Math 3D Summer 2016
Homework 3 Solutions
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Chapter 1 Section 5

Problem 1

Solve $y' + y(x^2-1) + xy^6 = 0,\ y(1) = 1$.

Solution. This is a Bernoulli eqn, so we use $v = y^{1-6} = y^{-5}$. Then $v' = -5y^{-6}y' \iff y' = -\frac{v'y^6}{5}$. Plugging in for y' first, we see

$$-\frac{y^6}{5}v' + y(x^2-1) + xy^6 = 0.$$

Next, divide by y^6 to get (the middle term becomes $\frac{1}{y^5} = v$!)

$$-\frac{v'}{5} + v(x^2-1) + x = 0 \iff v' + 5(1-x^2)v - 5x = 0.$$

which we can now solve with integrating factor,

$$R(x) = e^{\int 5(1-x^2)dx} = e^{5(x-\frac{x^3}{3})}.$$

Then, as we know what results from integrating factors,

$$\frac{d}{dx}(e^{5(x-\frac{x^3}{3})}v) = 5xe^{5(x-\frac{x^3}{3})}.$$

Again, I don't think the RHS has any fundamental antiderivative. You can follow the formula on Pg. 33 above Exercise 1.4.2, with that $v(1) = 1^{-5} = 1$ since $y(1) = 1$,

$$R(x) = e^{\int_1^x}5(1-t^2)dt = e^{5(x-\frac{x^3}{3})-10/3} \longrightarrow v(x) = e^{5(\frac{x^3}{3}-x)+10/3}\left[1 + \int_1^x 5te^{5(t-\frac{t^3}{3})-10/3}dt\right]$$

But here is another perspective:

As an integral function,

$$e^{5(x-\frac{x^3}{3})}v = \int_1^x 5te^{5(t-\frac{t^3}{3})}dt + C.$$

If $y(1) = 1$, then $v(1) = 1$ too. Hence at $x = 1, v = 1$, we have $e^{5(2/3)} = 0 + C,\ C = e^{10/3}$. Thus,

$$v(x) = e^{-5(x-\frac{x^3}{3})}\left[\int_1^x 5te^{5(t-\frac{t^3}{3})}dt + e^{10/3}\right]$$

*****This v is the same as if we used that formula on Page 33!!*****

Since $v(x) = y(x)^{-5}$, we raise everything to the $-1/5$ power,

$$y(x) = e^{(x-\frac{x^3}{3})}\left[e^{10/3} + \int_1^x 5te^{5(t-\frac{t^3}{3})}dt\right]^{-1/5}.$$

We lastly note the domain is the whole real line. It turns out we have no issues of division by zero. (The integral term can never be less than $-e^{10/3}$.) We also see that y is never zero since exp is never zero, so our Bernoulli substitution is valid.

Problem 2

Solve $2yy' + 1 = y^2 + x$ with $y(0) = 1$.

Solution. With $v = y^2$, we have $v' = v + x - 1$. We need to do another substitution of $w = v + x - 1$ so $v' = w' - 1$. Then,

$$w' - 1 = w \iff w' = w + 1.$$

So,

$$\ln|w+1| = x + C, \implies \ln|v + x - 1 + 1| = x + C, \implies \ln|y^2 + x| = x + C.$$

We choose the positive branch of $|y^2 + x|$ because $y(0) = 1$, we need the positive case to make sense of natural log. Also solving for C,

$$\ln|1| = 0 + C, \quad C = 0.$$

And undoing the natural log, we get

$$y^2 + x = e^x, \quad y = \pm\sqrt{e^x - x}.$$

Because $y(0) = 1$, we choose the $+$ case and also we need $e^x - x > 0$ but this holds for all x. So,

$$y(x) = \sqrt{e^x - x}, \quad x \in \mathbb{R}.$$

Problem 3

Solve $y' + xy = y^4$ with $y(0) = 1$.

