Math 3D Practice Final (2014)

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Problem 1

General solution to $y'''' + y'' = t^2$ with undetermined coefficients.

Solution. First, we get the homogeneous solution. The auxiliary equation is

$$r^4 + r^2 = 0 \iff r^2(r^2+1) = 0, \quad r = 0 \text{ (mult. 2) }, \pm i.$$

Thus,

$$y_c = C_0 + C_1 t + C_2 \cos t + C_3 \sin t.$$

Now we use undetermined coefficients to get y_p, and since the RHS is t^2 while up to linear terms are complementary solutions, we actually have to account for two extra powers. (Another way to see this is to see the lowest order derivative of y, here it's 2nd order, so we have to go to t^2 (from the RHS) * t^2 (from the 2nd order derivative) = t^4).

Thus, we should actually guess

$$y_p = At^4 + Bt^3 + Ct^2.$$

With this,

$$y_p'' = 12At^2 + 6Bt + 2C, \quad y_p'''' = 24A.$$

Plugging into the ODE and matching coefficients for powers of t,

$$(24A) + (12At^2 + 6Bt + 2C) = t^2.$$

We read off that we need:

$$\begin{cases} constant: & 24A + 2C = 0 \\ t: & 6B = 0 \\ t^2: & 12A = 1. \end{cases}$$

Thus, we see $A = 1/12,\ B = 0$, and $\ 2 + 2C = 0 \iff C = -1$. We get all the coefficients, so

$$y_p = \frac{t^4}{12} - t^2.$$

Thus, the general solution is $y_c + y_p$ which is

$$y(t) = C_0 + C_1 t + C_2 \cos t + C_3 \sin t + \frac{t^4}{12} - t^2.$$

Problem 2

General solution to $3y'' + 4y' + y = e^{-t}\sin(t)$ with variation of parameters.

Solution. Again, first solve the homogeneous case, so begin with the auxiliary equation

$$3r^2 + 4r + 1 = 0 \iff (3r+1)(r+1) = 0 \iff r = -1, -1/3.$$

So, the complementary solution is

$$y_c = C_1 e^{-t} + C_2 e^{-t/3}$$

where in particular, $y_1 = e^{-t}$ and $y_2 = e^{-t/3}$ are our two functions to use for variation of parameters. Proceeding with that method for the particular solution, we assume that

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where the functions u_1, u_2 satisfy

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = f(t).$$

Be Careful!! See page 74 right under where these equations are boxed. We need the ODE to actually read as $L(y) = y'' + p(x)y' + q(x) \ \rightarrow$ mainly, the coefficient of y'' must be 1!
Thus, we need to rewrite our ODE as

$$y'' + \frac{4}{3}y' + \frac{y}{3} = \frac{e^{-t}}{3}\sin t.$$

Specifically plugging in for y_1, y_2 and *this* $f(t)$ now, our system is

$$\begin{cases} u_1'e^{-t} + u_2'e^{-t/3} = 0 \\ -u_1'e^{-t} - \frac{u_2'}{3}e^{-t/3} = \frac{e^{-t}\sin t}{3}. \end{cases}$$

Adding the two lines yields

$$\frac{2}{3}u_2'e^{-t/3} = \frac{e^{-t}\sin t}{3} \iff u_2' = \frac{1}{2}e^{-2t/3}\sin t.$$

Using this in the 1st line yields

$$u_1'e^{-t} = -\frac{1}{2}e^{-3t/3}\sin t \iff u_1' = -\frac{1}{2}\sin t.$$

Integrating, we get that

$$u_1(t) = \int \frac{-1}{2}\sin t dt = \frac{1}{2}\cos t.$$

Similarly, we have to do this for u_2, where now,

$$u_2(t) = \frac{1}{2}\int e^{-2t/3}\sin t dt$$

where in particular,

$$\int e^{-2t/3}\sin t dt = -e^{-2t/3}\cos t - \int +\frac{2e^{-2t/3}}{3}\cos t dt$$
$$= -e^{-2t/3}\cos t - \frac{2e^{-2t/3}}{3}\sin t + \int -\frac{4e^{-2t/3}}{9}\sin t dt.$$

Thus, we get that

$$\frac{13}{9}\int e^{-2t/3}\sin t dt = e^{-2t/3}[-\frac{2\sin t}{3} - \cos t] \iff \int e^{t/3}\sin t dt = \frac{9}{13}e^{-2t/3}\,[-\frac{2\sin t}{3} - \cos t].$$

