Weyl group representations and signatures of intertwining operators

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Abstract

We discuss the non-unitarity of a spherical principal series of a real split group by means of Weyl group calculations.

For a good choice of the parameters, a spherical principal series admits a non-degenerate invariant Hermitian form. To discuss unitarity, one needs to compute the signature of such a form. Computations can be done separately on the isotypic of each K-type and when the K-type is “petite”, they can be reduced to Weyl group calculations. It follows that, to compute the signature on the isotypic of a petite K-type μ, one only needs to understand the representation ψ_μ of the Weyl group on the space of M-invariants in μ.

For type $\mathcal{A}$, the set of Weyl group representations ψ_μ that arise from petite K-types can be identified with the set of partitions of n in at most two parts. We give an inductive algorithm to extend each Weyl group representation in this class to a petite K-type. This construction produces all the K-types used by Barbasch to detect unitarity in $SL(n, \mathbb{R})$.

Our inductive algorithm appears to be generalizable to other split groups with one root-length. The final result will be a list of petite K-types on which the signature can be computed with Weyl group calculations; hence, a non-unitarity test for a spherical principal series for split groups of types $\mathcal{D}$, E_6, E_7 and E_8.

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To my Mom

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Chapter 1

Introduction

One of the most interesting problems in representation theory is the study of the unitary dual of a real reductive Lie group G. It is a theorem by Harish-Chandra that every irreducible unitary representation of G is admissible, and that two irreducible unitary representations are unitarily equivalent if and only if they are infinitesimally equivalent. Therefore, the problem of classifying the unitary dual is equivalent to:

1. describing the set of irreducible admissible representations of G, up to infinitesimal equivalence;

2. understanding which irreducible admissible representations of G admit an invariant Hermitian form;

3. deciding whether the invariant Hermitian form on an admissible irreducible representation of G is definite.

The first problem was solved in the early 1970s by Langlands, who showed that every irreducible admissible representation of G is infinitesimally equivalent to a Langlands quotient $J_P(\delta \otimes \nu)$.

Let $P = MAN$ be a parabolic subgroup of G, let δ be an irreducible tempered unitary representation of M and let ν be a complex linear functional of A, with real part in

the open dominant Weyl chamber corresponding to N. Let $I_P(\delta\otimes\nu)$ be the principal series obtained by inducing the representation $\delta\otimes\nu$ of P to G. The assumptions on the parameters δ and ν guarantee that $I_P(\delta\otimes\nu)$ has a unique irreducible quotient, $J_P(\delta\otimes\nu)$, which we call a Langlands quotient with data $(P,\,\delta,\,\nu)$.
We can describe $J_P(\delta\otimes\nu)$ as the quotient of $I_P(\delta\otimes\nu)$ modulo the kernel of the intertwining operator

$$A(\bar{P}:P:\delta:\nu)\colon I_P(\delta\otimes\nu)\to I_{\bar{P}}(\delta\otimes\nu),\;\; F\mapsto\int_{\Theta(N)}F(x\bar{n})\,d\bar{n}.$$

Please, refer to chapter 2 for more details.

A few years later, Knapp and Zuckerman understood that a Langlands quotient $J_P(\delta\otimes\nu)$ is Hermitian if and only of there exists an element ω in K satisfying the formal symmetry condition

$$\omega\cdot\delta\simeq\delta,\quad \omega P\omega^{-1}=\bar{P},\quad \omega\cdot\nu=-\bar{\nu}. \tag{1.1}$$

The existence of a non-composition on the rangedegenerate invariant Hermitian form on $J_P(\delta\otimes\nu)$ is equivalent to the existence of a Hermitian $(\mathfrak{g},\,K)$-map that intertwines $J_P(\delta\otimes\nu)$ and its Hermitian dual $J_{\bar{P}}(\delta\otimes(-\bar{\nu}))$. The symmetry condition (1.1) follows from the fact that two Langlands quotients $J_P(\delta\otimes\nu)$ and $J_{P'}(\delta'\otimes\nu')$ can only be equivalent if there exists an element ω of K such that $\omega P\omega^{-1}=P'$, $\omega\cdot\delta=\delta'$ and $\omega\cdot\nu=\nu'$.

For each Hermitian Langlands quotient, Knapp and Zuckerman constructed a Hermitian intertwining operator B on the principal series $I_P(\delta\otimes\nu)$ whose kernel coincides with the kernel of the operator $A(\bar{P}:P:\delta:\nu)$. Such an operator induces a non-degenerate Hermitian form on the Langlands quotient $J_P(\delta\otimes\nu)=\frac{I_P(\delta\otimes\nu)}{Ker(A(\bar{P}:P:\delta:\nu))}$.
Please, refer to chapter 3 for details.

We are left with the problem of deciding whether a Hermitian Langlands quotient $J_P(\delta \otimes \nu)$ is unitary. This is equivalent to understanding whether the Hermitian intertwining operator B (on $I_P(\delta \otimes \nu)$) is semi-definite.

Because the representation is admissible, the signature of B can be computed separately on the isotypic component of each K-type appearing in the principal series, which is a finite-dimensional vector space. Of course there are infinitely many K-types, so the problem remains complicated.

Please, refer to chapter 5 for a short description of how to study the signature of an Hermitian intertwining operator defined on an admissible representation.

For each K-type $(\mu,\, E_\mu)$ appearing in the principal series, we obtain an intertwining operator

$$I_\mu(\omega,\, \nu)\colon\ \mathrm{Hom}_K(E_\mu,\, I_P(\delta \otimes \nu)) \to \mathrm{Hom}_K(E_\mu,\, I_P(\delta \otimes \omega \cdot \nu))$$

(induced from B by composition on the range) whose signature equals the signature of B w.r.t. the K-type μ. Using Frobenius reciprocity, we can interpret $I_\mu(\omega,\, \nu)$ as an endomorphism of $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_\mu,\, V^\delta)$.

Problem Compute the signature of $I_\mu(\omega,\, \nu)$, for each K-type $(\mu,\, E_\mu)$ appearing in the principal series $I_P(\delta \otimes \nu)$.

The example of $SL(2,\, \mathbb{R})$ (chapter 4) shows that it is fairly easy to solve this problem when G is a *split* group of real rank one.

For $G = SL(2,\, \mathbb{R})$, we let $P = MAN$ be the minimal parabolic consisting of upper triangular matrices. We assume $\delta = \delta^+$ to be the trivial representation of M and ν to

be a real dominant character of A satisfying the symmetry condition $\omega \cdot \nu = -\nu$. We identify the irreducible representations of $K = SO(2)$ with integers, and the characters of A with complex numbers. In particular, since the long Weyl group element ω is the negative of the identity, we identify ν with a real positive number number.
For all $\lambda > 0$, we construct the spherical principal series $I_P(\delta^+ \otimes \lambda)$, and we observe that $I_P(\delta^+ \otimes \lambda)$ contains every even K-type ξ_{2m} with multiplicity one, and no odd K-type at all.
For all integers m, the operator B acts on the the isotypic of ξ_{2m} by a scalar (because this isotypic is one-dimensional), and we denote this scalar by c_{2m}. A long but classical computation shows that

$$c_{2m} = c_{-2m} = \frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)} \qquad \forall\ m \geqslant 0.$$

By definition, B is positive semi-definite if and only if $c_{2m} \geqslant 0$, for all integers m, and this happens if and only of $0 < \lambda \leqslant 1$. We notice that this condition is equivalent to the one we obtain by requiring the signature to be positive *only* on the K-types ξ_0, ξ_{+2} and ξ_{-2} (which are petite).

When the group G is split[1], and P is a minimal parabolic we can always reduce the computations for the construction of the operator $I_\mu(\omega, \nu)$ to rank-one calculations. For simplicity, we assume that δ is the trivial representation of M and that ν is a real character of A. The element ω that appears in the symmetry condition 1.1 is a Weyl group element, hence it admits a minimal decomposition as a product of simple reflections. We decompose $I_\mu(\omega, \nu)$ accordingly, and notice that the intertwining operator corresponding to a simple root β can be interpreted an operator for a principal series of a rank-one subgroup $L^{(\beta)}$ (which is essentially a product of M and the $SL(2)$

[1]Write $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the Cartan decomposition of Lie(G), and choose a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$. The group G is called "split" if the centralizer of $\mathfrak{a}_0$ inside $\mathfrak{k}_0$ is zero. As a consequence, the group M is finite (and abelian).

attached to the root β), and can therefore be computed using the results already known for $SL(2,\,\mathbb{R})$. Here are more details.

If $\omega = s_{\alpha_r}\cdots s_{\alpha_2}s_{\alpha_1}$ is a minimal decomposition of ω_0 as a product of simple reflections, we can write:

$$I_\mu(\omega,\,\nu) = I_\mu(s_{\alpha_r}s_{\alpha_{r-1}}\cdots s_{\alpha_2}s_{\alpha_1},\,\nu) =$$

$$= I_\mu(s_{\alpha_r},\,(s_{\alpha_{r-1}}\cdots s_{\alpha_2}s_{\alpha_1})\cdot\nu)\cdots I_\mu(s_{\alpha_2},\,s_{\alpha_1}\cdot\nu)I_\mu(s_{\alpha_1},\,\nu). \tag{1.2}$$

Each factor of this decomposition is an operator of the form $I_\mu(s_\beta,\,\gamma)$, with s_β a simple reflection and γ a real character of A satisfying $\langle\gamma,\,\beta\rangle \geqslant 0$.

For β and γ as above, and for each integer l, we let φ_l be the isotypic component of the character ξ_l of $K^{(\beta)} \simeq SO(2)$ inside μ, so that $E^\mu = \bigoplus_{l\in\mathbb{Z}}\varphi_l$ is the decomposition of μ in $K^{(\beta)}$-types, and

$$\mathrm{Hom}_M(\mathrm{Res}^K_M\,E_\mu,\,V^\delta) = \underbrace{\bigoplus_{m\in\mathbb{N}}\mathrm{Hom}_M(\mathrm{Res}^{MK^{(\beta)}}_M(\varphi_{2m}+\varphi_{-2m}),\,\mathbb{C})}_{U_m}$$

is a decomposition of $\mathrm{Hom}_M(\mathrm{Res}^K_M\,E_\mu,\,V^\delta)$ in $MK^{(\beta)}$-stable subspaces. The operator

$$I_\mu(s_\beta,\,\gamma)\colon\ \mathrm{Hom}_M(\mathrm{Res}^K_M\,E_\mu,\,V^\delta) \to \mathrm{Hom}_M(\mathrm{Res}^K_M\,E_\mu,\,V^\delta)$$

preserves the decomposition of $\mathrm{Hom}_M(\mathrm{Res}^K_M\,E_\mu,\,V^\delta)$ in $MK^{(\beta)}$-stable subspaces. Precisely, for all $m \geqslant 0$, $I_\mu(s_\beta,\,\gamma)$ acts on U_m as scalar multiplication by

$$c_{2m,\gamma} = \frac{(1-\langle\gamma,\,\check\beta\rangle)(3-\langle\gamma,\,\check\beta\rangle)\cdots(2m-1-\langle\gamma,\,\check\beta\rangle)}{(1+\langle\gamma,\,\check\beta\rangle)(3+\langle\gamma,\,\check\beta\rangle)\cdots(2m-1+\langle\gamma,\,\check\beta\rangle)}. \tag{1.3}$$

Please, refer to chapter 6 for details.

