Math 3D Practice for the Final (2017)

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Problem 1

Review past quizzes / midterms / homework solutions on the discussion website for material prior to Chapter 2. [Also review them for the material after Chapter 2, too.]
****Especially the Quizzes!****

Problem 2 (Ch 2.5)

(a) General solution to $mx'' + cx' + kx = 0$ where $m = 3\ kg,\ c = 4\ kg/s,\ k = 1\ N/m$.
(b) Classify the damping.
(c) Reduce the order of the equation and solve the system using the Eigenvalue method.

Solution. (a) The 2nd order equation we get is $3x'' + 4x' + x = 0$.
Again, begin with the auxiliary equation

$$3r^2 + 4r + 1 = 0 \iff (3r+1)(r+1) = 0 \iff r = -1, -1/3.$$

So, the general solution is

$$y_c = C_1 e^{-t} + C_2 e^{-t/3}.$$

(b) Since we get two distinct decays, our system is overdamped.
(c) To reduce the order, first isolate x'' as $x'' = -\frac{4}{3}x' - \frac{1}{3}x$. Then our system after reducing the order is

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1/3 & -4/3 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

or if you like to set $u_1 = x$ so $u_2 = x'$ to get the equations

$$\begin{cases} u_1' = u_2 \\ u_2' = -\frac{1}{3}u_1 - \frac{4}{3}u_2 \end{cases}$$

the system is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1/3 & -4/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Solving the eigenvalues,

$$\det \begin{bmatrix} -\lambda & 1 \\ -1/3 & -4/3 - \lambda \end{bmatrix} = \lambda^2 + \frac{4}{3}\lambda + \frac{1}{3} = 0 \iff 3\lambda^2 + 4\lambda + 1 = 0.$$

So, $\lambda = -1, -1/3$ like the roots from the auxiliary equation in (a). Next get the eigenvectors from solving $(A - \lambda I)\vec{v} = 0$,

$$\lambda = -1 : \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1/3 & -1/3 & 0 \end{array}\right] \to \begin{array}{l} v_1 = -v_2 \\ v_2 = free \end{array}, \quad \vec{v} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda = -1/3 : \left[\begin{array}{cc|c} 1/3 & 1 & 0 \\ -1/3 & -1 & 0 \end{array}\right] \to \begin{array}{l} v_1 = -3v_2 \\ v_2 = free \end{array}, \quad \vec{v} = v_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

so our general solution is

$$\vec{x} = \begin{bmatrix} x \\ x' \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-t/3} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

which is consistent componentwise with what we got in (a), with some rescaling of constants C_1, C_2.

Problem 3 (Ch 3.4) - Rather Difficult Example.

General solution to the system $\vec{x}\,' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \vec{x}$.

Solution. First we need the eigenvalues and eigenvectors, cofactor along the 2nd column,

$$c_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ -2 & 0 & -1-\lambda \end{bmatrix} = (1-\lambda)\cdot(-1)^{2+2}\cdot \det \begin{bmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix} = (1-\lambda)(\lambda^2+1)$$

so thus $\lambda = 1$ and $\pm i$. Now to find their eigenvectors (let them be of the form $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$)

$$\lambda = 1: \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \overset{Reduce\ Col\ 3}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} a = 0 \\ b = free \\ c = 0 \end{array}, \quad \vec{v} = b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so thus we have our one eigenvector of $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. You may have also noticed this that it was the 2nd column and used it from observation.

For the complex pairs, we only need to do work on one of them, lets say $\lambda = i$:

$$\left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ -2 & 0 & -1-i & 0 \end{array}\right] \overset{R3=R3+(1+i)R1}{\sim} \left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ 0 & 0 & 0 & 0. \end{array}\right]$$

where notably, $(1-i)(1+i) = 1 - i^2 = 2$. We should also make the 1st and 2nd pivot columns real numbers, so multiply by $(1+i)$ to all the terms now, (again using that $(1-i)(1+i) = 2$)

$$\left[\begin{array}{ccc|c} 2 & 0 & 1+i & 0 \\ 0 & 2 & -(1+i) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} a = -\frac{1+i}{2}c \\ b = \frac{1+i}{2}c \\ c = free \end{array}, \quad \vec{v} = \tilde{c} \begin{bmatrix} -1-i \\ 1+i \\ 2 \end{bmatrix} \quad \text{(rescaled by 2)}$$

so our eigenvector is $\vec{v} = \begin{bmatrix} -1-i \\ 1+i \\ 2 \end{bmatrix}$. The corresponding complex solution then looks like

$$e^{it}\begin{bmatrix} -1-i \\ 1+i \\ 2 \end{bmatrix} = (\cos t + i\sin t)\left(\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\cos t + \sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + i\begin{bmatrix} -\cos t - \sin t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}.$$

With this, we take the real and imaginary parts as our 2nd and 3rd independent solutions.

