Solns

Math 3D Last Quiz Afternoon - March 14th

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(10 pts)

Show all of your work. Happy π-Day! *There is a question on the back side.*

1. Find the Laplace Transform $\mathcal{L}(f(t))$ where $f(t) = \begin{cases} t & t \geq 1. \\ 0 & t < 1. \end{cases}$

- 1st note $f(t) = t \cdot u(t-1)$ where u is the Heaviside fn
- Now there's two approaches: Both work,

1) Use the Definition,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Since f is zero if $t<1$

$$= \int_1^\infty e^{-st} \cdot t\, dt \qquad (+3)$$

$u = t$ | $dv = e^{-st}$

$du = dt$ | $v = \frac{e^{-st}}{-s}$

By Parts

$$= -\frac{te^{-st}}{s}\Big|_{1=t}^{\infty=t} + \int_1^\infty \frac{e^{-st}}{-s}\, dt \qquad (+4)$$

$$= +\frac{e^{-s}}{s} + \frac{e^{-st}}{s^2}\Big|_{t=1}^{\infty}$$

$$= \boxed{\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}} \qquad (+3)$$

✓

2) Use the table

↳ Needs Rewriting!

↳ Since $u(t-1)$ shifted,

need to rewrite t as

$$t = (t-1) + 1(t-1)^0 \qquad (+3)$$

So that

$$f(t) = (t-1)\,u(t-1) + u(t-1)$$

and thus,

$$\mathcal{L}(f) = F(s) \qquad (+3)$$

Heaviside's property

$$= e^{-s} \cdot \mathcal{L}(t) + \mathcal{L}(u(t-1))$$

$$= \boxed{e^{-s} \cdot \frac{1}{s^2} + \frac{e^{-s}}{s}} \qquad (+4)$$

✓

Solus cont'd

// A

(10 pts)

2. (a) Use Matrix Exponential to find the general solution to $\vec{x}\,' = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}\vec{x}$.

(Know that $\vec{X}_{gen} = e^{tA}\vec{c}$)

(b) Find the exact solution if $\vec{x}(2) = e^2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\boxed{\lambda = 1, 5}$ +1

(a) E-vectors: $\lambda = 1$: $(A - I)\vec{v} = \vec{0}$ is $\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 4 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} v_1 = \text{free} \\ v_2 = 0 \end{array}$, $\boxed{\vec{v} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\lambda = 5$: $(A - 5I)\vec{v} = 0$ is $\left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} v_1 = \frac{3}{4} v_2 \\ v_2 = \text{free} \end{array}$, $\vec{v} = v_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ +1

or, $\boxed{\vec{v} = \tilde{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$ +1

+1 D, E

So $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$ is

one way to diagonalize A, because no defects.

So, $e^{tA} = E e^{tD} E^{-1}$, and $E^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$ +1 E^{-1}

So $e^{tA} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{5t} \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}\frac{1}{4}$

$= \frac{1}{4}\begin{bmatrix} e^t & 3e^{5t} \\ 0 & 4e^{5t} \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4e^t & 3e^{5t} - 3e^t \\ 0 & 4e^{5t} \end{bmatrix} \cdot \frac{1}{4}$

Thus, $\boxed{\vec{X}_{gen} = \frac{1}{4}\begin{bmatrix} 4e^t & 3e^{5t} - 3e^t \\ 0 & 4e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$ +2

(b) Know that $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e^{-2A}\vec{x}(2)$ +1 then, so

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4e^{-2} & 3e^{-10} - 3e^{-2} \\ 0 & 4e^{-10} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^2 = \frac{1}{4}\begin{bmatrix} 4e^{-2} \\ 0 \end{bmatrix} e^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ +1

So $\boxed{\vec{X}_{exact} = \frac{1}{4}\begin{bmatrix} 4e^t & 3e^{5t} - 3e^t \\ 0 & 4e^{5t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ +1