Math 3D 2014 Practice Midterm Solutions

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Problem 1: Solve $x^2y' = y - xy$ satisfying $y(-1) = -1$.

Solution. First rewrite it as

$$y' = y\frac{1-x}{x^2}$$

so then this becomes separable, using Leibniz notation just for clarity,

$$\frac{dy}{y} = \frac{(1-x)}{x^2}dx$$

where we keep in mind that $x = 0$ and $y = 0$ are bad points. Integrating,

$$\ln|y| = -\frac{1}{x} - \ln|x| + C \iff \ln|y| = -\frac{1}{x} + \ln\frac{1}{|x|} + C.$$

Due to the I.C, we take the negative branch of absolute value to make sure the natural log is well defined. Also we take exponent,

$$-y = D\frac{1}{-x}e^{-\frac{1}{x}}$$

and with the initial condition, we see $1 = De^1, \quad D = e^{-1}$. So,

$$y(x) = \frac{1}{x}e^{-\frac{1}{x}-1}.$$

There are only issues when x or y are 0, so hence our domain of validity is $(-\infty, 0)$.

Problem 2: General solution to $xy^2y' = y^3 - x^3$.

Solution. This has 2 substitutions which both work.

1) Sub $v = y^3$: Then $v' = 3y^2y'$. The equation becomes

$$x\frac{v'}{3} - v = -x^3, \quad v' - \frac{3v}{x} = -3x^2.$$

This is solved with an integration factor of $R(x) = e^{-3\ln|x|} = |x|^{-3} = \pm\, x^{-3}$ depending on the initial condition. Thus, we know from integration factors, after we multiply $R(x)$ to both sides,

$$\frac{d}{dx}(\pm x^{-3}v) = \mp\frac{3}{x}, \quad \pm x^{-3}v = \mp 3\ln|x| + C.$$

Since $v = y^3$,

$$y = \sqrt[3]{Dx^3 - 3x^3\ln|x|}$$

2) Note that the degree of all the terms is 3. Thus, divide xy^2 to obtain $y' = \frac{y}{x} - \frac{x^2}{y^2}$. So, we sub $v = y/x$. Then $y' = xv' + v$, and we obtain

$$xv' + v = v - 1/v^2 \iff xv' = -1/v^2, \quad v^2dv = -\frac{dx}{x}.$$

Solving, we get

$$\frac{v^3}{3} = -\ln|x| + C, \quad \frac{y^3}{3x^3} = -\ln|x| + C.$$

Thus, renaming $D = 3C$, our general solution is

$$y = \sqrt[3]{Dx^3 - 3x^3 \ln|x|}.$$

Comments: We see we get the same answer either way. Also, note that there are issues with the domain of definition such as $x \neq 0$ in our substitutions, and choosing branches of absolute value.

Problem 3: General solution to $y' + 3x^2y = x^2$.

Solution. This is directly an integrating factor setup. Here $R(x) = e^{\int 3x^2 dx} = e^{x^3}$. Then, we know we multiply both sides by $R(x)$,

$$e^{x^3}y' + 3x^2e^{x^3}y = x^2e^{x^3}$$

where we know the integration factor makes the LHS a product rule,

$$\frac{d}{dx}(e^{x^3}y) = x^2e^{x^3} \quad \longrightarrow \quad e^{x^3}y = \int x^2e^{x^3}dx + C.$$

The integral we can actually calculate,

$$\int x^2e^{x^3}dx \overset{u=x^3}{=} \int e^u\frac{du}{3} = \frac{e^u}{3} = \frac{e^{x^3}}{3}.$$

Thus,

$$e^{x^3}y = \frac{e^{x^3}}{3} + C \iff y = Ce^{-x^3} + \frac{1}{3}.$$

Problem 4: Solve $4y'' + 4y' + 17y = 0$ satisfying $y(0) = -1,\ y'(0) = 2$.

Solution. First we look at the auxiliary (aka characteristic equation), from assuming $y = e^{rx}$,

$$4r^2 + 4r + 17 = 0.$$

We can find the roots in one of two ways:
1) Factor as $(2r+1)^2 + 16 = 0$, implying that $(2r+1) = \pm 4i \iff r = -\frac{1}{2} \pm 2i$.
2) Quadratic formula:

$$r = \frac{-4 \pm \sqrt{16 - 16 \cdot 17}}{8} = \frac{-4 \pm \sqrt{-16 \cdot 16}}{8} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i.$$

Either way, we get the correct roots for r. We also see these are complex, so we know the general solution must be of the form

$$y(x) = e^{-x/2}[A\cos(2x) + B\sin(2x)].$$

We solve for A, B with the initial conditions,

$$y(0) = -1: \quad -1 = e^0[A + 0] \iff A = -1.$$

For the $y'(0) = 2$ initial condition, we need y' first, (and plug in $A = -1$)

$$y'(x) = -\frac{1}{2}e^{-x/2}[-\cos(2x) + B\sin(2x)] + e^{-x/2}[+2\sin(2x) + 2B\cos(2x)]$$

$$\text{So } y'(0) = 2: \quad 2 = -\frac{1}{2}e^{0}[-1+0] + e^{0}[0+2B] \iff 2B = 2 - \frac{1}{2} = \frac{3}{2} \iff B = \frac{3}{4}.$$

$$\text{Thus, } \; y(x) = e^{-x/2}[-\cos(2x) + \frac{3}{4}\sin(2x)].$$