Ch 2.3 and 2.5 Worksheet - Math 3D Feb. 7th

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Chapter 2.3 - Higher Order Linear ODEs

Summary: Still assume $y = e^{rx}$ and solve for r. Now there can be higher multiplicities, where we multiply by respective powers of x. We thus have to know linear independence for 3+ functions, too.

1. Are $\{0,\ e^x + \cos x,\ e^x - \sin x,\ e^x\}$ linearly independent? What about $\{e^{rx}, xe^{rx}, x^2e^{rx}\}$?

2. (2.3.102)
(a) Find the general solution to $y''' - 3y'' + 4y' - 12y = 0$.
(b) Find the general solution to an ODE with characteristic equation $r(r-1)^3(r^2 - 4r + 13)^2$.

3. (2.3.5,6)
If $y(x) = 2xe^{4x}\cos x$, what is the 4th order ODE it came from? What is its general solution?
What are the initial conditions imposed to get this specific solution of $y(x) = 2xe^{4x}\cos x$?

Chapter 2.5 - Nonhomogeneous (aka Inhomogeneous) Equations

Summary: Only for 2nd order equations, like $y'' + ay' + by = f(x)$.
The general solution is of the form $y = y_c + y_p$ where y_c solves the homogeneous equation (Right side $= 0$, $y_c'' + ay_c' + by_c = 0$) and y_p solves specifically for the Right side's inhomogeneity ($y_p'' + ay_p' + by_p = f$). They are called the Complementary and Particular pieces.

There are two methods for inhomogeneous equations:
1) Variation of Parameters: Harder to solve but always works.
2) Undetermined Coefficients: Easier to solve but needs a good guess.

For both, the 1st step is to find the form of $y_c(x) = A_1y_1(x) + A_2y_2(x)$ like in Ch 2.2. From there:

1) For Variation of Parameters: ** Make sure the coefficient of y'' is a 1 !! **
- Solve for the functions u_1', u_2' from the system

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x) \end{cases}$$

- Integrate to find u_1, u_2.
- The particular solution is given by $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.
- Then $y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$. If we have initial conditions, we can solve for C_1 and C_2.

** When solving for C_1, C_2, don't forget there is a $y_p(x)$ there, too! **

2) For Undetermined Coefficients: Make an educated "guess" for a general form of $y_p(x)$. Then, plug y_p into the differential equation and solve for the specific form of y_p.
Here are some guesses and some rules of guessing. Hopefully you can pick up the idea:
. If $f(x)$ is a polynomial of degree k: Guess $y_p(x) = A_0 + A_1 x + .. + A_k x^k$.
. If $f(x) = \sin kx$ or $\cos kx$ or both: Guess $y_p(x) = A_1 \sin kx + A_2 \cos kx$. We must include both!
. If $f(x) = e^{kx}$: Guess $y_p(x) = Ae^{kx}$.
. Solve for the constants by making $y_p'' + ay_p' + by_p = f(x)$ hold.
Rule 1: If $f(x)$ has a piece of the Complementary solution, rescale the guess by x^d where d is the multiplicity of the complementary solution.
Rule 2: If $f(x)$ is a sum or product of these types of functions, y_p is similarly a sum or product of the types of guesses.
. After we've solved for y_p, then $y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$. Again, we can solve for C_1, C_2 if we have initial conditions. Don't forget there's a $y_p(x)$ there!

If you can't see the guess for undetermined coefficients, or don't want to do undetermined coefficients, you have to use Variation of Parameters.

1. Consider $y'' - 5y' + 6y = e^{3x}$. Find a particular solution with Variation of Parameters.
If you were to use Undetermined coefficients, what would you guess for the form of y_p?

2. (2.5.102)
If $y'' - 2y' + y = e^x + x^3$, find a particular solution to the equation.

3. What would be a good guess for y_p with Undetermined Coefficients for the following equations:
i. $y'' - 2y' + y = e^{3x} + x^2$.
ii. $y'' + 4y = \cos 2x + \sin x$.
iii. $y^{(5)} - y^{(4)} = x^2$.
For practice, try to solve for y_p in each case.