Hence, we have our u_1, u_2:

$$u_1(t) = \frac{-\sin t}{2}, \qquad u_2(t) = \frac{1}{2} \cdot \frac{9}{13} e^{-2t/3} \left[-\cos t - \frac{2\sin t}{3} \right].$$

Thus we can just plug in now,

$$y_p = u_1 e^{-t} + u_2 e^{-t/3}$$

$$y_p = \frac{1}{2} e^{-t} \cos t + \frac{1}{2} \cdot \frac{9}{13} e^{-t} \cdot [-\frac{2\sin t}{3} - \cos t]$$

because $-9/26 + 1/2 = (13-9)/26$ for the $\cos t$ coefficient,

$$= e^{-t} \left[-\frac{3}{13} \sin t + \frac{2}{13} \cos t \right]$$

(Now this is the same answer as in Wolframalpha!)
Anywho, the final solution is $y_p + y_c$,

$$y = y_c + y_p = C_1 e^{-t} + C_2 e^{-t/3} + e^{-t} \left[\frac{2}{13} \cos t - \frac{3}{13} \sin t \right]$$

Problem 3

General solution to the system $\vec{x}\,' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \vec{x}$.

Solution. First we need the eigenvalues and eigenvectors,

$$c_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ -2 & 0 & -1-\lambda \end{bmatrix} = (1-\lambda)(\lambda^2 - 1) + 1(2*(1-\lambda))) = (1-\lambda)(\lambda^2+1)$$

so thus $\lambda = 1, \pm i$. Now to find their eigenvectors (let them be of the form $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$)

$$\lambda = 1 : \begin{bmatrix} 0 & 0 & 1 & \vdots 0 \\ 0 & 0 & -1 & \vdots 0 \\ -2 & 0 & -2 & \vdots 0 \end{bmatrix} \overset{clear\ Col\ 3}{\sim} \begin{bmatrix} 0 & 0 & 1 & \vdots 0 \\ 0 & 0 & 0 & \vdots 0 \\ -2 & 0 & 0 & \vdots 0 \end{bmatrix} \longrightarrow a = c = 0, b = free$$

so thus we have our one eigenvector of $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. You may have also noticed this that it was the 2nd column and used it from observation.

For the complex pairs, we only need to do work on one of them, lets say $\lambda = i$:

$$\begin{bmatrix} 1-i & 0 & 1 & \vdots 0 \\ 0 & 1-i & -1 & \vdots 0 \\ -2 & 0 & -1-i & \vdots 0 \end{bmatrix} \overset{R3=R3+(1+i)R1}{\sim} \begin{bmatrix} 1-i & 0 & 1 & \vdots 0 \\ 0 & 1-i & -1 & \vdots 0 \\ 0 & 0 & 0 & \vdots 0. \end{bmatrix}$$

where notably, $(1-i)(1+i) = 1-i^2 = 2$. We should also make the 1st and 2nd pivot columns real numbers, so multiply by $(1+i)$ to all the terms now, (again using that $(1-i)(1+i) = 2$)

$$\left[\begin{array}{ccc:c} 2 & 0 & 1-i & 0 \\ 0 & 2 & -(1-i) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow a = \frac{i-1}{2}c, \; b = \frac{1-i}{2}c, \; c = free$$

so our eigenvector is $\vec{v} = \begin{bmatrix} i-1 \\ 1-i \\ 2 \end{bmatrix}$ We have that this would give us terms like

$$e^{it}\begin{bmatrix} i-1 \\ 1-i \\ 2 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} i-1 \\ 1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -\cos t - \sin t \\ \cos t + \sin t \\ 2\cos t \end{bmatrix} + i\begin{bmatrix} \cos t - \sin t \\ -\cos t + \sin t \\ 2\sin t \end{bmatrix}$$

and with this, hence, our general solution is of the form

$$\vec{x}(t) = C_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} -\cos t - \sin t \\ \cos t + \sin t \\ 2\cos t \end{bmatrix} + C_3\begin{bmatrix} \cos t - \sin t \\ -\cos t + \sin t \\ 2\sin t \end{bmatrix}.$$

Problem 4

General solution with undetermined coefficients for $\vec{x}\,' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution. First we need the complementary solution - find the eigenvalues/vectors. First,

$$c_A(\lambda) = (3-\lambda)(-1-\lambda) + 4 = \lambda^2 - 2\lambda + 1 = (1-\lambda)^2, \quad \lambda = 1 \text{ (mult. 2)}$$

so thus we need to check if it's defective, too. First getting the eigenvector(s) of the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\left[\begin{array}{cc:c} 2 & -4 & 0 \\ 1 & 2 & 0 \end{array}\right] \longrightarrow a = 2b, \quad \vec{v}_1 = b\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

