Math 3D Quiz 1 Morning - January 24th

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Show all of your work. *There is a question on the back side.*

1. Consider the following ODE with initial condition: $x^2y' = y - xy$ and $y(-1) = 1$.

(a) [3pts] Prove that this ODE has a unique solution. (Hint: Section 1.2 material).

Proof. First, rewrite it as $y' = \frac{1-x}{x^2}y$.
We need to check Picard's Theroem. Here,

$$f(x,y) = \frac{(1-x)}{x^2}y, \quad f_y(x,y) = \frac{1-x}{x^2}, \quad and \quad (x_0, y_0) = (-1, 1).$$

• f is continuous as long as x is away from 0. It is fine for any y value. Since $x_0 = -1$, f is continuous near the initial condition so this part of Picard's Theorem is satisfied.
• f_x exists and is also continuous as long as we are away from $x = 0$ so similarly, f_x exists and is continuous near the initial condition.
Since the assumptions of Picard's Theorem are satisfied, we conclude there exists a solution, and that it is unique by Picard's Theorem.

□

(b) [7pts] Find the exact solution for y. Reminder: Address what the domain of validity is.

Solution. From (a) how we wrote $y' = \frac{1-x}{x^2}y$, we separate and integrate. With Leibniz notation,

$$\frac{dy}{y} = \left(\frac{1}{x^2} - \frac{1}{x}\right)dx \implies \ln|y| = -\frac{1}{x} - \ln|x| + C. \quad Note: \; x \neq 0, \; y \neq 0.$$

We can actually solve for C now, to obtain

$$\ln|1| = 1 - \ln|-1| + C \iff C = -1.$$

Also from the initial condition, we choose the positive branch of absolute value for $|y|$ but negative branch for $|x|$, in that

$$\ln(y) = -\frac{1}{x} - \ln(-x) - 1 \overset{exp}{\implies} y = e^{-\frac{1}{x} - \ln(-x) - 1}.$$

Writing this a little cleaner, and including the domain of validity, we have that

$$y = -\frac{1}{x}e^{-1-\frac{1}{x}}, \quad x \in (-\infty, 0) \qquad (or \; x < 0).$$

Why $x < 0$: We note that we can't have $x = 0$ so our domain is either $(-\infty, 0)$ or $(0, \infty)$. Since our initial condition $x = -1$ lies in $(-\infty, 0)$, our domain of validity is $(-\infty, 0)$ which is the same as saying $x < 0$.

2. (a) [9pts] A twist on Newton's law of cooling. Find the general solution to

$$\frac{dT}{dt} = A_0 \cos(t) - T.$$

Conceptually, this is like if the ambient temperature is oscillating, say, between night and day.

Solution. First, this cannot be separated so we have to use an integration factor. Rewriting,

$$\frac{dT}{dt} + T = A_0 \cos(t) \quad \Longrightarrow \quad R(t) = e^{\int 1 dt} = e^t.$$

Using this, we multiply $R(x)$ to both sides and recall / see it induces a product rule on the left side,

$$e^t T' + e^t T = A_0 e^t \cos(t) \iff \frac{d}{dt}[e^t \cdot T] = A_0 e^t \cos(t).$$

We integrate both sides now with respect to dt, and we see we have a hard integral to compute:

$$e^t \cdot T = A_0 \int e^t \cos(t) dt.$$

This requires two integration by parts. Set $u = \cos t$ and $dv = e^t$ so $du = -\sin t dt$ and $v = e^t$,

$$\int e^t \cos(t) dt = C + e^t \cos t - \int -e^t \sin t dt$$

where we integrate by parts again, $u = \sin t$, $dv = e^t$ so $du = \cos t dt$ and $v = e^t$,

$$\int e^t \cos(t) dt = C + e^t \cos t + e^t \sin t - \int e^t \cos t dt$$

so we see that

$$2 \int e^t \cos t dt = C + e^t (\cos t + \sin t) \iff \int e^t \cos t dt = \frac{e^t}{2} (\cos t + \sin t) + \hat{C}.$$

Therefore,

$$e^t \cdot T = \frac{A_0}{2} e^t (\cos t + \sin t) + \hat{C}$$

so our general solution is

$$T(t) = \frac{A_0}{2} (\cos t + \sin t) + \hat{C} e^{-t}.$$

For good measure, you may check that T indeed satisfies the differential equation :)

(b) [1pt] If we imposed an initial condition, we would determine the constant in the general solution in (a) and find an exact solution. But, in the long term, as $t \to \infty$, will the initial condition make much of a difference? Explain why or why not.

Solution. It will not make a difference in the long term. The initial condition will ultimately determine what the constant $\hat{C}$ is. But, $\hat{C} e^{-t}$ is going to go to zero as $t \to \infty$, that is $\lim_{t \to \infty} \hat{C} e^{-t} \to 0$ so it doesn't do anything in the long term.