Solutions

Math 3D Quiz 2 Afternoon - February 2nd

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Show all of your work. *There is a question on the back side.*

1. (a) [8pts] Solve $xy' + y + x = 0$ where $y(1) = 1$.

Homogeneous: Degree of all terms is 1 (one).

$\hookrightarrow \quad xy' = -y - x \; ; \quad y' = -\frac{y}{x} - 1$.

Let $v = \frac{y}{x}$, $\quad y' = xv' + v$.

+3

$\Rightarrow \quad xv' + v = -v - 1 \; ; \quad \boxed{v' = \frac{-2v-1}{x}, \; v(1) = 1}$

Separate: $\frac{dv}{2v+1} = -\frac{dx}{x}$ and integrate,

$\Rightarrow \quad \frac{1}{2}\ln|2v+1| = -\ln|x| + c,$

$\ln|2v+1| = -2\ln|x| + \hat{c}$ +2

$= \ln|x^{-2}| + \hat{c}$;

$\overset{\exp}{\Rightarrow} \quad |2v+1| = \frac{D}{x^2}$. $\overset{\text{IoCo}}{\Rightarrow} \quad 3 = \frac{D}{1}, \; \boxed{D = 3}$ +1

IoCo $\Rightarrow$ for branch $\quad 2v+1 = \frac{3}{x^2}$; $\overset{v = y/x}{\Rightarrow} \quad \frac{2y}{x} + 1 = \frac{3}{x^2}$ +1

so, $\boxed{y = \frac{1}{2}\left(\frac{3}{x} - x\right)}$ +1, and need $\boxed{x > 0}$

Check w/ int. factor:

$y' + \frac{y}{x} = -1$;

$R(x) = e^{\ln x} = x$,

$\frac{d}{dx}(xy) = -x$;

$xy = -\frac{x^2}{2} + c$;

$y = -\frac{x}{2} + \frac{c}{x}$,

$y(1) = 1 \Rightarrow 1 = -\frac{1}{2} + c$,

so $\boxed{c = 3/2}$,

$y = \frac{3}{2x} - \frac{x}{2}$

$= \frac{1}{2}\left(\frac{3}{x} - x\right)$ ✓

(b) [2pts] Prove that the functions $\{\cos(3t), \; \sin(3t)\}$ are linearly independent.

Short Way: $\frac{\cos 3t}{\sin 3t} = \cot 3t \neq$ constant $\Rightarrow$ L.I ✓

Formal Way: For $C_1 \cos 3t + C_2 \sin 3t = 0$ for all t,

Try $t = 0 \Rightarrow \boxed{C_1 = 0}$

$3t = \frac{\pi}{2}, \; t = \frac{\pi}{6} \Rightarrow C_2 \sin\frac{\pi}{2} = \boxed{C_2 = 0}$

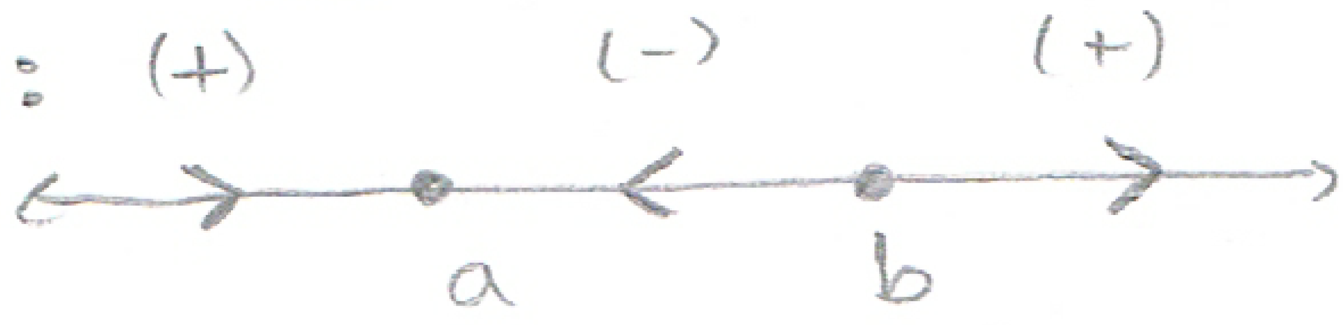
$\Rightarrow$ Needs $\boxed{C_1 = C_2 = 0}$ for 0-eqn for all t

$\Rightarrow$ L.I ✓

$\boxed{f(y)}$

2. [5pts] Let $y' = (y-a)(y-b)$ where a, b are real numbers (constants) and $a < b$.
(a) Find and classify all critical points. Sketch the phase diagram, too.
(b) i. If $y(0) = b + 0.001$ and if ii. $y(0) = \frac{1}{2}(a+b)$, what is $\lim_{t\to\infty} y(t)$ for each case, i. and ii.?

a) $y^* = a, b$ are crit pts.



so, $y = a$ stable, $y = b$ unstable

+1 +2

b) • If $y(0) = b + 0.001 \Rightarrow$ Goes to $+\infty$ +1

• If $y(0) = \frac{a+b}{2} \Rightarrow$ Is between a and $b \Rightarrow$ Goes to $y = a$.
(ie. $2a \leq a+b \leq 2b$) +1

3. [5pts] Consider the equation $y'' + 4y' + 13y = 0$. (Here, y is a function of t).
(a) Find the general solution to the equation.
(b) Find the exact solution if $y(0) = 1,\ y'(0) = -2$.

a) Aux. Eqn $\Rightarrow r^2 + 4r + 13 = 0$, $r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$ ($\sqrt{-36} = 6i$)

+1

$\boxed{r = -2 \pm 3i}$ +1

so, $\boxed{y = e^{-2x}(c_1 \sin 3x + c_2 \cos 3x)}$ +1

is the general solution

b) Also $y' = -2e^{-2x}(c_1 \sin 3x + c_2 \cos 3x) + e^{-2x}(3c_1 \cos 3x - 3c_2 \sin 3x)$

$y(0) = 1$: $\boxed{1 = c_2}$ +1

+1

$y'(0) = -2$: $-2 = -2c_2 + 3c_1$; Since $c_2 = 1$, $3c_1 = 0$, $\boxed{c_1 = 0}$

so, $\boxed{y(x) = e^{-2x}\cos(3x)}$

• It's good to check IoC:

• $y(0) = e^0 \cos 0 = 1$ ✓

• $y'(0) = -2e^0 \cos 0 - 3e^0 \sin 0 = -2$ ✓