Math 3D Quiz 1 Afternoon - January 24th

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Show all of your work. *There is a question on the back side.*

1. Consider the following ODE with initial condition: $x' = \frac{e^{-t^2}}{x}$ and $x(0) = -2$.

(a) [3pts] Prove that this has a unique solution. (Hint: Section 1.2 material).

Proof. We need to check Picard's Theroem. Here,

$$f(t,x) = \frac{e^{-t^2}}{x}, \quad f_x(t,x) = -\frac{e^{-t^2}}{x^2}, \quad and \quad (t_0, x_0) = (0, 2).$$

• f is continuous as long as x_0 is away from 0. It is also fine for any t. Since $x_0 = 2$, this f is continuous near the initial condition.

• f_x exists and is also continuous as long as we are away from $x_0 = 0$ so similarly, f_x exists and is continuous near the initial condition.

Since the assumptions of Picard's Theorem are satisfied, we conclude there exists a solution, and that it is unique by Picard's Theorem. □

(b) [7pts] Find the exact solution for x. Don't worry about the domain of definition for this problem. You may leave a definite integral in the answer.

Solution. Separating, and using Leibniz notation, we have that

$$x \, dx = e^{-t^2} \, dt \implies \int x dx = \int e^{-t^2} dt.$$

This gives us

$$\frac{x^2}{2} = C + \int e^{-t^2} dt.$$

We cannot find an antiderivative to e^{-t^2} so we have to make it a definite integral,

$$x^2 = \hat{C} + 2\int_0^t e^{-s^2} ds.$$

The initial condition implies

$$(-2)^2 = \hat{C} + 2\int_0^0 e^{-s^2} ds \iff 4 = \hat{C} + 0 \iff \hat{C} = 4.$$

Therefore we have

$$x^2 = 4 + 2\int_0^t e^{-s^2} ds.$$

Solving for x, we read

$$x = \pm\sqrt{4 + 2\int_0^t e^{-s^2} ds}$$

where since $x(0) = -2$, we take the negative branch,

$$x = -\sqrt{4 + 2\int_0^t e^{-s^2} ds}.$$

2. (a) [7pts] Find the general solution to the differential equation $xy' + y = x\sin(x^2)$.

Solution. First, this is an integrating factor problem. Dividing out by x first (we then note $x \neq 0$),

$$y' + \frac{1}{x}y = \sin(x^2).$$

The integrating factor is thus

$$R(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|.$$

Note: It does not matter which branch of absolute value we take for $R(x)$, because we multiply this to both sides! For instance, when we multiply to both sides, it is either

$$x(y' + y) = x\sin(x^2)$$

or

$$-x(y' + y) = -x\sin(x^2)$$

which are the same thing. Now, knowing that the integrating factor induces a perfect derivative from product rule, (or, you may have seen it immediately from starting the problem), we have that

$$\frac{d}{dx}[xy] = x\sin(x^2).$$

Integrating both sides,

$$xy = \int x\sin(x^2)dx$$

where the right side is computed with a u-sub of $u = x^2, du = 2xdx$:

$$\int x\sin(x^2)dx = \int \frac{\sin u}{2}du = -\frac{1}{2}\cos(u) + C = -\frac{\cos(x^2)}{2} + C.$$

This means that

$$y = \frac{C}{x} - \frac{\cos(x^2)}{2x}$$

is our general solution.

(b) [3pts] Find the exact solution if we are given the initial condition $y(\sqrt{\pi/2}) = 0$.
Remember to include the domain of validity. (You should always do this without being asked).

Solution. From the initial condition, we have

$$0 = \frac{C}{\sqrt{\pi/2}} - \frac{\cos(\pi/2)}{2\sqrt{\pi/2}} \iff C = 0$$

Also, since we had from part (a) that $x \neq 0$, this implies our domain is either $(0, \infty)$ or $(-\infty, 0)$. Since the initial condition is on the positive side, we take $(0, \infty)$ which is the same as saying $x > 0$. Thus, our exact solution is

$$y = -\frac{\cos(x^2)}{2x}, \quad x \in (0, \infty).$$