Math 3D Quiz 1 Afternoon - April 13th

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Show all of your work. *There is a question on the back side.*

1. Consider the differential equation

$$x' = 3xt^2 - 3t^2, \quad x(0) = 2.$$

(a) [3pts] Prove that this has a unique solution. (Hint: Use Picard's Theorem).

Here, $f(t,x) = 3t^2(x-1)$, +1 $t_0 = 0$, $x_0 = 2$.

+1 (i) f is well defined everywhere and is a product of polynomials so it is continuous everywhere

+1 (ii) $\frac{\partial f}{\partial x} = 3t^2$ is, likewise, continuous everywhere

Thus, Picard's Thm is satisfied, There exists a unique solution

(b) [7pts] Find the explicit solution to the differential equation.

↳ Turns out it is separable or can use int. factor.

Separable: $\frac{dx}{dt} = 3t^2(x-1)$,

$\frac{dx}{x-1} = 3t^2\, dt$,

$\ln|x-1| = t^3 + c$. +3

↳ I.C. ⇒ $\ln(1) = 0 + c$,

$\boxed{c = 0}$ +1

And $|x-1| = |1|$ initially, keep (+) Branch, +1

$\ln(x-1) = t^3$,

+1 $x - 1 = e^{t^3}$;

$\boxed{x = e^{t^3} + 1}$ → This is defined for all t,

$\boxed{\text{Domain: } (-\infty, \infty)}$ +1

$\longleftrightarrow$ $\boxed{\text{SAME}}$

Integrating Factor: First, rewrite,

$x' - 3t^2 x = -3t^2$, $P(t) = -3t^2$

$R(t) = e^{-t^3}$ then. +2

Multiply to both sides,

$\frac{d}{dt}\left[e^{-t^3} \cdot x\right] = -3t^2 e^{-t^3}$ +2

Integrate, $e^{-t^3} \cdot x = c + e^{-t^3}$.

Solve for C with I.C.: $2 = C + 1$; $\boxed{C = 1}$

↳ $\boxed{x = e^{t^3}\left[1 + e^{-t^3}\right]}$ +2

$\boxed{\text{And } t \in (-\infty, \infty)}$ (domain) +1

2. (a) [8pts] Using an integrating factor, solve

$$y' + 3x^2 y = e^{-x^3}\sin(x), \quad y(0) = 1.$$

1^{st} way, general solution + solve for C :

$R(x) = e^{\int 3x^2\,dx} = e^{x^3}.$ +2

Multiply Both sides,

$\hookrightarrow \frac{d}{dx}\left[e^{x^3} \cdot y \right] = \sin x$ +1

Integrate, $\underline{e^{x^3} \cdot y = C - \cos x}.$ +2

Solve for C using I.C. $y(0) = 1$,

$e^0 \cdot 1 = C - \cos(0)$, $1 = C - 1$,

+2 $\boxed{C = 2}$, and isolate y,

So, $\boxed{y = e^{-x^3}\left[2 - \cos x\right]}$

+1

2^{nd} way w/ Formula :

$R_{sp}(x) = e^{\int_0^x 3\hat{x}^2\,d\hat{x}} = e^{x^3}$ +2 still

Then, with $R_{sp}(x) = e^{x^3}$,

+2 $y = \frac{1}{R_{sp}(x)}\left[y_0 + \int_{x_0}^{x} R_{sp} \cdot f \, d\hat{x} \right]$

$= \frac{1}{e^{x^3}}\left[1 + \int_0^x \cancel{e^{\hat{x}^3}} \cdot \cancel{e^{-\hat{x}^3}} \sin\hat{x}\,d\hat{x} \right]$

