Solutions

Math 3D Quiz 7 Morning - May 25th

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Show all of your work. *There is a question on the back side.*

1. (a) [2pts] Fill in the blank. What are the following shifting properties? (Use $F(s)$ to denote the Laplace Transform of $f(t)$.)

$$\mathcal{L}\{e^{-at}f(t)\} = \underline{F(s+a)} \quad +1$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = \underline{e^{-as}F(s)} \quad +1$$

(b) [8pts] Using the Laplace Transform, solve $x'' + 6x' + 8x = 0$ with $x(0) = 1$, $x'(0) = 4$.

Transform the Eqn: $\left(s^2 X(s) - s - 4\right) + 6\left(sX(s) - 1\right) + 8X(s) = 0$

Isolate $X(s)$: $X(s)\cdot(s^2+6s+8) = s-4-6$; $\boxed{X(s) = \dfrac{s+10}{s^2+6s+8}}$ +2

To get $x(t) = \mathcal{L}^{-1}(X(s))$, there's a couple ways.

1) Partial Fractions:

$$\frac{s+10}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4},$$ multiply by LHS denominator,

$$s+10 = A(s+4) + B(s+2)$$

Plugin: $s=-4$: $6 = -2B$, $\boxed{B=-3}$

$s=-2$: $8 = 2A$, $\boxed{A=4}$

+3

So $X(s) = 4\cdot\dfrac{1}{s+2} - 3\cdot\dfrac{1}{s+4}$,

$$x(t) = 4\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$$

$$\boxed{x(t) = 4e^{-2t} - 3e^{-4t}} \quad +3$$

2) Use $\cosh(\omega t)$, $\sinh(\omega t)$:

$$X(s) = \frac{s+10}{(s+3)^2-1} \quad \text{// complete the square}$$

$$= \frac{(s+3)+7}{(s+3)^2-1} \quad \text{// Impose shift}$$

$$= \frac{s+3}{(s+3)^2-1} + \frac{7}{(s+3)^2-1} \quad +3$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2-1}\right) + \mathcal{L}^{-1}\left(\frac{7}{(s+3)^2-1}\right)$$

$$= e^{-3t}\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) + e^{-3t}\mathcal{L}^{-1}\left(\frac{7}{s^2-1}\right)$$

So $\boxed{x(t) = e^{-3t}\cosh(t) + 7e^{-3t}\sinh(t)}$ +3

Undetemined Coefs

$\Leftarrow \vec{f}(t) = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

2. [10pts] Find the general solution to $\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \boxed{\begin{bmatrix} 0 \\ e^t \end{bmatrix}}$

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now, and add it to $\vec{x}_c$.
If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.

1) We have $\vec{x}_c$ already ✓

2) For $\vec{f}(t) = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and since this function overlaps with $\vec{x}_c$ once, so,

Guess $\vec{x}_p = e^t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + te^t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. +2

To plug this in, $\vec{x}_p\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}_p + e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\vec{x}_p\,' = e^t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + (e^t + te^t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = e^t \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} + te^t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t \right)$$

$$= e^t \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} + te^t \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}.$$

So $e^t \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} + te^t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = te^t \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} + e^t \begin{bmatrix} a_2 \\ a_1 + 1 \end{bmatrix}$. +1

Row 1: $e^t (a_1 + b_1) + te^t (b_1) = te^t (b_2) + e^t (a_2)$

$\hookrightarrow$ e^t: $a_1 + b_1 = a_2 \rightarrow a_1 - a_2 + b_1 = 0$. *

+2 te^t: $b_1 = b_2$ ✓

Row 2: $e^t (a_2 + b_2) + te^t (b_2) = te^t (b_1) + e^t (a_1 + 1)$.

$\hookrightarrow$ e^t: $a_2 + b_2 = a_1 + 1 \rightarrow -a_1 + a_2 + b_1 = 1$. *
(since $b_2 = b_1$).

te^t: $b_2 = b_1$ ✓

Add these,
$2b_1 = 1$,
$\boxed{b_1 = \frac{1}{2}}$ +2
so $\boxed{b_2 = \frac{1}{2}}$ too.

Then with $b_1 = 1/2$, we have
$a_1 - a_2 + 1/2 = 0 \rightarrow a_1 - a_2 = -\frac{1}{2}$
$-a_1 + a_2 + 1/2 = 1 \rightarrow -a_1 + a_2 = 1/2$
same eqn, free var!

same eqn $\Rightarrow$ let $a_1 = a_2 - \frac{1}{2}$, a_2 = free +2
$\Rightarrow$ set $a_2 = 0$, $\boxed{\begin{matrix} a_1 = -\frac{1}{2} \\ a_2 = 0 \end{matrix}}$, Plug in a_i, b_i for $\vec{x}_p$,

so, $$\boxed{\vec{x} = \vec{x}_c + \vec{x}_p = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$$ +1

Eigenvector Decomposition

1) From $\vec{x}_c$, we see eigenvalues are $\lambda = \pm 1$ with eigenvectors $\vec{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

2) Build $E = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ so $E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

+1

$\lambda = 1$, $\vec{v}_+$; $\lambda = -1$, $\vec{v}_-$.

3) Then $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\vec{f}(t) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} e^t \\ e^t \end{bmatrix}}$ (same Result) ✓ +2

Or, you could do $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & e^t \end{array}\right] \overset{R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & e^t \end{array}\right] \Rightarrow \boxed{\begin{array}{l} g_1 = g_2 = e^t/2 \\ g_2 = e^t/2 . \end{array}}$

Thus the eqns we need to solve are:

$\varepsilon_1' = \varepsilon_1 + e^t/2 \Leftrightarrow \boxed{\varepsilon_1' - \varepsilon_1 = \frac{e^t}{2}}$ +1

$R(t) = e^{-t}$ would be our int. factor,

$\hookrightarrow \frac{d}{dt}[e^{-t}\varepsilon_1] = \frac{1}{2}e^{t-t} = \frac{1}{2}$

so $e^{-t}\varepsilon_1 = \frac{t}{2} + c_1$, +2

$\boxed{\varepsilon_1 = c_1 e^t + \frac{1}{2}te^t}$

$\varepsilon_2' = -\varepsilon_2 + e^t/2 \Leftrightarrow \boxed{\varepsilon_2' + \varepsilon_2 = \frac{e^t}{2}}$ +1

$R(t) = e^t$ is this one's int. factor,

$\hookrightarrow \frac{d}{dt}[e^t \varepsilon_2] = \frac{e^{2t}}{2}$,

so $e^t\varepsilon_2 = \frac{1}{4}e^{2t} + c_2$, +2

$\boxed{\varepsilon_2 = c_2 e^{-t} + \frac{1}{4}e^t}$

4) Thus, $\vec{x} = \varepsilon_1\vec{v}_1 + \varepsilon_2\vec{v}_2$ // 1^{st} component uses $\lambda = 1$, $\vec{v}_+$
2^{nd} component uses $\lambda = -1$, $\vec{v}_-$

$$\boxed{\vec{x} = \left(c_1e^t + \frac{te^t}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_2e^{-t} + \frac{1}{4}e^t\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$ +1

Note: Here, it looks like $\vec{x}_p = te^t\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \underline{\frac{1}{4}e^t\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ ← Looks Different!

But it's okay, $\frac{1}{4}e^t\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \frac{1}{4}e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{-\frac{1}{2}e^t\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ ← Is the same!

↑ Just added a complementary piece!