We obtain a decomposition of $I_\mu(\omega,\,\nu)$ as a product of operators corresponding to simple reflections, and for each of these operators an explicit formula exists. The difficulty is that this formula depends on the decomposition of E_μ in irreducible $K^{(\beta)}$-types (with $K^{(\beta)}$ the $SO(2)$ attached to β) and this decomposition changes when β varies. It is very hard to keep track of these different decompositions when you multiply the various rank-one operators to obtain $I_\mu(\omega,\,\nu)$. Therefore, we look for an alternative approach to construct the operator $I_\mu(\omega,\,\nu)$.

Strategy For *some* K-types μ, construct the operator $I_\mu(\omega,\,\nu)$ by means of Weyl group calculations.

Definition 1. *Let G be a real reductive group, and let K be a maximal compact subgroup of G. A K-type $(\mu,\,E_\mu)$ is called petite if the $SO(2)$ subgroup attached to every restricted root (of G) only acts with characters $0,\ \pm1,\ \ \pm2$.*

When μ is a petite K-type, the operator $I_\mu(s_\beta,\,\gamma)$ only depends on the representation ψ_μ of the Weyl group on the space of M-fixed vectors in E_μ.
We can construct $I_\mu(\omega,\,\nu)$ in a purely algebraic fashion, as follows:

1. We factorize $I_\mu(\omega,\,\nu)$ as a product of rank-one operators (as in 1.2)

2. For each simple root β, and for each real character γ of A satisfying $\langle\gamma,\,\beta\rangle \geqslant 0$, we write:[2]

$$I_\mu(s_\beta,\,\gamma) = \begin{cases} +1 & \text{on the } (+1)\text{-eigenspace of } \psi_\mu(s_\beta) \\ \frac{1-\langle\gamma,\,\check\beta\rangle}{1+\langle\gamma,\,\check\beta\rangle} & \text{on the } (-1)\text{-eigenspace of } \psi_\mu(s_\beta). \end{cases}$$

[2]Recall that $I_\mu(s_\beta,\,\gamma)$ is defined on the space of M-fixed vectors in E_μ. Because μ is petite

$$\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_\mu,\,V^\delta) = \underbrace{\mathrm{Hom}_M(\mathrm{Res}^{MK^{(\beta)}}_M\,\varphi_0,\,\mathbb{C})}_{U_0} \bigoplus \underbrace{\mathrm{Hom}_M(\mathrm{Res}^{MK^{(\beta)}}_M(\varphi_2\oplus\varphi_{-2}),\,\mathbb{C})}_{U_2}.$$

We can identify U_0 with the $(+1)$-eigenspace of $\psi_\mu(s_\beta)$, and U_2 with the (-1)-eigenspace of $\psi_\mu(s_\beta)$. It follows from 1.3, that $I_\mu(s_\beta,\,\gamma)$ acts by $c_{0,\gamma}=1$ on U_0, and by $c_{2m,\gamma} = \frac{1-\langle\gamma,\,\check\beta\rangle}{1+\langle\gamma,\,\check\beta\rangle}$ on U_2.

3. We multiply the various rank-one operators $I_\mu(s_\beta, \gamma)$ to obtain $I_\mu(\omega, \nu)$.

Please, refer to chapter 6 for details, and to chapter 7 for an application of these considerations to $SL(3, \mathbb{R})$.

Final Results: When G is a real split group, and $X(\nu)$ is a unitary spherical Langlands quotient, the Hermitian operator $I_\mu(\omega, \nu)$ must be semi-definte for all K-types μ. When μ is petite, we can compute $I_\mu(\omega, \nu)$ by Weyl group calculations, so it should not be too hard to understand its signature.
The result is a ***non-unitarity test*** for a spherical principal series of G, in the sense that this test can be used to show that some representations are not unitary: if the "algebraic" operator $I_\mu(\omega, \nu)$ fails to be semi-definite for some petite K-type μ, then we can conclude that $X(\nu)$ is not unitary.

It is quite an amazing fact that this test also detects unitarity when the group G is a classical group. This beautiful result is due to Barbasch and can can be stated as follows:

Theorem. *If G is a classical (real or p-adic) split group, then the spherical Langlands quotient $X(\nu)$ is unitary if and only if the invariant Hermitian form is positive semi-definite on the petite K-types.*

Inspired by this result, we decide to explore the relation between petite K-types and Weyl group representations. The main result is an inductive algorithm that extends a certain class of representations of the Weyl group to petite K-types. We have formulated this result for the group $SL(n)$, but it appears to be generalizable to other split groups whose root-system admits only one root-length (this includes the non-classical groups E_6, E_7, E_8).

In chapter 8 we give an explicit description of the petite spherical K-types of $SL(2n)$, and we compute the corresponding representations of the Weyl group on the space of M-fixed vectors. The following table summarizes the results of the chapter

petite K-type	**Weyl group repr.**	**dim.**
$0 = 0\psi_1 + \cdots + 0\psi_n$	$S^{(2m)} = trivial$	1
$2\psi_1 + \cdots + 2\psi_k$ $0 < k < n$	$S^{(2n-k,k)}$	$\binom{2n}{k} - \binom{2n}{k-1}$
$2\psi_1 + \cdots + 2\psi_{n-1} \pm 2\psi_n$	$S^{(n,n)}$	$\binom{2n}{n} - \binom{2n}{n-1}$

With standard notations, we have identified the Weyl group of $SO(2n)$ with the symmetric group in $2n$ letters, and the irreducible representations of $\mathcal{S}_{2n}$ with the partitions of $2n$. We notice that the Weyl group representations ρ arising from the petite spherical representations are exactly the Specht modules corresponding to the partitions in at most two parts. This condition is equivalent to requiring that ρ do not contain any copy of the sign representation of $\mathcal{S}_3$.

In chapter 9 we run this process backwards, and for each Weyl group representation ρ not containing the sign of $\mathcal{S}_3$, we construct a spherical petite K-type μ_ρ such that ρ is a submodule of the representation of the Weyl group on the space of M-fixed vectors of μ_ρ. This construction produces *all* the petite spherical K-types.

In chapter 10 we determine the highest weight decomposition of μ_ρ and provide a number of examples.

In chapter 11 we generalize this construction to non-spherical representations of M', obtaining an inductive algorithm to construct petite K-types.

For type $\mathcal{A}$, our algorithm provides a list of petite K-types on which the intertwining operator for a spherical principal series can be tested by means of Weyl group computations; in other words, a list of K-types on which we should be able to explicitly calculate the signature.

Generalizing the algorithm, we will obtain a similar output for other groups with one root-length. The final result will be a non-unitarity test for a spherical principal series for split groups of type A, D, E_6, E_7 and E_8.

Chapter 2

Classifications of Irreducible Admissible Representations

2.1 Principal Series Representations

In this section we introduce the notion of a Principal Series Representation induced from a *minimal* parabolic subgroup[1], and we list its main properties. A detailed description of the material can be found in Knapp[21].

Let $P = MAN$ be a minimal parabolic subgroup of G. Fix an irreducible representation (δ, V_δ) of M and a character ν of A. We consider $\delta \otimes \nu$ as a representation of P, with N acting trivially.

The **Principal Series Representation**

$$\pi_P(\delta \otimes \nu) = \mathrm{Ind}_P^G(\delta \otimes \nu)$$

is obtained by inducing $\delta \otimes \nu$ from P to G. We give two different realizations of this

[1]There is a more general notion of Principal Series Representation, in which the parabolic subgroup is not required to be minimal. We will discuss it later.

representation:

▶ ***Induced Picture.*** The representation space is the completion of the vector space

$$\{F\colon G \to V_\delta \ \ continuous \mid \ F(gman) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(g),$$

$$\forall\, man \in P, \forall\, g \in G\}$$

with respect to the L^2 norm

$$\|f\|^2 = \textstyle\int_K \|f(k)\|^2_{V_\delta} dk,$$

where dk is the normalized Haar measure on K. It can be identified with the Hilbert space

$$\mathcal{H}^P_{\delta\otimes\nu} = \{F\colon G \to V_g\colon F\mid_K \in L^2(K, V^\delta) \text{ and } F(gman) = e^{-(\nu+\rho)\log(a)}\delta(m)^{-1}F(g),$$

$$\forall\, m \in M,\, a \in A,\, n \in N,\, g \in G\}.$$

This space is invariant by left translation in G, so we can define a representation of G on $\mathcal{H}^P_{\delta\otimes\nu}$ by

$$[\pi_P(\delta\otimes\nu)(g)\cdot F](x) = F(g^{-1}x) \qquad \forall\, x,\, g \in G.$$

We call $\pi_P(\delta\otimes\nu)$ the **principal series representation** with parameters P, δ and ν. We also denote it by $\mathrm{Ind}_P^G(\delta\otimes\nu)$, to emphasize the fact that it is induced from the representation $\delta\otimes\nu$ of P.

Notice that the restriction of $\pi_P(\delta\otimes\nu)$ to the maximal compact subgroup K is independent of the character ν, and it is given by:

$$\mathrm{Res}^G_K(\mathrm{Ind}^G_P(\delta\otimes\nu)) = \mathrm{Ind}^K_M\,\delta.$$

► ***Compact Picture.*** The representation space is obtained from $\mathcal{H}^P_{\delta\otimes\nu}$ by restriction to K:

$$\mathcal{H}^P_\delta = \{f \in L^2(K, V_\delta) \mid f(km) = \delta(m)^{-1} f(k), \quad \forall\, m \in M, \forall\, k \in K\}.$$

The action of G on $\mathcal{H}_\delta$ is defined by

$$[\pi_P(\delta \otimes \nu)(g) \cdot f](k) = e^{-(\nu+\rho)H(g^{-1}k)} f(\kappa(g^{-1}k)) \qquad \forall\, x \in G, \forall\, k \in K,$$

where $x = \kappa(x)e^{H(x)}n(x)$ denotes the Iwasawa decomposition of an element x of $G = KAN$.

These two pictures of the principal series representation are isomorphic. Indeed, restriction to K defines a Hilbert space isomorphism from $\mathcal{H}^P_{\delta\otimes\nu}$ onto $\mathcal{H}_\delta$, which intertwines the action of G.