We get the general solution, (don't forget our eigenvector for $\lambda = 1$ from the beginning)

$$\vec{x}(t) = C_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -\cos t + \sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + C_3 \begin{bmatrix} -\cos t - \sin t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}.$$

Problem 3 and 1/2

Review Exercises 3.3 # 3-5 in Homework 5 on Linear Independence of Vector Valued Functions. Reminder: 3.3.5 was a graded homework problem and 3.3.4 was one we may have done in discussion.

Thus, here's the solution to 3.3.3: Both ways are valid.

Way 1: The books hint. If we plug in $t = 0$, then just at this time, our vector functions become the constant vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

These two vectors are independent (collumn 1 is not a scalar multiple of collumn 2) so if our vector functions are independent just at one time $t = 0$ (or in general, you could try any other time like $t = 1$ or $t = -1$ or ...etc...), they are considered independent vector functions overall.

Way 2: Definition approach. We want to solve for C_1, C_2 such that

$$C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} C_1 e^t \\ C_1 e^t \end{bmatrix} + \begin{bmatrix} C_2 e^t \\ -C_2 e^t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the first row, we read

$$R1: \quad C_1 e^t + C_2 e^t = 0 \longrightarrow C_1 = -C_2.$$

Then in the second row, we have

$$R_2: \quad C_1 e^t - C_2 e^t = 0 \overset{C_1 = -C_2}{\longrightarrow} -2C_2 e^t = 0, \quad \text{has to hold for ALL times } t.$$

The only way we can have $2C_2 e^t = 0$ for all times t is to have $C_2 = 0$. That means $C_1 = 0$ too. Thus, the only solution to the zero equation is the trivial solution, meaning that the two vector functions

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t, \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t \right\}$$

are linearly independent by definition.

Problem 4 (Ch 3.7,8)

Exact solution to $\vec{x}\,' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$, with initial condition $\vec{x}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e$ using matrix exponent. Check your answer by using methods in 3.7.

Solution. First we need the general solution, which will be $\vec{x}_{gen} = e^{tA} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$. Begin by finding the eigenvalues/vectors.

$$c_A(\lambda) = (3-\lambda)(-1-\lambda) + 4 = \lambda^2 - 2\lambda + 1 = (1-\lambda)^2, \quad \lambda = 1 \text{ (mult. 2)}$$

so thus we need to check if it's defective, too. First getting the eigenvector(s) of the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\begin{bmatrix} 2 & -4 & \vdots 0 \\ 1 & -2 & \vdots 0 \end{bmatrix} \longrightarrow a = 2b, \quad \vec{v}_1 = b \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

There's only one, so we need to decompose the matrix as $A = \mu I + N$ where $\mu = 1$ our repeated eigenvalue. Therefore, $A = I + N$ where

$$N = A - I = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

So now, we know that N commutes with I identity so we use $e^{tA} = e^{tI} \cdot e^{tN}$. To compute e^{tN},

$$e^{tN} = I + tN + \frac{t^2}{2!} N^2 + ...$$

and computing N^2, we get $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This means all higher powers of N are also zero so the series for e^{tN} is only two terms. We get that

$$e^{tN} = I + tN = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}.$$

Therefore

$$e^{tA} = e^{tI} e^{tN} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} = e^t \cdot I \cdot \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}$$

so,

$$e^{tA} = e^t \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}$$

and hence

$$\vec{x}_{gen} = e^t \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now to satisfy the initial condition, we have to solve for c_1, c_2, but from matrix exponents, we know that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{-t_0 A} \vec{x}(t_0)$$

so plugging in our initial time and initial values,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{-tA}\Big|_{t=1} \vec{x}(1) = e^{-1} \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Therefore,

$$\vec{x}_{exact} = e^t \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = e^t \begin{bmatrix} 3-2t \\ 2-t \end{bmatrix}$$

where we get the last equality if you want to multiply it out, which you don't have to but it helps with checking the answer using 3.7 eigenvalue method.

