+1 +1

If one of these is 0 $\Rightarrow$ $V = 0$, minimizing Volume!

(c) [5pts] Now we can assume none of the above equations read as $0 = 0$. Solve the system in (a) for the values of x, y, z where hopefully you should get $x = y = z = 5$ cm.

$$(i) \div (ii) \Rightarrow \frac{y}{x} = \frac{y+z}{x+z} \Rightarrow \cancel{xy} + zy = \cancel{xy} + xz \Rightarrow zy = zx$$

so, $\boxed{y = x}$ +2

$$(ii) \div (iii) \Rightarrow \frac{z}{y} = \frac{x+z}{x+y} \Rightarrow zx + \cancel{zy} = xy + \cancel{zy} \Rightarrow zx = xy$$

so, $\boxed{z = y}$ too! +2

Thus, $x = y = z$. Using this in (iv) the constraint,

$$\Rightarrow 2(3x^2) = 150\ (\text{cm}^2) \Rightarrow \boxed{x^2 = \frac{150}{6} = 25\ (\text{cm}^2)}$$ +1

Thus, $x = 5$ cm and $x = y = z$ $\Rightarrow$ $\boxed{x = y = z = 5\text{ cm}}$
($x \geq 0$ assumed)