The inverse mapping associates to an element $f\colon K \to V_\delta$ of $\mathcal{H}_\delta$ the unique extension $F\colon G \to V_\delta$ endowed with the desired transformation property under right multiplication by elements of P:

$$F(g) = F(kan) = e^{-(\nu+\rho)\log(a)} f(k)$$

(the unicity of the extension follows from the unicity of the Iwasawa decomposition of $G = KAN$).

We will continuously switch between the induced picture and the compact one. The compact picture has the advantage that the representation space is independent of ν, but the action of G is much simpler in the induced picture.

Now that the definition of a principal series representation is understood, we describe its main properties. For clarity, we subdivide the results in three categories:

Admissibility Any principal series representation $\pi_P(\delta \otimes \nu)$ is admissible.

Indeed, by Frobenius reciprocity, the multiplicity of a K-type μ in

$$\operatorname{Res}^G_K(\operatorname{Ind}^G_P(\delta \otimes \nu)) = \operatorname{Ind}^K_M \delta$$

equals the multiplicity of δ in $\operatorname{Res}^K_M(\mu)$, which is finite.

We denote by $X(\delta \otimes \nu)$ the corresponding Harish-Chandra module.

It is a theorem by Harish-Chandra that every irreducible admissible representation of G is infinitesimally equivalent to a composition factor of a principal series representation.

This is the famous "Harish-Chandra subquotient theorem". It states that an irreducible representation of G is admissible if and only if the corresponding Harish-Chandra module is equivalent to an irreducible quotient M_1/M_2, where $M_1 \supseteqq M_2$ are two submodules of a principal series induced from a minimal parabolic.

Equivalence Two principal series representations $\pi_P(\delta \otimes \nu)$ and $\pi_{P'}(\delta' \otimes \nu')$ have equivalent composition series if and only if there exists a Weyl group element ω such that

$$(\delta', \nu') = (\omega \cdot \delta, \omega \cdot \nu).$$

We make the Weyl group act on equivalence classes of irreducible representations of M and A as follows:

$$(\omega \cdot \delta)(m) = \delta(x^{-1}mx) \quad \forall\, m \in M,$$

with x a representative for ω in the normalizer of $\mathfrak{a}_0$ in K, and

$$(\omega \cdot \nu)(e^H) = \nu(e^{\operatorname{Ad}(\omega^{-1})H}) \quad \forall\, H \in \mathfrak{a}_0.$$

Unitarity If ν is imaginary, the principal series representation $\pi_P(\delta \otimes \nu)$ is unitary.

Notice this condition on ν makes $\delta \otimes \nu$ unitary, for each irreducible representation δ of M. Indeed, when $P = MAN$ is minimal parabolic, M is included in K and therefore is compact.

2.2 The Standard Intertwining Operator

In this section we discuss the existence of an intertwining operator between two principal series representations $\pi_{MAN}(\delta \otimes \nu)$ and $\pi_{MA\tilde{N}}(\delta \otimes \nu)$, that are obtained by inducing the same representation $\delta \otimes \nu$ of MA from two different minimal parabolic subgroups $P = MAN$ and $\tilde{P} = MA\tilde{N}$. Results are stated without a proof. A good reference is Knapp[21].

Let $P = MAN$ be a minimal parabolic subgroup of G, with

$$\mathrm{Lie}(N) = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\circ\alpha}.$$

The choice of a new positive system $\tilde{\Delta}^+ \subset \Delta(\mathfrak{g}_\circ, \mathfrak{a}_\circ)$ determines a new minimal parabolic subgroup $\tilde{P} = MA\tilde{N}$, with

$$\mathrm{Lie}(\tilde{N}) = \bigoplus_{\alpha \in \tilde{\Delta}^+} \mathfrak{g}_{\circ\alpha}.$$

Fix an irreducible representation (δ, V_δ) of M and a character ν of A. The representation $\delta \otimes \nu$ of MA can be regarded both as a representation of P and as a representation of $\tilde{P}$ (with N and $\tilde{N}$ acting trivially).

Induction to G gives rise to two different principal series representations $\pi_{MAN}(\delta \otimes \nu)$ and $\pi_{MA\tilde{N}}(\delta \otimes \nu)$, defined on Hilbert spaces $\mathcal{H}^P_{\delta\otimes\nu}$ and $\mathcal{H}^{\tilde{P}}_{\delta\otimes\nu}$.

Consider the operator

$$A(\tilde{P} \colon P \colon \delta \colon \nu) \colon \mathcal{H}^P_{\delta\otimes\nu} \longrightarrow \mathcal{H}^{\tilde{P}}_{\delta\otimes\nu}$$

defined *formally*[2] by

$$[A(\tilde{P}\colon P\colon \delta\colon \nu)F](x) = \int_{\tilde{N}\cap\bar{N}} F(x\bar{n})\, d\bar{n},$$

where $\bar{N} = \Theta(N)$ denotes the analytic subgroup of G with Lie algebra

$$\mathrm{Lie}(\bar{N}) = \vartheta(\mathrm{Lie}(N)) = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_{\mathfrak{o}\alpha} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{\mathfrak{o}-\alpha}.$$

It is not too hard to check that

- for each F in $\mathcal{H}^P_{\delta\otimes\nu}$, the function

$$A(\tilde{P}\colon P\colon \delta\colon \nu)F\colon G \to V_\delta$$

  has the proper behavior under right multiplication by elements of $\tilde{P} = MA\tilde{N}$:

$$A(\tilde{P}\colon P\colon \delta\colon \nu)F(xm) = \delta(m)^{-1}(A(\tilde{P}\colon P\colon \delta\colon \nu)F(x)) \quad \forall\, m \in M$$

$$A(\tilde{P}\colon P\colon \delta\colon \nu)F(xa) = e^{-(\nu+\tilde{\rho})}A(\tilde{P}\colon P\colon \delta\colon \nu)F(x) \qquad \forall\, a \in A$$

$$A(\tilde{P}\colon P\colon \delta\colon \nu)F(x\tilde{n}) = A(\tilde{P}\colon P\colon \delta\colon \nu)F(x) \qquad \forall\, \tilde{n} \in \tilde{N}$$

  (here $\tilde{\rho}$ denotes the semi-sum of the roots in $\tilde{\Delta}^+$, counting multiplicity);

- for each F in $\mathcal{H}^P_{\delta\otimes\nu}$ and each g in G,

$$A(\tilde{P}\colon P\colon \delta\colon \nu)[\pi_{MAN}(\delta\otimes\nu)(g)\cdot F] = \pi_{MA\tilde{N}}(\delta\otimes\nu)(g)\cdot[A(\tilde{P}\colon P\colon \delta\colon \nu)F].$$

So for each choice of the parameters δ and ν, $A(\tilde{P}\colon P\colon \delta\colon \nu)$ is a *formal* intertwining operator from $\pi_{MAN}(\delta\otimes\nu)$ to $\pi_{MA\tilde{N}}(\delta\otimes\nu)$.
It is an *actual* intertwining operator only if the integral

$$A(\tilde{P}\colon P\colon \delta\colon \nu)F(x) = \int_{\tilde{N}\cap\bar{N}} F(x\bar{n})\, d\bar{n}$$

[2] At this stage, we only consider $A(\tilde{P}\colon P\colon \delta\colon \nu)$ as a formal operator; the attribute "formal" means that we do not worry about convergence.

converges, for each F in $\mathcal{H}^P_{\delta\otimes\nu}$.

Unfortunately, convergence happens only for certain values of ν.
If $P = MAN$ and $\tilde{P} = MA\tilde{N}$ are minimal parabolic, a sufficient condition for the convergence is that $\langle \mathrm{Re}(\nu), \beta\rangle > 0$, for each β which is positive for N and negative for $\tilde{N}$:

Theorem 1. *Let $P = MAN$ and $\tilde{P} = MA\tilde{N}$ be minimal parabolic subgroups, and suppose that ν satisfies the condition*

$$\langle \mathrm{Re}(\nu), \beta\rangle > 0 \quad \forall\, \beta \in \Delta^+ \cap \tilde{\Delta}^-.$$

Then, the integral defining $A(\tilde{P}\colon P\colon \delta\colon \nu)F(x)$ is convergent for every continuous function F in $\mathcal{H}^P_{\delta\otimes\nu}$ and for all x in G.

In this case, we refer to $A(\tilde{P}\colon P\colon \delta\colon \nu)$ as the **standard intertwining operator** from $\pi_{MAN}(\delta\otimes\nu)$ to $\pi_{MA\tilde{N}}(\delta\otimes\nu)$.

We also mention a result about the factorization of standard intertwining operators:

Theorem 2. *Let $P = MAN$, $P' = MAN'$ and $P'' = MAN''$ be minimal parabolic subgroups of G. Suppose that*

$$\mathrm{Lie}(N'') \cap \mathrm{Lie}(N) \subseteqq \mathrm{Lie}(N') \cap \mathrm{Lie}(N)$$

and that $\langle \mathrm{Re}(\nu), \beta\rangle > 0$, for each β which is positive for N and negative for N''. Then

$$A(P''\colon P\colon \delta\colon \nu) = A(P''\colon P'\colon \delta\colon \nu) \circ A(P'\colon P\colon \delta\colon \nu).$$

2.3 A Langlands Quotient

We use the standard intertwining operator $A(\bar{P}\colon P\colon \delta\colon \nu)$, with $\bar{P}$ equals the minimal parabolic opposite to P, to construct an irreducible representation of G. Such representation is called a Langlands quotient, because it was obtained by Langlands as a quotient of the principal series representation $\pi_P(\delta \otimes \nu)$ module the kernel of $A(\bar{P}\colon P\colon \delta\colon \nu)$. The main reference for this section is [9].

With the same notations used the previous section, we consider the case in which $\tilde{P}$ is the minimal parabolic opposite to P:

$$\bar{P} = \Theta(P) = MA\bar{N}.$$

Since every root positive for N is automatically negative for $\bar{N}$, the convergence condition for $A(\bar{P}\colon P\colon \delta\colon \nu)$ is

$$\langle \mathrm{Re}(\nu),\, \beta\rangle > 0 \quad \forall\, \beta \in \Delta^+.$$

As a result, we obtain the following theorem:

Theorem 3. *Let $P = MAN$ be a minimal parabolic subgroup.*
For each irreducible representation δ of M, and for those characters ν of A such that $\mathrm{Re}(\nu)$ *is strictly dominant, there exists an intertwining operator*

$$A(\bar{P}\colon P\colon \delta\colon \nu)$$

from $\pi_P(\delta \otimes \nu)$ to $\pi_{\bar{P}}(\delta \otimes \nu)$, with $\bar{P}$ the opposite parabolic.

The intertwining operator $A(\bar{P}\colon P\colon \delta\colon \nu)$ was the object of many studies in the '70s and '80's, because of its intimate connection with the irreducibility of the principal series representation $\pi_P(\delta \otimes \nu)$.

The following results are mainly due to Langlands (and Miličić), and can be found in [9]:

- The operator $A(\bar{P}\colon P\colon \delta\colon \nu)$ is not identically zero.