Check the initial condition is satisfied? Yup!

$$\vec{x}_{exact}(1) = e^1 \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = e \begin{bmatrix} 9-8 \\ 3-2 \end{bmatrix} = e \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad :)$$

Check with 3.7 Eigenvalue Method:

There was only one eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so we need another generalized eigenvector.

So, we solve $(A - I)\vec{v}_2 = \vec{v}_1$, letting $\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ as its components,

$$\begin{bmatrix} 2 & -4 & \vdots & 2 \\ 1 & -2 & \vdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \vdots & 1 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \longrightarrow \begin{matrix} a = 2b+1 \\ b = free \end{matrix}, \; \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\vec{v}_1 \;\; (\text{let } b = 0).$$

We get a complementary solution of

$$\vec{x}_c = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right).$$

At time $t = 1$, we need (we can cancel the e's)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e \iff \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\iff \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution to this is $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ so plugging these values in,

$$\vec{x} = 2e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = e^t \begin{bmatrix} 4-1-2t \\ 2-t \end{bmatrix}$$

so

$$\vec{x} = e^t \begin{bmatrix} 3-2t \\ 2-t \end{bmatrix}$$

which is the same solution we had before! :)

Problem 5 (Ch 6.1,2)

Solve $y'' + y = te^t$ with $y(0) = y'(0) = 0$ by using Laplace transform.

Solution. Since the I.C. are zero, the transform is reasonably nice:

$$s^2 Y(s) + Y(s) = \frac{1}{(s-1)^2}$$

where recall that $\mathcal{L}(e^{at} f(t)) = F(s-a)$. We see that we have

$$Y(s) = \frac{1}{(s-1)^2(s^2+1)}$$

so decomposing into partial fractions,

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}.$$

In other words, we need

$$1 = A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)(s-1)^2.$$

Plugging in $s = 1$ tells us that $2B = 1 \iff B = 1/2$. Now we can go through powers of s: (first expand out the above line and then read off that:)

$$\begin{cases} s^3: & 0 = A + C \\ s^2: & 0 = -A + 1/2 - 2C + D \\ s: & 0 = A + C - 2D \\ 1: & 1 = -A + 1/2 + D. \end{cases}$$

Since $A + C = 0$ we infer $D = 0$ for s. This means for the 1 - constant equation, $A = -1/2$. Lastly, $C = 1/2$ then. So, we have that

$$Y(s) = \frac{1}{(s-1)^2(s^2+1)} = \frac{-1/2}{(s-1)} + \frac{1/2}{(s-1)^2} + \frac{s/2}{s^2+1}$$

where we can now invert this using the same shifting property $\mathcal{L}(e^{at} f(t)) = F(s-a)$ in reverse :

$$y(t) = -\frac{e^t}{2} + \frac{te^t}{2} + \frac{\cos t}{2}.$$

We can/should check that the I.C. are satisfied:

$$\begin{cases} y(0) = -e^0/2 + \cos(0)/2 = 0 \quad :) \\ y'(0) = -e^0/2 + e^0/2 - \sin(0)/2 = 0 \quad :) \end{cases}$$

(And you could even moreover check that $y'' + y = te^t$ too).

Problem 6 (Ch 6.2,3)

(This is Exercise 6.3.9) Solve $x'' - 2x = e^{-t^2}$ with $x(0) = 0,\ x'(0) = 0$.
Leave the answer as a definite integral. Hint: Use a convolution in the inverse Laplace transform.

Solution. First transform the equation,

$$s^2 X(s) - 2X(s) = \mathcal{L}(e^{-t^2}).$$

So,

$$X(s)(s^2 - 2) = \mathcal{L}(e^{-t^2}) \iff X(s) = \mathcal{L}(e^{-t^2}) \cdot \frac{1}{s^2 - 2}.$$

To invert these, we use Theorem 6.3.1 with how $\mathcal{L}(f * g) = F(s)G(s)$, so $\mathcal{L}^{-1}(F(s)G(s)) = f * g(t)$,

$$x(t) = \mathcal{L}^{-1}(\mathcal{L}(e^{-t^2})) * \mathcal{L}^{-1}\left(\frac{1}{s^2 - 2}\right)$$

$$\overset{Table\ 6.1}{=} (e^{-t^2}) * \frac{1}{\sqrt{2}} \sinh(\sqrt{2}\ t)$$

where using the definition of convolution, we get

$$x(t) = \frac{1}{\sqrt{2}} \int_0^t e^{-y^2} \sinh(\sqrt{2}(t - y))dy$$

as our solution. Don't forget that $f * g$ really means convolution, not multiplication.