Solutions

Math 3D Quiz 3 Afternoon - February 23rd

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Show all of your work. *There is a question on the back side.*

1. [10pts] Find the general solution to $\vec{y}\,' = \begin{bmatrix} 0 & 4 & -2 \\ -1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \vec{y}$ using the eigenvalue method.

① Find Eigenvalues. $\det \begin{bmatrix} -\lambda & 4 & -2 \\ -1 & -4-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (-2-\lambda)(-1)^{3+3} \cdot \det \begin{bmatrix} -\lambda & 4 \\ -1 & -4-\lambda \end{bmatrix}$

2 pts

$= (-2-\lambda)(\lambda^2+4\lambda+4) = -(\lambda+2)^3$ $\boxed{\lambda = -2 \text{ (algebraic multiplicity 3)}}$

②a Eigenvectors: $(A+2I)\vec{x} = \vec{0}$

$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $-x_1 - 2x_2 + x_3 = 0$

So, we can write $x_1 = -2x_2 + x_3$, $x_2 = \text{free}$, $x_3 = \text{free}$ $\rightsquigarrow$ $\boxed{\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

4 pts

- 2 e-vectors
- Defective Once.

②b Generalized Eigenvector: Note: If we tried $(A+2I)\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, no solution!

2 pts

↳ $\left[\begin{array}{ccc|c} 2 & 4 & -2 & 1 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ Has $0=1$ in 3rd row (X)

So solve for $(A+2I)\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ ↗ $\left[\begin{array}{ccc|c} 2 & 4 & -2 & -2 \\ -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $-x_1 - 2x_2 + x_3 = 1$. (similar to before)

So, $x_1 = -1 - 2x_2 + x_3$, $x_2 = \text{free}$, $x_3 = \text{free}$ $\rightsquigarrow$ $\boxed{\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

(set x_2, x_3 to 0 because free)

↳ Have 3 vectors for algebraic multiplicity 3 ✓

③ General soln: $\boxed{\vec{x}_{general} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-2t} \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)}$

2 pts

or, using Fundamental Matrix, $\vec{x}_{general} = \begin{bmatrix} e^{-2t} & -2e^{-2t} & e^{-2t}(-2t-1) \\ 0 & e^{-2t} & te^{-2t} \\ e^{-2t} & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Afternoon #1 Eigenvector Remark:

Following from the solutions, $\lambda = -2$ (mult. 3).

When getting eigenvectors: Solving $(P-(-2)I)\vec{x} = \vec{0}$ had

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

Some people had $x_3 = x_1 + 2x_2$ so x_1, x_2 free.

$$\Rightarrow \vec{x} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v_1}} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{v_2}}$$

which is still correct!

↳ Just a different basis of e-vects

⚠ But, neither of these gets the ~~3rd~~ generalized eigenvector!

The Best Fix: 1st we can't get the 3rd eigenvector due to the last row

↳ Need new 2nd eigenvector, say $\vec{v_2}_{new} = \vec{v_2} - 2\vec{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

because $\vec{v_2}$ new has consistent last row for solvability.

↳ Can solve

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -2 \\ -1 & -2 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} -x_1 - 2x_2 + x_3 = 1, \\ x_1 = \text{free} \\ x_2 = \text{free} \\ x_3 = 1 + x_1 + 2x_2 \end{array}$$

$$(P+2I)\vec{v_3} = \vec{v_2}$$

$$\Rightarrow \vec{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v_3}} + x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

So that $\vec{X}_{general} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right.$

(Used $\vec{v}_{2,new}$)

Solutions Cont'd

2. (a) [3pts] Find the general solution to $mx'' + cx' + kx = 0$ where $m = 1kg$, $c = 7kg/s$ and $k = 12N/m$. Is the physical system underdamped, critically damped, or overdamped?

↳ $x'' + 7x' + 12x = 0$, $r^2 + 7r + 12 = 0$, $r = \frac{-7 \pm \sqrt{49-48}}{2} = \frac{-7 \pm 1}{2}$

$(r+4)(r+3) = 0$

+1 $\boxed{r = -3, -4}$, and $\boxed{x_{general} = c_1 e^{-3t} + c_2 e^{-4t}}$ +1

↳ $\boxed{\text{Overdamped}}$ +1

(b) [7pts] Reduce the order and write the equivalent 1st order matrix-vector equation. Then, find its general solution by using the eigenvalue method.

+2 ↳ $x'' = -7x' - 12x$ so $\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$ is our 1st order system.

① Eigenvalues: $\det \begin{bmatrix} -\lambda & 1 \\ -12 & -7-\lambda \end{bmatrix} = \lambda^2 + 7\lambda + 12$, $\boxed{\lambda = -3, -4}$ from (a)

+1

② Eigenvectors: $\lambda = -3$: $\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -12 & -4 & 0 \end{array}\right] \xrightarrow{(R_2 = -4R_1)} 3x_1 + x_2 = 0$; $x_1 = \text{free}$, $x_2 = -3x_1$,

↳ $(A - \lambda I)\vec{x} = \vec{0}$.

+3

Gets $\boxed{\vec{x} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}}$

$\lambda = -4$: $\left[\begin{array}{cc|c} 4 & 1 & 0 \\ -12 & -3 & 0 \end{array}\right] \xrightarrow{(R_2 = -3R_1)} 4x_1 + x_2 = 0$; $x_1 = \text{free}$, $x_2 = -4x_1$

Gets $\boxed{\vec{x} = x_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix}}$

+1

③ General Solution: $\boxed{\vec{x}_{general} = \begin{bmatrix} x \\ x' \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}}$

Note: Row 1 gets the general solution from (a) ✓