- If $F \in \mathcal{H}^P_{\delta\otimes\nu}$ but $F \notin \mathrm{Kernel}(A(\bar{P}\colon P\colon \delta\colon \nu))$, then F is a cyclic vector for $\pi_P(\delta \otimes \nu)$.

- The image of the operator $A(\bar{P}\colon P\colon \delta\colon \nu)$ is an irreducible submodule of $\pi_{\bar{P}}(\delta \otimes \nu)$.

- $\pi_P(\delta \otimes \nu)$ has a unique non-zero irreducible quotient:

$$J_P(\delta \otimes \nu) = \frac{\mathcal{H}^P_{\delta\otimes\nu}}{\mathrm{Kernel}(A(\bar{P}\colon P\colon \delta\colon \nu))},$$

  which is isomorphic to $A(\bar{P}\colon P\colon \delta\colon \nu)(\mathcal{H}^P_{\delta\otimes\nu})$.

The irreducible admissible representation $J_P(\delta \otimes \nu)$ is called a **Langlands Quotient** with data (P, δ, ν).

If we allow the parabolic subgroup P to be non-minimal, we basically obtain all the irreducible admissible representations of G in the form of Langlands quotients. The next step is therefore a generalization of the principal series to the case "P non-minimal parabolic".

2.4 Generalization to non-minimal parabolic

Under some additional conditions on δ, all the constructions carried out in the previous sections (principal series representation, standard intertwining operator, Langlands quotient) can be generalized to the case in which the parabolic subgroup P is not a minimal parabolic. A good reference is again Knapp[21].

Let $P^1 = M^1 A^1 N^1$ be a parabolic subgroup of G, containing a minimal parabolic $P = MAN$ (so $A^1 \subsetneqq A$, $N^1 \subsetneqq N$ and $M^1 \supsetneqq M$).

Let $\Delta^1 = \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0^1)$ be the set of roots:

$$\Delta^1 = \{\beta \in (\mathfrak{a}_0^1)' : (\mathfrak{g}_0)^1_\beta \neq \emptyset\},$$

where we have denoted by $(\mathfrak{g}_0)^1_\beta$ the root space corresponding to β:

$$(\mathfrak{g}_0)^1_\beta = \{X \in \mathfrak{g}_0 \colon [H,\, X] = \beta(H)X,\ \forall\, H \in \mathfrak{a}_0^1\}.$$

The roots in $\Delta^1 = \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0^1)$ are basically [3] obtained from $\Delta = \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0)$ by restriction to $\mathfrak{a}_0^1$. When P^1 is not a *minimal* parabolic, they do not necessarily form a root system.

We introduce a notion of positivity in Δ^1 by calling a root β *positive* if the corresponding root space $(\mathfrak{g}_0)^1_\beta$ is included in $\mathfrak{n}_0$. We denote by ρ^1 the semi-sum (with multiplicities) of the positive roots in Δ^1.

The construction of the principal series mimics the one done in the case of a minimal parabolic:

- We fix a character ν^1 of $\mathfrak{a}_0^1$ and a unitary[4] representation δ^1 of M^1, and we regard $\nu^1 \otimes \delta^1$ as a representation of $P^1 = M^1A^1N^1$ (with N^1 acting trivially);
- We describe the *induced picture* of the principal series representation $\pi_{P^1}(\delta^1 \otimes \nu^1)$ as the action of G by left translation on the Hilbert space:

$$\mathcal{H}^{P^1}_{\delta^1\otimes\nu^1} = \{F \colon G \to V^{\delta^1} :\ F\,|_K \in L^2(K,\, V^{\delta^1}) \text{ and}$$

$$F(xman) = e^{-(\nu^1+\rho^1)\log(a)}\delta^1(m)^{-1}F(x),\, \forall\, m \in M^1 \cap K,\, \forall\, an \in A^1N^1, \forall\, x \in G\}.$$

- In the *compact picture* of $\pi_{P^1}(\delta^1 \otimes \nu^1)$, the Hilbert space is simply

$$\mathcal{H}^{P^1}_{\delta^1|_{M^1\cap K}} = \{f \in L^2(K,\, V^{\delta^1}) \colon f(xm) = \delta^1(m)^{-1}f(x),\, \forall\, m \in M^1 \cap K\}$$

[3] Just make sure to disregard the roots of Δ that are identically zero on a_0^1.

[4] In the minimal parabolic case, we did not require δ be unitary. Indeed, this was automatically true, given the compactness of M.

and we define the action of G by:

$$[\pi_{P^1}(\delta^1 \otimes \nu^1)(g) \cdot f](k) = e^{-(\nu^1+\rho^1)H'(g^{-1}k)} \delta^1(\mu'(g^{-1}k))^{-1} f(\kappa'(g^{-1}k))$$

(where $\kappa'(x)\nu'(x)e^{H'(x)}n'(x)$ denotes the decomposition of an element x of G under $G = KM^1A^1N^1$).

Restriction to K is an isomorphism from the induced picture to the compact one. The inverse mapping sends $f\colon K \to V^{\delta^1}$ to the unique extension $F\colon G \to V^{\delta^1}$ satisfying the desired transformation properties under right multiplication by elements of P^1:

$$F(g) = F(\kappa'(g)\nu'(g)e^{H'(g)}n'(g)) = e^{-(\nu^1+\rho^1)H'(g)} \delta^1(\mu'(g))^{-1} f(\kappa'(g)).$$

Just like in the minimal parabolic case, the principal series representation $\pi_{P^1}(\delta^1 \otimes \nu^1)$ is admissible for any value of the parameters δ^1 (unitary) and ν^1, and it is unitary if ν^1 is purely imaginary.

Encouraged by these strong similarities, we introduce the *formal* intertwining operator

$$A(\bar{P}^1\colon P^1\colon \delta^1\colon \nu^1)\colon \mathcal{H}^{P^1}_{\delta^1\otimes\nu^1} \longrightarrow \mathcal{H}^{\bar{P}^1}_{\delta^1\otimes\nu^1} \tag{2.1}$$

defined by:

$$A(\bar{P}^1\colon P^1\colon \delta^1\colon \nu^1)F(x) = \int_{\bar{N}^1} F(xn)\,dn.$$

For convergence (on K-finite vectors) we need $\mathrm{Re}(\nu^1)$ to be in the open positive Weyl chamber relative to N^1, and also an additional condition: we need δ^1 to be tempered.

Theorem 4. *Let $P^1 = M^1A^1N^1$ be a (not necessarily minimal) parabolic subgroup of G. Let δ^1 be an irreducible tempered unitary representation of M^1, and let ν^1 be a characters of A^1 such that $\mathrm{Re}(\nu^1)$ is in the open positive Weyl chamber determined by N^1. Define the parameters (P^2, δ^2, ν^2) is a similar fashion. Then*

1. *there exists an intertwining operator*

$$A(\bar{P^1} : P^1 : \delta^1 : \nu^1)$$

   from $\pi_{P^1}(\delta^1 \otimes \nu^1)$ to $\pi_{\bar{P^1}}(\delta^1 \otimes \nu^1)$, with $\bar{P}$ the opposite parabolic.
   $A(\bar{P^1} : P^1 : \delta^1 : \nu^1)$ is defined on the space of K-finite vectors, and since $\pi_{P^1}(\delta^1 \otimes \nu^1)$ and $\pi_{\bar{P^1}}(\delta^1 \otimes \nu^1)$ are both admissible, it gives rise to an intertwining operator for the corresponding actions of $\mathfrak{g}_0$.

2. *$\pi_{P^1}(\delta^1 \otimes \nu^1)$, as a representation of $\mathfrak{g}_0$ on the space of K- finite vectors, has a unique irreducible quotient which we denote by $J(P^1, \delta^1, \nu^1)$.*

3. *$J(P^1, \delta^1, \nu^1)$ is infinitesimally equivalent with the image of the intertwining operator $A(\bar{P^1} : P^1 : \delta^1 : \nu^1)$ defined above.*

4. *$J(P^1, \delta^1, \nu^1)$ is infinitesimally equivalent to $J(P^2, \delta^2, \nu^2)$ if and only if there is an element of K carrying P^1 to P^2, δ^1 to δ^2 (up to equivalence), and ν^1 to ν^2.*

Definition 2. *Let $P^1 = M^1A^1N^1$ be a parabolic subgroup of G. Let δ^1 be an irreducible tempered unitary representation of M^1, and let ν^1 be a characters of A^1 with real part in the open positive Weyl chamber determined by N. We call $J(P^1, \delta^1, \nu^1)$ the **Langlands quotient** with parameters (P^1, δ^1, ν^1).*

2.5 Langlands Classification of Irreducible Admissible Representations

We just report the main result, as stated in Theorem 8.54, Knapp[21].

Theorem 5. *Fix a minimal parabolic subgroup $P = MAN$ of G.*

The infinitesimal equivalence classes of irreducible admissible representations of G stand in one-one correspondence with all triples $(P^1, [\delta^1], \nu^1)$ such that

- *$P^1 = M^1A^1N^1$ is a parabolic subgroup of G containing P;*
- *δ^1 is an irreducible tempered unitary representation of M^1, and $[\delta^1]$ is its equivalence class;*
- *ν^1 is a complex linear functional on $\mathfrak{a}_0^1$, with $\mathrm{Re}(\nu^1)$ in the open positive Weyl chamber corresponding to N^1.*

Chapter 3

Classification of Unitary Irreducible Representations

It is a theorem by Harish-Chandra that each irreducible unitary representation is admissible and two irreducible unitary representations are unitarily equivalent if and only if they are infinitesimally equivalent. Therefore, classifying the irreducible unitary representations of a semi-simple group amounts to deciding which Langlands quotients are infinitesimally unitary[1].

This problem was solved by Knapp and Zuckerman in 1976. They proved that:

Theorem 6. *$J_{P=MAN}(\delta \otimes \nu)$ is infinitesimally unitary if and only if*

(i) the formal symmetry conditions hold: there exists ω in K normalizing $\mathfrak{a}_\mathfrak{o}$ with $\omega\, P\, \omega^{-1} = \bar{P}$, $\omega \cdot \delta \simeq \delta$ and $\omega \cdot \nu = -\bar{\nu}$

(ii) the Hermitian intertwining operator

$$B = \delta(\omega)\, R(\omega)\, A(\bar{P} \colon P \colon \delta \colon \nu)$$

is positive or negative semi-definite.

[1]An admissible representation π of G on a Hilbert space V is said to be *infinitesimally unitary* if its space V_0 of K-finite vectors admits an Hermitian inner product with respect to which $\pi(\mathfrak{g}_0^{\mathbb{C}})$ acts by skew Hermitian operators.

We direct the reader to [25] for a detailed proof of this result. We will only add a few comments.