There's only one, so we need another generalized eigenvector. For example,

$$\left[\begin{array}{cc:c} 2 & -4 & 2 \\ 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow a = 2b+1, \; \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\vec{v}_1 \quad (\text{let } b = 0).$$

We get a complementary solution of

$$\vec{x}_c = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right).$$

For the particular solution since our $\vec{f} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we should guess that $\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Plugging this into the ODE, we have that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

where solving this,

$$\begin{bmatrix} 3 & -4 & \vdots -1 \\ 1 & -1 & \vdots -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & \vdots 2 \\ 1 & -1 & \vdots -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & \vdots -2 \\ 1 & 0 & \vdots -3 \end{bmatrix} \iff \vec{x}_p = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

so the general solution is hence

$$\vec{x} = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Problem 5

Solve $y'' + y = te^t$ with $y(0) = y'(0) = 0$ by Laplace transform.

Solution. Since the I.C. are zero, the transform is reasonably nice:

$$s^2 Y(s) + Y(s) = \frac{1}{(s-1)^2}$$

where recall that $\mathcal{L}(e^{at} f(t)) = F(s-a)$. We see that we have

$$Y(s) = \frac{1}{(s-1)^2(s^2+1)}$$

so decomposing into partial fractions,

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}.$$

In other words, we need

$$1 = A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)(s-1)^2.$$

Plugging in $s = 1$ tells us that $2B = 1 \iff B = 1/2$. Now we can go through powers of s: (first expand out the above line and then read off that:)

$$\begin{cases} s^3: & 0 = A + C \\ s^2: & 0 = -A + 1/2 - 2C + D \\ s: & 0 = A + C - 2D \\ 1: & 1 = -A + 1/2 + D. \end{cases}$$

Since $A + C = 0$ we infer $D = 0$ for s. This means for 1, $A = -1/2$. Lastly, $C = 1/2$ then. So, we have that

$$Y(s) = \frac{1}{(s-1)^2(s^2+1)} = \frac{-1/2}{(s-1)} + \frac{1/2}{(s-1)^2} + \frac{s/2}{s^2+1}$$

where we can now invert this:

$$y(t) = -\frac{e^t}{2} + \frac{te^t}{2} + \frac{\cos t}{2}.$$

(We can/should check that the I.C. are satisfied:)

$$\begin{cases} y(0) = -e^0/2 + \cos(0)/2 = 0 \quad :) \\ y'(0) = -e^0/2 + e^0/2 - \sin(0)/2 = 0 \quad :) \end{cases}$$

(And you could even moreover check that $y'' + y = te^t$ too).

Problem 6

Solve $y'' + t^2 y + 2ty = 0$ with $y(0) = 1,\ y'(0) = 0$ by power series.

Solution. We are expanding about $t = 0$ so we assume $y(t) = \sum_{k=0}^{\infty} a_k t^k$. Thus,

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} + t^2 \sum_{k=0}^{\infty} a_k t^k + 2t \sum_{k=0}^{\infty} a_k t^k = 0.$$

i.e.

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} + \sum_{k=0}^{\infty} a_k t^{k+2} + 2\sum_{k=0}^{\infty} a_k t^{k+1} = 0.$$

The first series starts at a constant, the 2nd at quadratic, and the 3rd at linear, so our main series should start at $k = 2$ and we pull out the prior terms:

$$2a_2 + (6a_3 + 2a_0)t + \sum_{k=2}^{\infty} t^k[(k+2)(k+1)a_{k+2} + a_{k-2} + 2a_{k-1}] = 0.$$

Thus, we can extrapolate the recursion relation:

$$\begin{cases} a_0, a_1 = \text{ free (to be determined with I.C.} \\ a_2 = 0 \\ a_3 = -a_0/3 \\ a_{k+2} = \frac{-2a_{k-1} - a_{k-2}}{(k+2)(k+1)}, \quad k \geq 2. \end{cases}$$

In particular, we know that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$ so we can use this simplification. Thus, when we tabulate the coefficients,

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = -1/3 \\ a_4 = (-2a_1 - a_0)/(4 * 3) = -1/12 \\ a_5 = (-2a_2 - a_1)/(5 * 4) = 0 \\ a_6 = (-2a_3 - a_2)/(6 * 5) = 2/(6 * 5 * 3) \\ a_7 = (-2a_4 - a_3)/(7 * 6) = 1/(7 * 6 * 2) \\ \vdots \end{cases}$$

so our series looks like

$$y(x) = 1 - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^6}{45} + \frac{x^7}{84} - ...$$