+2 $= e^{-x^3}\left[1 + (-\cos\hat{x})\Big|_{\hat{x}=0}^{x} \right]$

$= e^{-x^3}\left[1 - (\cos x - 1) \right]$

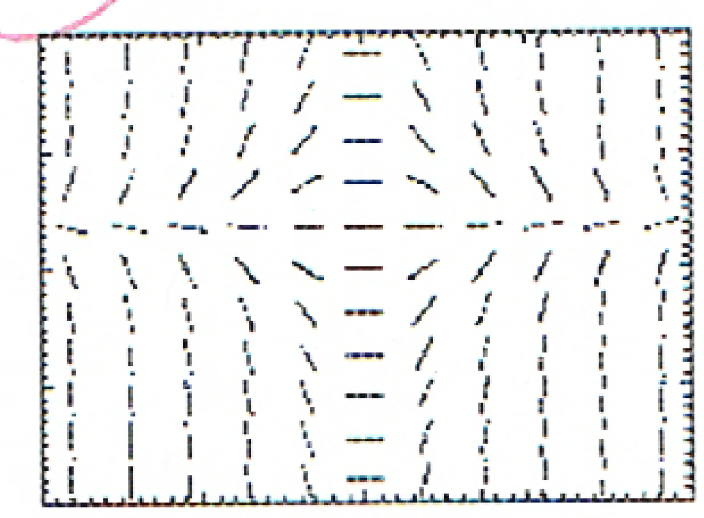
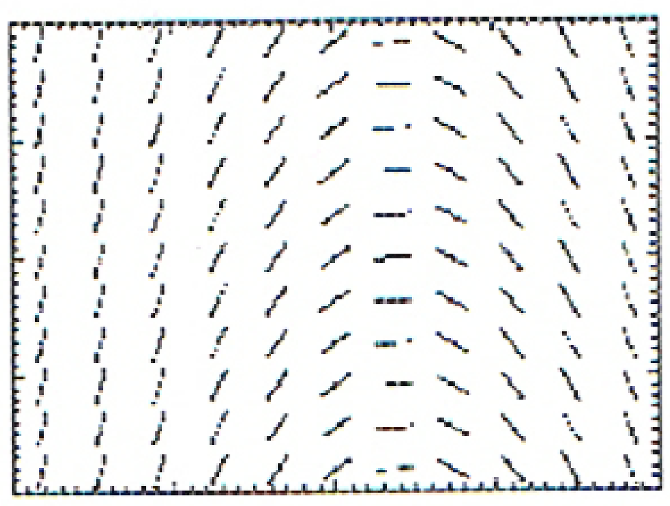
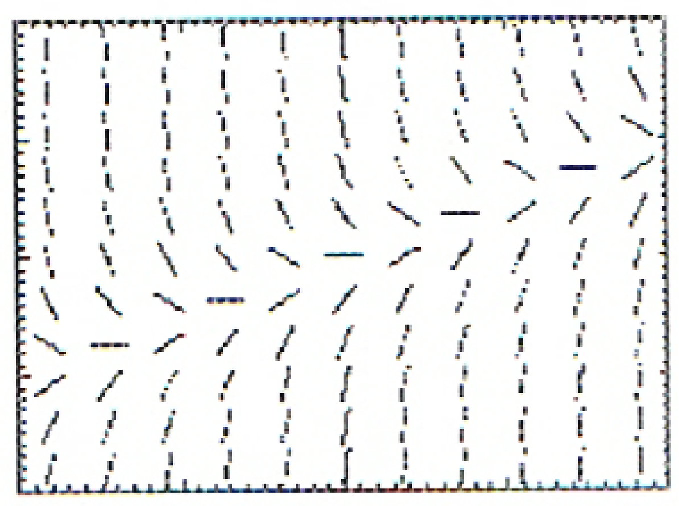
+2 So, $\boxed{y = e^{-x^3}(2 - \cos x)}$

$\Longleftrightarrow$

$\boxed{\text{same}}$

✓

(b) [2pts] Match the equations to their slope fields: (Fill in the blank with the letter)

a)  b)  c) 

$[y' = 1 - x]$: b

↑ slope = 0 on $x = 1$

$[y' = x - 2y]$: c

↑ slope = 0 on $x = 2y$, $y = x/2$

$[y' = x(1 - y)]$: a

↑ slope = 0 on $x = 0$ and $y = 1$