3.1 The formal symmetry condition

The aim of this section is to explain why the formal symmetry condition

$$\omega\, P\, \omega^{-1} = \bar{P} \qquad \omega \cdot \delta \simeq \delta \qquad \omega \cdot \nu = -\bar{\nu} \tag{3.1}$$

(for some ω in K) is necessary for the unitarity of the Langlands quotient $J_P(\delta \otimes \nu)$.

Let $J_P(\delta \otimes \nu)$ be infinitesimally unitary, and let $\langle\, ,\, \rangle$ be a non degenerate (positive definite) invariant form on $J_P(\delta \otimes \nu)$. The mapping

$$T\colon J_P(\delta \otimes \nu) \to (J_P(\delta \otimes \nu))^h,\; x \mapsto \langle x,\, \cdot \rangle$$

defines an invertible $(\mathfrak{g}, K)$-module map from $J_P(\delta \otimes \nu)$ to its Hermitian dual.[2]
Because δ is unitary, the Hermitian dual of $J_P(\delta \otimes \nu)$ is the Langlands quotient $J_{\bar{P}}(\delta \otimes -\bar{\nu})$.[3] We conclude that $J_P(\delta \otimes \nu)$ and $J_{\bar{P}}(\delta \otimes -\bar{\nu})$ are infinitesimally equivalent. It then follows from 4 that the formal symmetry condition 3.1 holds.
Finally, we notice that the element ω in (3.1) has the property that ω^2 lies in $M^1 \cap K$.

[2] As a vector space, the Hermitian dual of an admissible $(\mathfrak{g}, K)$-module X is the set of K-finite conjugate linear maps $X \to \mathbb{C}$. The actions of K and $\mathfrak{g}$ are defined by:

$$(k \cdot f)(x) = f(k^{-1} \cdot x) \qquad (Z \cdot f)(x) = -f(\bar{Z} \cdot x)$$

for all k in K, Z in $\mathfrak{g}$ and x in X. Notice that if X is the Harish-Chandra module of an admissible representation π of G on a Hilbert space, then X^h is the Harish-Chandra module of π^h. We have denoted by π^h the representation of G defined by: $\pi^h(g) = (\pi(g^{-1}))^*$. Please, refer to [24] for more information on the Hermitian dual.

[3] See [24], Proposition 11.26, or [26], lemma 20.

3.2 The operator B

Let P be a (not necessarily minimal) parabolic subgroup of G, with Langlands decomposition $P = MAN$. Fix a unitary irreducible tempered representation δ of M, and a character ν of A in the open positive Weyl chamber determined by N. These assumptions on δ and ν guarantee the convergence of the standard intertwining operator (2.1)

$$A(\bar{P}\colon P\colon \delta\colon \nu)\colon \mathcal{H}_P(\delta \otimes \nu) \longrightarrow \mathcal{H}_{\bar{P}}(\delta \otimes \nu).$$

Suppose that there exists an element ω in K satisfying the formal parity condition (3.1). By construction, $\omega \cdot \delta$ is unitarily equivalent to δ, and ω^2 belongs to $M \cap K$. Hence, it is possible to define $\delta(\omega)$ in exactly two ways (differing by a minus sign) so that δ extends to a unitary representation of the subgroup of K generated by M and ω.[4] Because ω normalizes $\mathfrak{a}_0$, the operator $\delta(\omega)$ is a well defined intertwining operator from $\pi_P(y \cdot \delta \otimes \nu)$ to $\pi_P(\delta \otimes \nu)$.
We also consider the operator $R(\omega)$ of right multiplication by ω, which is an intertwining operator from $\pi_P(\delta \otimes \nu)$ to $\pi_{\omega P \omega^{-1}}(\omega \cdot \delta \otimes \omega \cdot \nu) = \pi_{\bar{P}}(\omega \cdot \delta \otimes -\bar{\nu})$.

Composing the standard intertwining operator $A(\bar{P}\colon P\colon \delta\colon \nu)$ with the operator $R(\omega)$, we obtain an operator

$$R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu)\colon \mathcal{H}_P(\delta \otimes \nu) \longrightarrow \mathcal{H}_{\omega \bar{P} \omega^{-1} = P}(\omega \cdot \delta \otimes \omega \cdot \nu)$$

which intertwines the representations $\pi_P(\delta \otimes \nu)$ and $\pi_P(\omega \cdot \delta \otimes -\bar{\nu})$.

We then compose $[R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu)]$ with $\delta(\omega)$ to obtain an intertwining operator

$$\boxed{B = \delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu)\colon \mathcal{H}_P(\delta \otimes \nu) \longrightarrow \mathcal{H}_P(\delta \otimes -\bar{\nu})} \qquad (3.2)$$

from $\pi_P(\delta \otimes \nu)$ to $\pi_P(\delta \otimes -\bar{\nu})$.

[4]See [21], lemma 14.22 or [22], lemma 18.

It is important to notice that the operator $B = \delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu)$ only depends on the equivalence class of ω modulo M. Indeed, for each m_0 in M, the element ωm_0 also satisfies the formal symmetry condition (3.1) and

$$\delta(\omega m_0) \circ R(\omega m_0) \circ A(\bar{P}\colon P\colon \delta\colon \nu) = \delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu).$$

Since ω^2 lies in $M \cap K$, this implies that

$$\delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu) = \delta(\omega^{-1}) \circ R(\omega^{-1}) \circ A(\bar{P}\colon P\colon \delta\colon \nu).$$

The other fundamental remark is that the operator B is Hermitian (we use the compact picture for the principal series representations $\pi_P(\delta \otimes \nu)$ and $\pi_P(\delta \otimes -\bar{\nu})$, so that domain and codomain of B are identifies with $\mathcal{H}^P_\delta$). This result follows from the following lemma:[5]

Lemma 1. *Let G be a semi-simple Lie group, and let $P_1 = MAN_1$ and $P_2 = MAN_2$ be associated minimal parabolic subgroups of G.*
Fix an irreducible (tempered) unitary representation δ of M, and a character ν of A satisfying

$$\langle \mathrm{Re}(\nu),\, \alpha \rangle > 0$$

for each α positive for P_1 and negative for P_2. Let $A(P_2\colon P_1\colon \delta\colon \nu)$ be the standard intertwining operator from $\pi_{P_1}(\delta \otimes \nu)$ to $\pi_{P_2}(\delta \otimes \nu)$. Then

$$A(P_2\colon P_1\colon \delta\colon \nu)^* = A(P_1\colon P_2\colon \delta\colon -\bar{\nu})$$

with the adjoint computed K-type by K-type in the compact picture.

We now prove that

[5] Please, refer to [23] (proposition 7.1 (*iv*)) for a proof.

Proposition 1. *The operator $B\colon \mathcal{H}_\delta^P \mapsto \mathcal{H}_\delta^P$ is Hermitian.*

Proof. This is not hard to prove:

$$B^* = [\delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu)]^*$$

$$= A(\bar{P}\colon P\colon \delta\colon \nu)^* \circ R(\omega)^* \circ \delta(\omega)^*$$

$$= A(P\colon \bar{P}\colon \delta\colon -\bar{\nu}) \circ R(\omega^{-1}) \circ \delta(\omega^{-1}) \qquad (\star)$$

$$= R(\omega^{-1}) \circ [R(\omega) \circ A(P\colon \bar{P}\colon \delta\colon -\bar{\nu}) \circ R(\omega^{-1})] \circ \delta(\omega^{-1})$$

$$= R(\omega^{-1}) \circ A(\bar{P}\colon P\colon \omega \cdot \delta\colon \nu) \circ \delta(\omega^{-1}) \qquad (\star\star)$$

$$= R(\omega^{-1}) \circ \delta(\omega^{-1}) \circ [\delta(\omega) \circ A(\bar{P}\colon P\colon \omega \cdot \delta\colon \nu) \circ \delta(\omega^{-1})]$$

$$= R(\omega^{-1}) \circ \delta(\omega^{-1}) \circ A(\bar{P}\colon P\colon \delta\colon \nu) \qquad (\star\star\star)$$

$$= \delta(\omega) \circ R(\omega) \circ A(\bar{P}\colon P\colon \delta\colon \nu) = B.$$

Just a few remarks:

- $(\star)$ follows from the fact that the operators $R(\omega)$ and $\delta(\omega)$ are unitary, and from the the previous lemma;
- $(\star\star)$ follows from the identity[6]

$$A(S_2\colon S_1\colon \sigma\colon \lambda) = R(y^{-1})A(yS_2y^{-1}\colon yS_1y^{-1}\colon y \cdot \sigma\colon y \cdot \lambda)R(y)$$

[6]See [21], p.182.

for $y = \omega^{-1}$, $S_2 = \bar{P}$, $S_1 = P$, $\sigma = \omega \cdot \delta$, and $\lambda = \nu$;

- $(\star\star\star)$ is the easy identity

$$\delta(\omega) \circ A(\bar{P} \colon P \colon \omega \cdot \delta \colon \nu) \circ \delta(\omega^{-1}) = A(\bar{P} \colon P \colon \delta \colon \nu).$$

□

3.3 Unitary Langlands Quotients

In this section we sketch the proof of theorem 6. A detailed proof can be found in [25].

Theorem (6). *$J_{P=MAN}(\delta \otimes \nu)$ is infinitesimally unitary if and only if*

(i) the formal symmetry conditions hold: there exists ω in K normalizing $\mathfrak{a}_\circ$ with $\omega\, P\, \omega^{-1} = \bar{P}$, $\omega \cdot \delta \simeq \delta$ and $\omega \cdot \nu = -\bar{\nu}$

(ii) the Hermitian intertwining operator

$$B = \delta(\omega)\, R(\omega)\, A(\bar{P} \colon P \colon \delta \colon \nu)$$

is positive or negative semi-definite.

Proof. Let $J_P(\delta \otimes \nu)$ be a Langlands quotient and suppose that there exists an element ω in K satisfying the formal symmetry condition 3.1. Construct the Hermitian intertwining operator

$$B = \delta(\omega) \circ R(\omega) \circ A(\bar{P} \colon P \colon \delta \colon \nu) \colon \mathcal{H}^P_\delta \longrightarrow \mathcal{H}^P_\delta$$

as instructed in the previous section. Because $\delta(\omega)$ and $R(\omega)$ are invertible, we can identify the kernel of B with the kernel of the standard intertwining operator $A(\bar{P} \colon P \colon \delta \colon \nu)$. By construction, B gives rise to an invariant Hermitian form on $\mathcal{H}_P(\delta \otimes \nu)$, and to a non degenerate Hermitian form $\langle\ ,\ \rangle$ on the quotient space:

$$J_P(\delta \otimes \nu) = \frac{\mathcal{H}_P(\delta \otimes \nu)}{\ker(A(\bar{P}\colon P\colon \delta\colon \nu))}.$$

Because $J_P(\delta \otimes \nu)$ is irreducible, the form $\langle\ ,\ \rangle$ is unique up to a non-zero real factor. We conclude that the Langlands quotient $J_P(\delta \otimes \nu)$ is infinitesimally unitary if and only if the invariant Hermitian form $\langle\ ,\ \rangle$ is definite, i.e. if and only if the operator B is semi-definite. $\square$

Chapter 4

The example of $SL(2, \mathbb{R})$

4.1 The data for $SL(2, \mathbb{R})$

In this section we briefly recall the data for the group $SL(2, \mathbb{R})$, and we fix the notations that will be used throughout the chapter.

