Solns

Math 3D Last Quiz Morning - March 14th

Please put name on front & ID on back for redistribution!

(10pts) Show all of your work. Happy π-Day! *There is a question on the back side.*

1. Find the Inverse Laplace Transform of

$$F(s) = \frac{s}{(s+4)(s^2+s+2)}$$

Has Roots $s = \frac{-1 \pm \sqrt{1-8}}{2}$ (complex) (+1)

With Partial Fractions: For $\frac{A}{s+4} + \frac{Bs+C}{s^2+s+2} = \frac{s}{(s+4)(s^2+s+2)}$,

$\Longleftrightarrow$ $s = A(s^2+s+2) + (Bs+C)(s+4)$ after common denominator.

Eqns: (coefficients)

s^2: $A + B = 0 \Rightarrow B = -A$ so $A+4B+C=1$ becomes

s: $A + 4B + C = 1$ $\quad -\frac{7}{2}A = 1$; $\boxed{A = -\frac{2}{7},\ B = \frac{2}{7},\ C = \frac{1}{7}}$ (+3)

1: $2A + 4C = 0 \Rightarrow C = -\frac{A}{2}$

Alternate way to get A, B, C.

(or, as a system, $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, solve $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 4 & 1 & 1 \\ 2 & 0 & 4 & 0 \end{array}\right]$ to get ↑.)

So, $F(s) = -\frac{2}{7}\cdot\frac{1}{s+4} + \frac{1}{7}\cdot\frac{2s+1}{s^2+s+2}$ ☆ Need to Rewrite Denominator (+1)

$= -\frac{2}{7}\cdot\frac{1}{s+4} + \frac{1}{7}\cdot\frac{2s+1}{(s+\frac{1}{2})^2+\frac{7}{4}}$ ☆ Now need to rewrite numerator. (+1)

$= -\frac{2}{7}\cdot\frac{1}{s+4} + \frac{2}{7}\cdot\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+7/4}$

Now invert,

$$\boxed{f(t) = \mathcal{L}^{-1}(F) = -\frac{2}{7}\cdot e^{-4t} + \frac{2}{7}e^{-\frac{t}{2}}\cos\left(\sqrt{\tfrac{7}{4}}\,t\right)}$$

(+2) (each term) (+2)

Soln contd

$$\left(\vec{X}_{gen} = e^{tA} \vec{c} \right)$$

(10pts) **2.** (a) Using Matrix Exponential, find the general solution to $\vec{x}\,' = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \vec{x}$. // A

(b) Find the exact solution if $\vec{x}(1) = e^5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = 5$, mult. 4. +1

(a)

Eigenvectors: $(A - 5I)\vec{v} = \vec{0}$,

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} v_1 = \text{free} \\ v_2 = 0 \\ v_3 = 0 \\ v_4 = \text{free} \end{array} \Rightarrow \boxed{\vec{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$ +1

~ Twice Defective ~

Defects $\Rightarrow$ Split $A = \mu I + N$ where $\mu = \lambda = 5$, $e^{tA} = e^{t\mu I} \cdot e^{tN}$.

So, $N = A - 5I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Squaring this,

$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $N^3 =$ zero matrix finally

Thus $e^{tN} = I + tN + \frac{t^2}{2!} N^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ +2 for e^{tN}

and $e^{t \cdot 5I} = \begin{bmatrix} e^{5t} & 0 & 0 & 0 \\ 0 & e^{5t} & 0 & 0 \\ 0 & 0 & e^{5t} & 0 \\ 0 & 0 & 0 & e^{5t} \end{bmatrix} = e^{5t} \cdot I$, (+1 for $e^{t\mu I}$) so

$e^{tA} = e^{t \cdot 5I} \cdot e^{tN} = e^{5t} \cdot e^{tN} \Rightarrow \boxed{\vec{X}_{gen} = e^{5t} \begin{bmatrix} 1 & t & t^2/2! & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}}$ +2

(b) Know $\vec{c} = e^{-1A} \vec{x}(1)$ then. (+1) defn

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \cancel{e^{-5}} \begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cancel{e^{5}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ +1 $\leadsto$ $\boxed{\vec{X}_{exact} = e^{5t} \begin{bmatrix} 1 & t & t^2/2! & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$ +1