Solution. This is a Bernoulli equation so we use $v = y^{-3}$. Thus $y = v^{-1/3}$, $y' = \frac{-v'}{3v^{4/3}}$. Plugging in,

$$\frac{-v'}{3v^{4/3}} + xv^{-1/3} = v^{-4/3}$$

and multiplying everything by $-3v^{4/3}$,

$$v' - 3xv = -3.$$

Solving this by using integrating factor, $R(x) = e^{-3x^2/2}$,

$$e^{-3x^2/2}v = \int -3e^{-3x^2/2}dx + C \iff y^{-3} = e^{3x^2/2}\left[\int -3e^{-3x^2/2}dx + C\right].$$

The integral isn't computable by hand, so we use instead that

$$y^{-3} = e^{3x^2/2}\left[\int_0^x -3e^{-3t^2/2}dt + C\right].$$

Since $y(0) = 1$, we have $1 = e^0[0 + C]$, $\quad C = 1$, and we undo the power of y,

$$y(x) = e^{-x^2/2}\left[1 - 3\int_0^x e^{-3t^2/2}dt\right]^{-1/3}.$$

As for the domain, we just need that $1 - 3\int_0^x e^{-3t^2/2}dt$ is never zero. Numerically solving, I got that the domain is $(-\infty, 0.35)$ approximately.

Problem 4

Solve $yy' + x = \sqrt{x^2 + y^2}$.

Solution. The thing to note here is that $yy' + x$ looks like the half of the derivative of $y^2 + x^2$. So, we should substitute $u = y^2 + x^2$. Using this, our ODE is now

$$\frac{u'}{2} = \sqrt{u}$$

where separating,

$$\frac{u'}{\sqrt{u}} = 2 \implies 2\sqrt{u} = 2x + C.$$

(This is only okay if $u \gneq 0$). Then plugging in for u,

$$\sqrt{x^2 + y^2} = x + D, \quad (x^2 + y^2) = (x + D)^2.$$

Solving for y gets us

$$y = \pm\sqrt{(x + D)^2 - x^2}.$$

Notice there are a lot of issues with the domain of definition being well defined if we had to find a particular solution given some initial condition. We'd also have to choose between $\pm$ cases.

Remark. It may have been common to try to use $v = y^2$ like on the table of common substitutions because we see a yy' term. It is still fine! Then, we have

$$\frac{v'}{2} + x = \sqrt{x^2 + v}.$$

However we note that this is still not exactly an equation we can solve or separate. We still have to acknowledge that we should use another substitution $w = x^2 + v$ because we see that $\frac{v'}{2} + x = \frac{w'}{2}$. We then have the ODE $w' = 2\sqrt{w}$ which is the same as in the solution above. Also note now, $w = x^2 + v = x^2 + y^2$ is our overall substitution, same as in the solution above.

Problem 5

Solve $y' = (x + y - 1)^2$.

Solution. We first are inclined to set $w = x + y - 1$, so thus $y' = w' - 1$. Then

$$y' = (x + y - 1)^2 \implies w' - 1 = w^2 \iff w' = w^2 + 1.$$

Separating and integrating,

$$\int \frac{dw}{w^2 + 1} = \int dx + C \iff \arctan(w) = x + C.$$

Thus, with $w = x + y - 1$, the general solution is

$$(x + y - 1) = \tan(x + C) \iff y = 1 - x + \tan(x + C).$$

Problem 6

Solve $y' = \frac{x^2-y^2}{xy}$ with $y(1) = 2$.

Solution. Notice that this is just $y' = \frac{x}{y} - \frac{y}{x}$ so this is a homogeneous equation. Using $v = y/x$,

$$xv' + v = \frac{1}{v} - v \iff \frac{dv}{dx} = \frac{\frac{1}{v} - 2v}{x} \iff \frac{vdv}{1-2v^2} = \frac{dx}{x}.$$

Again we see this has problems if $x, y = 0$ just to keep in mind. Solving this, we use a u-sub of $u = 1 - 2v^2$ to get

$$-\frac{1}{4}\ln|1-2v^2| = \ln|x| + C \iff D = 4\ln|x| + \ln|1 - 2(y/x)^2|.$$

Since $y(1) = 2$, we actually have to take the following branches of the absolute value,

$$D = 4\ln(x) + \ln(2\frac{y^2}{x^2} - 1).$$

At $y(1) = 2$, we have $D = \ln(8-1) = \ln(7)$. Thus, taking exponential,

$$7 = x^4 \cdot (\frac{2y^2}{x^2} - 1) \iff 7 = 2y^2x^2 - x^4 \iff y = \pm\sqrt{\frac{x^2}{2} + \frac{7}{2x^2}}$$

where from the I.C., and recall that our substitution is only valid for $x, y \neq 0$,

$$y(x) = \sqrt{\frac{x^2}{2} + \frac{7}{2x^2}}, \quad x > 0.$$