- $G = SL(2, \mathbb{R})$

- $K = SO(2) = \left\{ k_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} : \vartheta \in \mathbb{R} \right\}$

- $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{a}_0 \oplus (\mathfrak{g}_0)_\alpha \oplus (\mathfrak{g}_0)_{-\alpha}$, with

$$\mathfrak{a}_0 = \mathbb{R}H_\alpha, \quad H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha = 2\varepsilon, \quad \rho = \varepsilon, \quad \varepsilon \colon \mathfrak{a}_0 \to \mathbb{R},\ tH_\alpha \mapsto t$$

$$(\mathfrak{g}_0)_\alpha = \mathbb{R}X_\alpha, \quad X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(\mathfrak{g}_0)_{-\alpha} = \mathbb{R}Y_\alpha, \quad Y_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

- $A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ a_y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y \in \mathbb{R},\, y > 0 \right\}$

- $N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

- $P = \left\{ \begin{pmatrix} b & r \\ 0 & b^{-1} \end{pmatrix} :\ b,\, r \in \mathbb{R},\, b \neq 0 \right\}$

- $\begin{pmatrix} b & r \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(b) & 0 \\ 0 & \operatorname{sgn}(b) \end{pmatrix} \begin{pmatrix} \mid b \mid & 0 \\ 0 & \mid b \mid^{-1} \end{pmatrix} \begin{pmatrix} 1 & r/b \\ 0 & 1 \end{pmatrix}$

  the Iwasawa decomposition of an element of P (where $\operatorname{sgn}(b) = b / \mid b \mid$)

- $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_\vartheta a_y n_x = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\text{with } e^{i\vartheta} = \frac{a - ic}{\sqrt{a^2 + c^2}} \text{ and } y = \sqrt{a^2 + c^2} \tag{4.1}$$

  the Iwasawa decomposition of an arbitrary element of G

- $\widehat{M} = \{\delta^{\pm}\}$, with δ^+ and δ^- respectively the trivial and the sign representation of M

- $(\mathfrak{a}_0')^{\mathbb{C}} = \mathbb{C}\varepsilon$, with $\varepsilon \colon \mathfrak{a}_0 \to \mathbb{R}$, $tH_\alpha \mapsto t$

- $\widehat{K} \simeq \mathbb{Z}$. For each integer n we denote by ξ_n the character:

$$\xi_n \colon K \to \mathbb{C},\ k_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mapsto e^{in\vartheta}.$$

4.2 Principal Series induced from the Minimal Parabolic

4.2.1 The Principal Series Representation $\pi_P(\delta \otimes \lambda)$

Fix an irreducible representation δ of M ($\delta = \delta^{\pm}$), and a complex linear functional $\lambda\varepsilon$ on $\mathfrak{a}_0$. We denote by $\pi_P(\delta \otimes \lambda)$ the principal series induced from the representation $(\delta \otimes \lambda\varepsilon)$ of the minimal parabolic subgroup $P = MAN$ (as usual, N acts trivially). The Hilbert space $\mathcal{H}^P_{(\delta\otimes\lambda)}$ of this representation is the completion with respect to the norm

$$\|F\|^2 = \int_K \mid f(k) \mid^2 \ dk$$

of the vector space of the continuous functions $F\colon G \to \mathbb{C}$ satisfying:

$$F(gman) = \delta(m)^{-1} e^{-(\lambda\varepsilon+\rho)\log(a)} F(g) \quad \forall\, g \in G, \forall\, man \in P.$$

In the notations introduced above, we can describe this transformation property by:

$$F\left(g\begin{pmatrix} b & r \\ 0 & b^{-1} \end{pmatrix}\right) = \delta\begin{pmatrix} \operatorname{sgn}(b) & 0 \\ 0 & \operatorname{sgn}(b) \end{pmatrix} \mid b \mid^{-\lambda-1} F(g)$$

for all b and r in $\mathbb{R}$, with b non-zero. G acts on $\mathcal{H}^P_{(\delta\otimes\lambda)}$ by left translation:

$$[g \cdot F](x) = F(g^{-1}x).$$

It is convenient to introduce also the compact picture for the principal series. In the compact picture, the Hilbert space is

$$\mathcal{H}^P_\delta = \{f \in L^2(K, \mathbb{C})\colon f(-g) = \delta\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} f(g)\}.$$

We notice that $\mathcal{H}^P_\delta$ is the space of all *even* functions in $L^2(K,\mathbb{C})$ when $\delta = \delta^+$, and it is the space of all *odd* functions in $L^2(K,\mathbb{C})$, when $\delta = \delta^-$.

Restriction to K is an isomorphism from $\mathcal{H}^P_{(\delta\otimes\lambda)}$ to $\mathcal{H}^P_\delta$. The inverse mapping carries an element f of $\mathcal{H}^P_\delta$ to the function

$$F\colon G \to \mathbb{C},\ g = k_\vartheta a_y n_x \mapsto F(k_\vartheta a_y n_x) = y^{-\lambda-1} f(k_\vartheta).$$

4.2.2 The restriction of $\pi_P(\delta \otimes \lambda)$ to K

By Frobenius reciprocity, the multiplicity of a K-type ξ_n in the restriction of $\pi_P(\delta\otimes\lambda)$ to K equals the multiplicity of δ in the restriction of ξ_n to M. Since

$$\xi_n \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = e^{in\pi} = \begin{cases} +1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases},$$

it follows immediately that

- if δ is the *trivial* representation of M, then the restriction of $\pi_P(\delta \otimes \lambda)$ to K contains only *even* characters, each occurring with multiplicity one:

$$\mathrm{Res}_K(\pi_P(\delta^+ \otimes \lambda)) = \bigoplus\nolimits_{n\in\mathbb{Z}} \xi_{2n}$$

- if δ is the *sign* representation of M, then the restriction of $\pi_P(\delta \otimes \lambda)$ to K contains only *odd* characters, each occurring with multiplicity one:

$$\mathrm{Res}_K(\pi_P(\delta^- \otimes \lambda)) = \bigoplus\nolimits_{n\in\mathbb{Z}} \xi_{2n+1}.$$

We now describe representatives for the various K-types in both the compact and the induced picture.

Assume that n has the same parity as δ. The function

$$f_n \colon K \to \mathbb{C},\ k_\varphi \mapsto f_n(k_\varphi) = e^{-in\varphi}$$

belongs to $\mathcal{H}^P_\delta$ and it transforms under K according to the character ξ_n:

$$[k_\vartheta \cdot f_n](k_\varphi) = f_n(k_\vartheta^{-1}k_\varphi) = e^{+in\vartheta} f_n(k_\varphi) = [\xi_n(k_\vartheta)\cdot f_n](k_\varphi).$$

In the induced picture, a generator for the K-type ξ_n is given by the function:

$$F_n \colon G \to \mathbb{C},\ g = k_\varphi a_y n_x \mapsto y^{-\lambda-1} e^{-in\varphi}.$$

4.3 Study of the irreducibility of $\pi_P(\delta \otimes \lambda)$

The trick is to look at the action of $\mathfrak{g}_0^{\mathbb{C}}$ in the compact picture of $\pi_P(\delta \otimes \lambda)$.
In $\mathfrak{g}_0^{\mathbb{C}}$ we pick the basis

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

which satisfies the bracket relations:

$$[H,\, X] = 2X \qquad [H,\, Y] = -2Y \qquad [X,\, Y] = H.$$

We let $\{H,\, X,\, Y\}$ act on the orthonormal basis $\{f_n\}$ of $\mathcal{H}_\delta^P$ (n runs over $2\mathbb{Z}$ when δ is the trivial representation of M, and over $2\mathbb{Z}+1$ when δ is the sign representation). We find that

1. H fixes each K-type: $\boxed{H \cdot f_n = n\, f_n}$

$$\begin{tikzcd} \cdots & \mathbb{C}f_{n-4} \arrow[loop above] & \mathbb{C}f_{n-2} \arrow[loop above] & \mathbb{C}f_{n} \arrow[loop above] & \mathbb{C}f_{n+2} \arrow[loop above] & \mathbb{C}f_{n+4} \arrow[loop above] & \cdots \end{tikzcd}$$

2. X pushes each K-type forward: $\boxed{X \cdot f_n = a_n\, f_{n+2} \qquad a_n = (\lambda + n + 1)/2}$

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{C}f_{n-4} \arrow[r] & \mathbb{C}f_{n-2} \arrow[r] & \mathbb{C}f_{n} \arrow[r] & \mathbb{C}f_{n+2} \arrow[r] & \mathbb{C}f_{n+4} \arrow[r] & \cdots \end{tikzcd}$$

3. Y pushes each K-type backwards: $\boxed{Y \cdot f_n = b_n\, f_{n-2} \qquad b_n = (\lambda - n + 1)/2}$

$$\begin{tikzcd} \cdots & \mathbb{C}f_{n-4} \arrow[l] & \mathbb{C}f_{n-2} \arrow[l] & \mathbb{C}f_{n} \arrow[l] & \mathbb{C}f_{n+2} \arrow[l] & \mathbb{C}f_{n+4} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

Proof. 1. $H \cdot f_n = n\, f_n$

The element $iH = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to the Lie algebra $\mathfrak{g}_0$ and the vector f_n is K-finite (hence smooth). So we can write:

$$(i\,H) \cdot f_n = \tfrac{d}{dt}\,|_{t=0}\, e^{itH} \cdot f_n = \tfrac{d}{dt}\,|_{t=0} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot f_n =$$

$$= \tfrac{d}{dt}\,|_{t=0}\, [e^{int} f_n] = inf_n.$$

The element H of $\mathfrak{g}_0^{\mathbb{C}}$ acts on each K-type ξ_n as scalar multiplication by n.

2. $X \cdot f_n = \frac{\lambda+n+1}{2}\, f_{n+2}$

From the commutator relation $[H,\, X] = 2X$ it follows immediately that $X \cdot f_n$ is an eigenvector of H of eigenvalue $n+2$. So $X \cdot f_n$ must belong to the isotypic component of the K-type ξ_{n+2}. Since this isotypic is one-dimensional and generated by f_{n+2}, there exists a constant a_n such that

$$X \cdot f_n = a_n f_{n+2}.$$

In particular, $X \cdot f_n(1) = a_n f_{n+2}(1) = a_n$.

