Math 3D PreQuiz - January 11th

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Show all of your work. *There is room and a question on the back side.*

Problem 1

a) Compute the indefinite integral $\int (\ln x)^2 dx$.
b) What does $\int_0^1 (\ln x)^2 dx$ equal based on (a)?

Solution. First let $u = \ln x$, so $du = dx/x$. Alternatively, this is like letting $x = e^t$ so that $dx = e^t dt$. With both perspectives,

$$\int (\ln x)^2 dx = \int u^2 \cdot e^u du$$

or

$$\int (\ln x)^2 dx = \int t^2 \cdot e^t dt.$$

To proceed, we need to integrate by parts, let $u = t^2, dv = e^t dt$ so that

$$= t^2 e^t - \int 2te^t dt$$

and do it one more time with $u = 2t$ and $dv = e^t dt$ again,

$$= t^2 e^t - 2te^t + \int 2e^t dt$$

$$= e^t(t^2 - 2t + 2) + C$$

We then have to resubstitute that $x = e^t \iff t = \ln x$ to get

$$\int (\ln x)^n dx = x((\ln x)^2 - 2\ln x + 2) + C$$

and moreover, if we made this a definite integral,

$$\int_0^1 (\ln x)^2 dx = 2.$$

For the upper bound, we can just plug in $x = 1$ to get 2. But, in the lower bound, since ln(0) is undefined, formally we have to take $\lim_{x \to 0}$ for the lower bound. Since x goes to 0 faster than $\ln x$ goes to $-\infty$, the lower limit of the integral goes to 0. (Don't worry too much about this, though).

Thanks for this alternate / easier solution mentioned! :
We can actually just let $u = (\ln x)^2$ right away. Then $dv = dx$. Using this, we get that $du = \frac{2lnx}{x} dx$ and $v = x$. So,

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2\ln x dx.$$

To integrate natural log, we have to do another integral by parts letting $u = \ln x$ and $dv = dx$ and we will wind up with the same answer.

Problem 2

a) Without solving for the partial fractions coefficients, what would be the general partial fraction decomposition of a function like

$$\frac{px^3 + qx - r}{(x-1)^3(x-3)(x^2+5)} \quad ?$$

(Do not solve for the coefficients).

Answer:

$$\frac{ax^3 + bx - c}{(x-1)^3(x-3)(x^2+5)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{A_4}{(x-3)} + \frac{A_5 x + A_6}{x^2+5}$$

b) Compute

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$$

Solution. First this is a partial fractions problem. Notably the polynomial numerator is a smaller degree than the bottom polynomial. Thus, we first factor the bottom,

$$x^3 + 4x = x(x^2 + 4).$$

This means that our partial fraction decomposition goes as

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

To solve for A, B, C, first find a common denominator, multiply by $x^3 + 4x = x(x^2+4)$ everywhere,

$$2x^2 - x + 4 = A(x^2 + 4) + x(Bx + C).$$

There's a couple ways to do this.
1st Way: Decompose the equation into powers of x. For instance,

$$\begin{cases} x^2: & 2 = A + B \\ x: & -1 = C \\ 1: & 4 = 4A \end{cases}$$

This way ends up being easier, as we can read off $A = 1,\ C = -1$ so that plugging in for the x^2 equation, $B = 1$ then. So,

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}.$$

2nd Way: Often the partial fractions are linear roots, so we can plug in those roots and isolate some constants. Here for example, $x = 0$ is a root, so if we plug in $x = 0$, we get

$$2(0^2) - 0 + 4 = A(0^2 + 4) + 0(B(0) + C) \iff 4 = 4A \iff A = 1.$$

However to solve for B and C, we have to go back to solving for the coefficients of x^2 and x as in the 1st Way.

Regardless, we'll get from partial fractions,

$$\int \frac{1}{x}dx + \int \frac{x-1}{x^2+4}dx = C + \ln|x| + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx.$$

For the middle term, let $u = x^2+4$, $du = 2xdx$ and for the last term, we let $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$ to obtain

$$= C + \ln|x| + \frac{1}{2}\ln|x^2+4| - \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta}$$

where we used the identity $4(\tan^2\theta+1) = 4\sec^2\theta$. The last term becomes just $\int \frac{1}{2}d\theta = \theta/2$ where since $x = 2\tan\theta$, we see that $\theta = \arctan(x/2)$ so finally,

$$\int \frac{2x^2-x+4}{x^3+4x}dx = C + \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan(x/2)$$

Problem 3

Find the eigenvalues (with multiplicity) and the corresponding eigenvector bases for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Optional: We can diagonalize A by writing it as $A = PDP^{-1}$. What are the matrices P and D?

Solution. One trick is that the eigenvalues of an upper or lower triangular matrix lie on its diagonal so we have $\lambda = 1$ (mult. 2) and $\lambda = 4$. Formally, we compute the characteristic polynomial

$$c_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 9/2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\cdot\det\begin{bmatrix} 4-\lambda & 9/2 \\ 0 & 1-\lambda \end{bmatrix}$$

$$c_A(\lambda) = (1-\lambda)^2(4-\lambda) \iff \lambda = 4, 1\ (mult. 2).$$

The eigenvectors are found by computing / solving for $(A - \lambda I)\vec{v} = 0$. So,

$$\lambda = 4,\ (A-4I)\vec{v} = 0 : \left[\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & 0 & 9/2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \overset{R_3 = -R_3/3}{\to} \left[\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 0 & 0 & 9/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{3rd\ Col}{\to} \left[\begin{array}{ccc|c} -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so the solution satisfies $-3v_1 = -2v_2 \iff v_1 = \frac{2}{3}v_2$ and $v_3 = 0$ so the eigenvector is of the form

$$\vec{v} = v_2 \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} \text{ so we can take the vector } \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} \text{ as the basis for the eigenspace of } \lambda = 4.$$

Checking, that $A\vec{v} = 4 \cdot \vec{v}$,

$$A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 0 \end{bmatrix} = 4 \cdot \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

Now for the other eigenvalue,

$$\lambda = 1, \ (A - I)\vec{v} = 0 : \begin{bmatrix} 0 & 2 & 3 & \vdots & 0 \\ 0 & 3 & 9/2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \overset{R_2 = R_2 - \frac{3}{2}R_1}{\rightarrow} \begin{bmatrix} 0 & 2 & 3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}, \quad \begin{aligned} v_1 &= \ free \\ 2v_2 &= -3v_3 \\ v_3 &= free. \end{aligned}$$

so we infer that the eigenspace consists of vectors of the form

$$\vec{v} = \begin{bmatrix} v_1 \\ -\frac{3}{2}v_3 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

where we see we have two distinct eigenvectors, so our eigenbasis for $\lambda = 1$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Checking, that $A\vec{v} = 1 \cdot \vec{v}$ now,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}.$$

Optional Part:
Lastly, to diagonalize A, we have that the matrices are respectively (and the order of collumns matters as the collumns of P are the corresponding eigenvectors to the eigenvalues)

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 3 \\ 0 & 2 & 0 \end{bmatrix}.$$

If we needed P^{-1} just to recall how to do it from Linear Algebra, consider the matrix

$$[\ P \vdots I\ ] = \begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -3 & 3 & \vdots & 0 & 1 & 0 \\ 0 & 2 & 0 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

and row reduce it until the left side is identity. The right side is then P^{-1}.
In other words, row reduce it to the form

$$[\ I \vdots P^{-1}\ ].$$