Math 3D Practice Quiz for Midterm

July 1st, 2016

Problem 1. (a) is 3pts, (b) is 7pts.

a) Recall from homework that a couple of solutions to the differential equation $[x'(t)]^2 + [x(t)]^2 = 4$ were $x_1(t) = 2\cos(t)$ and $x_2(t) = 2\sin(t)$. This equation is <u>not</u> linear (the x, x' terms are squared). Show that this equation does not satisfy the superposition principle. [Hint: Consider $x = x_1 + x_2$.]

Solution. Let us see if the sum of the solutions is a solution. Note that

$$x = 2(\cos t + \sin t), \quad x' = 2(-\sin t + \cos t).$$

Then,

$$\begin{aligned}(x')^2 + x^2 &= 4(\cos^2 t - 2\cos t \sin t + \sin^2 t) + 4(\cos^2 t + 2\cos t \sin t + \sin^2 t)\\ &= 8(\cos^2 t + \sin^2 t) = 8.\end{aligned}$$

But $8 \neq 4$ so we see the sum of these two solutions is NOT a solution - the superposition principle fails for this differential equation.

b) Suppose the population growth of mushrooms is given by $P' = (1-t)P$. If initially at time $t = 0$ we have 1000 mushrooms, find the number of mushrooms at $t = 2$. What happens as $t \to \infty$?

Solution. We can do this with either an integrating factor or by separation (separation is easier).

Separation: We have that $\frac{dP}{P} = (1-t)dt$ so integrating, $\ln|P| = t - \frac{t^2}{2} + C$. Population is never negative, and undoing the natural log,

$$P(t) = De^{t - \frac{t^2}{2}}, \quad \overset{P(0)=1000}{\Longrightarrow} \quad D = 1000, \quad P(t) = 1000e^{t-\frac{t^2}{2}}.$$

So, at $t = 2$, we have $1000e^{2-\frac{4}{2}} = 1000$ mushrooms still. But as $t \to \infty$, $P(t) \to 0$.

Integrating Factor: We see that $P' + (t-1)P = 0$ so $R(t) = e^{\frac{t^2}{2}-t}$ is the integrating factor. Multiplying this to both sides,

$$e^{\frac{t^2}{2}-t}P' + (t-1)e^{\frac{t^2}{2}-t}P = 0 \iff \frac{d}{dt}(e^{\frac{t^2}{2}-t}P) = 0.$$

Integrating, we have that

$$e^{\frac{t^2}{2}-t} \cdot P(t) = C \iff P(t) = Ce^{t-\frac{t^2}{2}}.$$

Then, continue from separation, we see $C = 1000$ and etc... (the rest is the same).

Problem 2. 10pts

a) Find the explicit solution to $xyy' + y^2 = -4x^2$ with $y(2) = -7$ and $x > 0$.
b) What is the interval of validity? Namely, the interval of validity is of the form $(0, x_F)$. I would like you to find x_F for the interval of validity.

Solution. a) This also has two substitutions that are possible.

Soln (i): Using $v = y^2$ this becomes $\frac{xv'}{2} + v = -4x^2 \implies v' + \frac{2v}{x} = -8x$. Solving this with an integration factor, $R(x) = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$, we have that

$$\frac{d}{dx}(x^2 v) = -8x^3, \quad x^2 v = -2x^4 + C \iff y^2 = -2x^2 + \frac{C}{x^2}, \quad y = \pm\sqrt{\frac{C}{x^2} - 2x^2}.$$

The condition $y(2) = -7$ tells us that $49 = -8 + \frac{C}{4}$, $C = 57 \cdot 4 = 228$. Also $y = -7$ so we actually choose the negative case of the square root. Hence

$$y(x) = -\sqrt{\frac{228}{x^2} - 2x^2}.$$

Soln (ii): If we divide by xy, this is a homogeneous equation $y' = \frac{-4x}{y} - \frac{y}{x}$. Using $v = y/x$, $y = xv$, we have

$$xv' + v = -\frac{4}{v} - v, \quad v' = \frac{-4v^{-1} - 2v}{x} = -\frac{\frac{4+2v^2}{v}}{x}$$

so separating and integrating, (we use a u-sub of $u = 2v^2 + 4$, $du = 4vdv$.)