To calculate $a_n = X \cdot f_n(1)$, we decompose X as a linear combination of elements of the Lie algebra:

$$X = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and we compute the action of each summand on the smooth vector f_n:

- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot f_n(1) = \tfrac{d}{dt}\,|_{t=0} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot f_n(1) = \tfrac{d}{dt}\,|_{t=0}\, a_t \cdot f_n(1)$

$$= \frac{d}{dt}\,|_{t=0}\, f_n(a_{-t}1) = \frac{d}{dt}\,|_{t=0}\, f_n(1a_{-t}) = \frac{d}{dt}\,|_{t=0}\, e^{-(\lambda+1)(-t)}\, f_n(1)$$

$$= +(\lambda+1) f_n(1) = (\lambda+1).$$

- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot f_n(1) = iH \cdot f_n(1) = in\, f_n(1) = in.$

- $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot f_n(1) = \frac{d}{dt}\,|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot f_n(1) = \frac{d}{dt}\,|_{t=0}\, f_n(1) = 0.$

This gives:

$$a_n = X \cdot f_n(1) = [\tfrac{1}{2}(\lambda+1) - \tfrac{i}{2} in] = \tfrac{\lambda+n+1}{2}$$

so $X \cdot f_n = \frac{\lambda+n+1}{2} f_{n+2}$. Finally, we prove that:

3. $Y \cdot f_n = \frac{\lambda-n+1}{2} f_{n-2}$

   Computations are similar to the previous case: since $[H, Y] = -2Y$, there exists a constant $b_n = Y \cdot f_n(1)$ such that $Y \cdot f_n = f_{n+2}$. Since

$$Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   we obtain:

$$b_n = Y \cdot f_n(1) = [\tfrac{1}{2}(\lambda+1) + \tfrac{i}{2} in] = \tfrac{\lambda+1-n}{2}.$$

$\square$

We now go back to the problem of discussing the irreducibility of the principal series representation $\pi_P(\delta \otimes \lambda)$. Of course, we need to distinguish between the cases" $\lambda =$ trivial" and" $\lambda =$ sign."

4.3.1 Irreducibility of $\pi_P(\delta^+ \otimes \lambda)$

The action of $\mathfrak{g}_0^{\mathbb{C}}$ on the Hilbert space $\mathcal{H}^P_{\delta^+} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C} f_{2m}$ is given by:

$$H \cdot f_{2m} = 2m\, f_{2m}$$

$$X \cdot f_{2m} = \tfrac{(\lambda+2m+1)}{2}\, f_{2m+2}$$

$$Y \cdot f_{2m} = \tfrac{(\lambda-2m+1)}{2}\, f_{2m-2}.$$

If λ is *not an odd integer*, the representation $\pi_P(\delta^+ \otimes \lambda)$ is clearly *irreducible.*

If λ is *a positive odd integer*, say $\lambda = 2m_0 - 1$ with $m_0 \geqslant 1$, the representation $\pi_P(\delta^+ \otimes \lambda)$ is *reducible.* There are *three* proper invariant subspaces:

- $\bigoplus_{m\geqslant m_0} \mathbb{C} f_{2m}$

  (invariance follows from $Y \cdot f_{2m_0} = 0$)

- $\bigoplus_{m\geqslant m_0} \mathbb{C} f_{-2m}$

  (invariance follows from $X \cdot f_{-2m_0} = 0$)

- $\left(\bigoplus_{m\geqslant m_0} \mathbb{C} f_{-2m}\right) \oplus \left(\bigoplus_{m\geqslant m_0} \mathbb{C} f_{2m}\right)$.

If λ is *a negative odd integer*, say $\lambda = -2m_0 - 1$ with $m_0 \geqslant 0$, then the representation $\pi_P(\delta^+ \otimes \lambda)$ is *reducible.* There are three proper invariant subspaces:

- $\bigoplus_{-m_0\leqslant m\leqslant m_0} \mathbb{C} f_{2m}$

  (invariance follows from $X \cdot f_{+2m_0} = Y \cdot f_{-2m_0} = 0$)

- $\bigoplus_{-m_0\leqslant m} \mathbb{C} f_{2m}$

- $\bigoplus_{m\leqslant m_0} \mathbb{C} f_{2m}$.

4.3.2 Irreducibility of $\pi_P(\delta^+ \otimes \lambda)$

The action of $\mathfrak{g}_0^{\mathbb{C}}$ on the Hilbert space $\mathcal{H}^P_{\delta^+} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}f_{2m-1}$ is given by:

$$H \cdot f_{2m-1} = (2m-1)\, f_{2m-1}$$

$$X \cdot f_{2m-1} = \tfrac{\lambda+2m}{2}\, f_{2m+1}$$

$$Y \cdot f_{2m-1} = \tfrac{\lambda-2(m-1)}{2}\, f_{2m-3}.$$

If λ is *not an even integer*, the representation $\pi_P(\delta^+ \otimes \lambda)$ is clearly *irreducible.*

If λ is *a positive even integer*, say $\lambda = 2m_0$ with $m_0 \geqslant 1$, then the representation $\pi_P(\delta^- \otimes \lambda)$ is *reducible.* There are *three* proper invariant subspaces:

- $\bigoplus_{m\geqslant m_0} \mathbb{C}f_{2m+1}$

  (invariance follows from $Y \cdot f_{2m_0+1} = 0$)

- $\bigoplus_{m\geqslant m_0} \mathbb{C}f_{-2m-1}$

  (invariance follows from $X \cdot f_{-2m_0-1} = 0$)

- $\left(\bigoplus_{m\geqslant m_0} \mathbb{C}f_{-2m-1}\right) \oplus \left(\bigoplus_{m\geqslant m_0} \mathbb{C}f_{2m+1}\right).$

If λ is *zero*, then the representation $\pi_P(\delta^- \otimes 0)$ is *reducible.* There are *two* proper invariant subspaces:

- $\bigoplus_{m\geqslant 0} \mathbb{C}f_{2m+1}$

  (invariance follows from $Y \cdot f_{+1} = 0$)

- $\bigoplus_{m\geqslant 0} \mathbb{C}f_{-2m-1}$

  (invariance follows from $X \cdot f_{-1} = 0$).

If λ is *a negative even integer*, say $\lambda = -2m_0$ with $m_0 \geqslant 0$, then the representation $\pi_P(\delta^- \otimes \lambda)$ is *reducible.* There are three proper invariant subspaces:

- $\bigoplus_{-m_0+1\leqslant m\leqslant m_0} \mathbb{C}f_{2m-1}$

  (invariance follows from $X \cdot f_{2m_0-1} = Y \cdot f_{-2m_0+1} = 0$)

- $\bigoplus_{-m_0+1\leqslant m} \mathbb{C}f_{2m-1}$

- $\bigoplus_{m\leqslant m_0} \mathbb{C}f_{2m-1}$.

4.4 Study of the Unitarity of $\pi_P(\delta \otimes \nu)$

Let $P = MAN$ be a minimal parabolic subgroup of G, let δ be an irreducible representation of M and let ν be a strictly dominant character of A. A principal series representation $\pi_P(\delta \otimes \nu)$ gives rise to a unitary Langlands quotient if and only if

(i) the formal symmetry conditions hold: there exists ω in K normalizing $\mathfrak{a}_o$ with $\omega\, P\, \omega^{-1} = \bar{P}$, $\omega \cdot \delta \simeq \delta$ and $\omega \cdot \nu = -\bar{\nu}$ and

(ii) the Hermitian intertwining operator

$$B = \delta(\omega)\, R(\omega)\, A(\bar{P} \colon P \colon \delta \colon \nu)$$

is positive or negative semi-definite.

When P is the minimal parabolic, the condition $\omega\, P\, \omega^{-1} = \bar{P}$ implies that ω is a representative in K for the long Weyl group element (the reflection s_α). So we can take

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = k_{\pi/2}.$$

It is easy to check that ω stabilizes both the trivial and the sign representation of M, so there are no conditions on δ. Let us now look at the conditions imposed on ν.

Write $\nu = \lambda\,\varepsilon$, so that

$$\nu\colon \mathfrak{a}_0 \to \mathbb{C},\ H_t = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mapsto \lambda\, t.$$

Since

$$\omega \cdot \nu(H_t) = \nu(k_{-\pi/2} H_t k_{+\pi/2}) = \nu(H_{-t}) = -\lambda\, t$$

it immediately follows that $\nu = \lambda\,\varepsilon$ satisfies the condition $\omega \cdot \nu = -\bar{\nu}$ if and only if λ is real. And of course λ must be strictly positive, because we want the character $\lambda\,\varepsilon$ of A to be in the open positive Weyl chamber determined by N.

For $\delta = \delta^+$ or δ^-, $\nu = \lambda\,\varepsilon$ (with $\lambda > 0$) and for $\omega = k_{\pi/2}$, we have to study the signature of the Hermitian intertwining operator

$$B = \delta(\omega)\, R(\omega)\, A(\bar{P}\colon P\colon \delta\colon \lambda).$$

Notice that $\omega^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\delta(\omega^2) = +1$ if δ is trivial, and $\delta(\omega^2) = -1$ if δ is the sign representation. We take the operator $\delta(\omega)$ to be the identity if δ is trivial, and scalar multiplication by i if δ is sign. In any case, B is a multiple of the operator:

$$A_\omega\colon \mathcal{H}^P_{\delta\otimes\lambda} \to \mathcal{H}^P_{\delta\otimes-\lambda}, F \mapsto A_\omega \cdot F$$

where $(A_\omega \cdot F)(x) = \int_{\bar{N}} F(x\,\omega\,\bar{n})\, d\bar{n}$.

The operator A_ω can be computed K-type by K-type, and since $\mathcal{H}^P_{\delta^+\otimes\lambda}$ and $\mathcal{H}^P_{\delta^-\otimes\lambda}$ have different K-types, it is appropriate to to distinguish between the spherical and the non-spherical case.

4.4.1 The Spherical Case

We use for both $\pi_P(\delta^+ \otimes \lambda)$ and $\pi_P(\delta^+ \otimes -\lambda)$ the compact picture, and we identify domain and codomain of A_ω with the Hilbert space:

$$\mathcal{H}^P_{\delta^+} = \bigoplus_{m\in\mathbb{Z}} \xi_{2m} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\, F_{2m}.$$

The intertwining operator A_ω preserves the decomposition of $\mathcal{H}^P_{\delta^+}$ in isotypical components of K-types, and since each isotypical is one-dimensional, A_ω acts as scalar multiplication on each K-type.

