Solutions

Math 3D Quiz 3 Morning - February 23rd

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Show all of your work. *There is a question on the back side.*

1. [10pts] Find the general solution to $\vec{y}\,' = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \vec{y}$ using the eigenvalue method.

Step 1: Find Eigenvalues. May note, upper-triangular so $\boxed{\lambda = 3, \text{ (algebraic) multiplicity } 3.}$

2pts

Check: $\det \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)\cdot(-1)^{3+3}\cdot \det\begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^3, \ \lambda = 3$ ✓

Step 2a: Eigenvectors: $(A-3I)\vec{x} = \vec{0}$

2pts

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \text{free} \\ x_2 = 0 \\ x_3 = 0 \end{array}, \quad \boxed{\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

- Only 1 e-vector,
- Defective twice.

Step 2b: Generalized eigenvectors. 1st one: $(A-3I)\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \text{free} \\ x_2 = 1 \\ x_3 = 0 \end{array}, \quad \boxed{\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

4pts

Need 2nd (twice defected): $(A-3I)\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \text{free} \\ x_2 = 0 \\ x_3 = 1 \end{array}, \quad \boxed{\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Set $x_1 = 0$ since free.

↳ Have 3 vectors for algebraic multiplicity 3 ✓

2pts

Step 3: General Solution. Because we have 2 further generalized eigenvectors,

$$\boxed{\vec{x}_{\text{general}} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{3t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)}$$

Using the Fundamental Matrix solution form, can also write

$$\boxed{\vec{x}_{\text{general}} = \begin{bmatrix} e^{3t} & te^{3t} & \frac{t^2 e^{3t}}{2} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}$$

Solns Contd

2. (a) [4pts] Find the general solution to $mx'' + cx' + kx = 0$ where $m = 1kg$, $c = 6kg/s$ and $k = 13N/m$. Is the physical system underdamped, critically damped, or overdamped?

$\hookrightarrow \quad x'' + 6x' + 13x = 0 \; ; \quad r^2 + 6r + 13 = 0, \quad r = \frac{-6 \pm \sqrt{36-52}}{2} \quad \sqrt{-16}$

So $\boxed{r = -3 \pm 2i}$ +1 , $\boxed{x_{general} = e^{-3t}\left(c_1 \cos 2t + c_2 \sin 2t\right)}$ +2

$\hookrightarrow$ $\boxed{\text{underdamped.}}$ +1

(b) [6pts] Reduce the order and write the equivalent 1st order matrix-vector equation. Then, find its general solution by using the eigenvalue method.

+2 write as $x'' = -6x' - 13x \rightsquigarrow \begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -13 & -6 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$, is our reduced order system.

1. Eigenvalues: +1 Let $\begin{bmatrix} -\lambda & 1 \\ -13 & -6-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 13 = 0 \rightsquigarrow \lambda = -3 \pm 2i$ like in (a)

2. Complex Eigenvector: +1 $(A - \lambda I)\vec{x} = \vec{0}$

Pick $\boxed{\lambda = -3+2i}$

$\left[\begin{array}{cc|c} +3-2i & 1 & 0 \\ -13 & -3-2i & 0 \end{array}\right]$, use 1st row: $(3-2i)x_1 = x_2$, $\begin{cases} x_1 = \text{free} \\ x_2 = (-3+2i)x_1 \end{cases}$

$\vec{x} = x_1 \begin{bmatrix} 1 \\ -3+2i \end{bmatrix}$

3. Split soln. into Real/IM: Solution is of the form $e^{(-3+2i)t} \cdot \begin{bmatrix} 1 \\ -3+2i \end{bmatrix}$

Split terms, $e^{-3t}\left(\cos 2t + i \sin 2t\right) \cdot \left(\begin{bmatrix} 1 \\ -3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$

+1 Real: $e^{-3t}\left(\cos 2t \begin{bmatrix} 1 \\ -3 \end{bmatrix} + i^2 \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$ +1 IM: $i e^{-3t}\left(\cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$

4. General Soln: $\boxed{\vec{x} = \begin{bmatrix} x \\ x' \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} \cos 2t \\ -3\cos 2t - 2\sin 2t \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} \sin 2t \\ 2\cos 2t - 3\sin 2t \end{bmatrix}}$

$\boxed{\text{Note:}}$ Row 1 is the general soln from (a) ✓