$$\frac{vdv}{2v^2 + 4} = -\frac{dx}{x}, \implies \frac{1}{4}\ln|2v^2 + 4| = -\ln|x| + C.$$

First let's rearrange / rewrite it, (you could just leave C to take place of $4C$ if you wish),

$$\ln|2v^2 + 4| = -4\ln|x| + 4C \iff \ln|2v^2 + 4| + \ln|x^4| = 4C \iff \ln|2v^2x^4 + 4x^4| = 4C.$$

Now substitute back that $v = y/x$. We also take the positive absolute value (all the terms are to even powers)

$$\ln(2y^2x^2 + 4x^4) = 4C, \quad 2y^2x^2 + 4x^4 = D, \quad y^2 = \frac{D}{2x^2} - 2x^2.$$

Solving, we have $y = \pm\sqrt{\frac{D}{2x^2} - 2x^2}$ where the initial condition this time gives us that $49 = \frac{D}{8} - 8 \iff D = 8 \cdot 56$. But when we plug into the equation, and choose the negative square root from the intitial condition, we get the same answer,

$$y = -\sqrt{\frac{56 \cdot 8}{2x^2} - 2x^2} = -\sqrt{\frac{228}{x^2} - 2x^2}.$$

b) The issue of domain arises from the square root. We need that $\frac{228}{x^2} - 2x^2 \geq 0$ so equivalently we need $228 \geq 2x^4$. This gives $114 \geq x^4$, so $x \leq (114)^{1/4} \approx 3.26$. This means $x_F = 114^{1/4} \approx 3.26$. In other words, the interval of validity is $(0, (114)^{0.25})$.

Problem 3. 10pts

a) What is the general solution for $2y'' + y' + y = 0$?
b) Solve $6y'' + 3y' + 3y = 0$ with initial conditions $y(0) = 1,\ y'(0) = -2$.

Solution. a) First, we assume that $y = e^{rx}$ for the characteristic/auxiliary equation,

$$2r^2 + r + 1 = 0, \quad r = \frac{-1 \pm \sqrt{1-8}}{4} = \frac{-1 \pm i\sqrt{7}}{4}$$

where you would use the quadratic formula as above, or may notice that for

$$2r^2 + r + 1 = 2\left(r^2 + \frac{r}{2} + \frac{1}{2}\right) = 2\left(r + \frac{1}{4}\right)^2 + \frac{7}{8} = 0, \quad \left(r + \frac{1}{4}\right)^2 + \frac{7}{16} = 0, \quad r = -\frac{1}{4} + \frac{i\sqrt{7}}{4}.$$

Either way since our roots are $r = \frac{-1 \pm i\sqrt{7}}{4}$ are complex, the general solution is

$$y(x) = e^{-x/4}\left(A\cos\left(\frac{\sqrt{7}}{4}x\right) + B\sin\left(\frac{\sqrt{7}}{4}x\right)\right).$$

b) Notice that if we just divide by 3, we have $6y'' + 3y' + 3y = 0 \iff 2y'' + y' + y = 0$, so the general solution for this ODE is the one from part (a). We just have to solve for A, B from the initial conditions. First,

$$y(0) = 1: \quad 1 = e^0(A), \quad A = 1.$$

Second, we now need y', in which (plug in $A = 1$)

$$y'(x) = -\frac{1}{4}e^{-x/4}\left(\cos\left(\frac{\sqrt{7}}{4}x\right) + B\sin\left(\frac{\sqrt{7}}{4}x\right)\right) + e^{-x/4}\left(-\frac{\sqrt{7}}{4}\sin\left(\frac{\sqrt{7}}{4}x\right) + \frac{B\sqrt{7}}{4}\cos\left(\frac{\sqrt{7}}{4}x\right)\right)$$

where now,

$$y'(0) = -2: \quad -2 = -\frac{1}{4} + \frac{B\sqrt{7}}{4}, \quad B = \frac{-7}{4} \cdot \frac{4}{\sqrt{7}} = -\sqrt{7}.$$

Thus, with $A = 1, B = -\sqrt{7}$, we have

$$y(x) = e^{-x/4}\left(\cos\left(\frac{\sqrt{7}}{4}x\right) - \sqrt{7}\sin\left(\frac{\sqrt{7}}{4}x\right)\right)$$