For each integer m, and for each k_φ in K, we want to compute the integral

$$A_\omega \cdot F_{-2m}(k_\varphi) = \int_{\bar{N}} F_{-2m}(k_\varphi\, \omega\, \bar{n})\, d\bar{n}$$

To proceed, we need to understand the Iwasawa decomposition of $(k_\varphi\, \omega\, \bar{n})$. Let $\bar{n} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, and let $\bar{n} = k_{\theta(t)} a_{y(t)} n$ be its Iwasawa decomposition. Then

$$y(t) = \sqrt{1+t^2} \quad \text{ and } \quad e^{i\,\theta(t)} = \tfrac{1-i\,t}{\sqrt{1+t^2}} \quad (\tan\theta(t) = -t).$$

Since $\omega = k_{\pi/2}$, we obtain for $(k_\varphi\, \omega\, \bar{n})$ the Iwasawa decomposition

$$k_\varphi\, \omega\, \bar{n} = k_{\varphi+\theta(t)+\pi/2}\, a_{y(t)}\, n.$$

Therefore

$$A_\omega \cdot F_{-2m}(k_\varphi) = \textstyle\int_{\mathbb{R}} y(t)^{-1-\lambda} f_{-2m}(k_{\varphi+\theta(t)+\pi/2})\, dt =$$

$$= \textstyle\int_{\mathbb{R}} \left(\sqrt{1+t^2}\right)^{-1-\lambda} \; e^{2m\, i(\varphi+\theta(t)+\pi/2)}\, dt$$

$$= (-1)^m f_{-2m}(k_\varphi) \int_{\mathbb{R}} (\sqrt{1+t^2})^{-1-\lambda}\, e^{2m\,i\,\theta(t)}\, dt$$

$$= (-1)^m F_{-2m}(k_\varphi) \int_{\mathbb{R}} (\sqrt{1+t^2})^{-1-\lambda} \left(\frac{1-i\,t}{\sqrt{1+t^2}}\right)^{2m} dt. \tag{4.2}$$

We perform the change of variable $t = -\tan\theta$, so that

$$\theta = -\arctan(t) \qquad (\text{with } -\pi/2 \leqslant \theta \leqslant +\pi/2).$$

This gives:

$$\sqrt{1+t^2} = \tfrac{1}{\cos\theta}$$

$$\tfrac{1-i\,t}{\sqrt{1+t^2}} = \cos\theta + i\,\tan\theta\,\cos\theta = e^{i\,\theta}$$

$$d\,t = -\tfrac{1}{(\cos\theta)^2}\, d\,\theta.$$

Finally, we get:

$$A_\omega \cdot F_{-2m}(k_\varphi) = (-1)^m F_{-2m}(k_\varphi) \int_{-\pi/2}^{\pi/2} (\cos\theta)^{1+\lambda}\, e^{2m\,i\,\theta}\, \tfrac{1}{(\cos\theta)^2}\, d\,\theta$$

$$= (-1)^m F_{-2m}(k_\varphi) \int_{-\pi/2}^{\pi/2} (\cos\theta)^{\lambda-1}\, e^{2m\,i\,\theta}\, d\,\theta.$$

With the change of variable $\theta \to x = \theta + \pi/2$, we get

$$A_\omega \cdot F_{-2m}(k_\varphi) = F_{-2m}(k_\varphi) \int_0^\pi (\sin\, x)^{\lambda-1}\, e^{2m\,i\,x}\, d\,x.$$

To conclude, we make use of the following result (see e.g. [10]):

$$\boxed{\int_0^\pi (\sin\, t)^a\, e^{i\,b\,t}\, d\,t = \frac{\pi\,\Gamma(1+a)\, e^{i\,\pi\,b/2}}{2^a\,\Gamma(1+\frac{a+b}{2})\,\Gamma(1+\frac{a-b}{2})}}$$

for each b in $\mathbb{R}$, and for each a in $\mathbb{C}$ such that $\mathrm{Re}(a) > -1$. We obtain:

$$A_\omega \cdot F_{-2m} = F_{-2m} \, \frac{\pi \, \Gamma(\lambda) \, e^{i\pi m}}{2^{\lambda-1} \, \Gamma\left(1+\frac{\lambda+2m-1}{2}\right) \Gamma\left(1+\frac{\lambda-2m-1}{2}\right)}$$

$$= F_{-2m} \, \frac{\pi \, \Gamma(\lambda) \, e^{i\pi m}}{2^{\lambda-1} \, \Gamma\left(\frac{\lambda+1}{2}+m\right) \Gamma\left(\frac{\lambda+1}{2}-m\right)}.$$

It is a convention to normalize the operator so that it acts trivially on the fine K-type $\mathbb{C} f_0$. Dividing by the constant

$$C = A_\omega \cdot F_0(1) = \frac{\pi \, \Gamma(\lambda)}{2^{\lambda-1} \, \Gamma\left(\frac{\lambda+1}{2}\right) \, \Gamma\left(\frac{\lambda+1}{2}\right)},$$

we get:

$$\frac{1}{C} A_\omega \cdot F_{-2m} = F_{-2m} \, \frac{e^{i\pi m} \, \Gamma\left(\frac{\lambda+1}{2}\right) \Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}+m\right) \Gamma\left(\frac{\lambda+1}{2}-m\right)}$$

$$= (-1)^m \, F_{-2m} \, \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}+m\right)} \, \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}-m\right)}.$$

So the operator $\frac{1}{C} A_\omega$ acts on the K-type ξ_{-2m} as scalar multiplication by

$$d_{-2m} = (-1)^m \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}+m\right)} \, \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}-m\right)}.$$

To simplify this expression we recall the factorization property of the Γ function:

$$\Gamma(z+1) = z \, \Gamma(z)$$

and we introduce the notation

$$(z)_n = z(z+1)(z+2) \cdots (z+n-1)$$

for each z in $\mathbb{C}$, and for every positive integer n. Then

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(z+n)\,\Gamma(z-n)} = \frac{\Gamma(z)\,(z-n)_n\,\Gamma(z-n)}{(z)_n\,\Gamma(z)\,\Gamma(z-n)} = \frac{(z-n)_n}{(z)_n} = \frac{(z-1)(z-2)\cdots(z-n)}{z(z+1)\cdots(z+n-1)}.$$

Setting $z = \frac{\lambda+1}{2}$ and $n = \mid m \mid$, we find:

$$d_{2m} = d_{-2m} = (-1)^m \frac{(\lambda-1)(\lambda-3)\cdots(\lambda-2m+1)}{(\lambda+1)(\lambda+3)\cdots(\lambda+2m-1)} = \frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)}.$$

Conclusions. Fix λ in $\mathbb{C}$ with positive real part. If we use the compact picture for both $\pi_P(\delta^+ \otimes \lambda)$ and $\pi_P(\delta^+ \otimes \omega \cdot \lambda)$, we can identify domain and codomain of the intertwining operator

$$A_\omega \colon \mathcal{H}^P_{\delta^+ \otimes \lambda} \longrightarrow \mathcal{H}^P_{\delta^+ \otimes \omega \cdot \lambda}$$

with the Hilbert space $\mathcal{H}^P_{\delta^+} = \bigoplus_{m \in \mathbb{Z}} \xi_{2m}$.

The operator A_ω preserves this decomposition and acts on the K-types ξ_{2m} and ξ_{-2m} as multiplication by the scalar

$$\boxed{C\,\frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)}}$$

where C is the value of A_ω on the fine K-type $\xi_0 = \mathbb{C}\,F_0$:

$$C = A_\omega \cdot F_0(1) = \frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda+1}{2}\right)\,\Gamma\left(\frac{\lambda+1}{2}\right)}.$$

Since C is real and positive when λ is real and positive, A_ω has the same signature as $\frac{1}{C}A_\omega$ (with respect to all K-types). The signature of the operator $B = A_\omega$ on a K-type ξ_{2m} is simply given by the sign of the scalar

$$d_{2m} = \frac{(1-\lambda)(3-\lambda)\cdots(2\mid m\mid -1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2\mid m\mid -1+\lambda)}.$$

We notice that

- if $0 < \lambda < 1$, then $d_{2m} > 0$ for all integers m:

  because $d_{2m} = d_{-2m}$, we just look at the case $m \geqslant 0$. Then it is clear that $d_{2m} = \frac{(1-\lambda)(3-\lambda)\cdots(2m-1-\lambda)}{(1+\lambda)(3+\lambda)\cdots(2m-1+\lambda)} > 0$ for $0 < \lambda < 1$.

- if $\lambda = 1$, then $d_{2m} \geqslant 0$ for all integers m:

  precisely, $d_0 = 1$ and $d_{2m} = 0$ otherwise.

- if $\lambda > 1$, then the integers $\{d_{2m}\}$ are neither all non-negative, nor all non-positive:

  for instance, $d_0 = 1 > 0$, while $d_2 = \frac{(1-\lambda)}{(1+\lambda)} < 0$.

So the Hermitian operator B on $\mathcal{H}_P(\delta^+ \otimes \lambda)$ is positive definite when $0 < \lambda < 1$, positive semi-definite when $\lambda = 1$ and indefinite when $\lambda > 1$. As a result we obtain the following theorem:

Theorem 7. *Let P be a minimal parabolic subgroup of $SL(2, \mathbb{R})$, let δ^+ be the trivial representation of M and let $\lambda > 0$ be a strictly dominant character of A. The Langlands quotient $J_P(\delta^+ \otimes \lambda)$ is infinitesimally unitary if and only if $0 < \lambda \leqslant 1$. When $\lambda = 1$, $J_P(\delta^+ \otimes 1)$ is the trivial representation of $SL(2, \mathbb{R})$.*

We conclude the analysis of the spherical case with a remark:

Remark 1. *The Hermitian operator B on $\mathcal{H}_P(\delta^+ \otimes \lambda)$ is positive semi-definite if and only if it has non negative signature with respect to the K-types ξ_0, ξ_2 and ξ_{-2}.*

Indeed, the previous analysis of the sign of the integers $\{d_{2m}\}$ shows that

$$d_{2m} \geqslant 0 \quad \forall m \in \mathbb{Z} \Leftrightarrow d_0,\, d_2,\, d_{-2} > 0.$$

4.4.2 The Non-Spherical Case

In the non-spherical case, $B = iA_\omega$ with

$$A_\omega \colon \mathcal{H}^P_{\delta^- \otimes \lambda} \to \mathcal{H}^P_{\delta^- \otimes -\lambda}$$

Using the compact picture for each principal series representation, we regard A_ω as an endomorphism on

$$\mathcal{H}^P_{\delta^-} = \bigoplus_{m \in \mathbb{Z}} \xi_{2m+1} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}\, F_{2m+1}.$$

Being an intertwining operator, A_ω acts on (the isotypical component of) each K-type as multiplication by a scalar. We want to compute this scalar explicitly.

$$A_\omega \cdot F_{-2m-1}(1) = \int_{\bar{N}} F_{-2m-1}(\omega\, \bar{n})\, d\bar{n}$$

$$= \int_{\mathbb{R}} y(t)^{-1-\lambda}\, f_{-2m-1}(k_{\theta(t)+\pi/2})\, dt$$

$$= \int_{\mathbb{R}} (\sqrt{1+t^2})^{-1-\lambda}\, e^{(2m+1)\, i(\theta(t)+\pi/2)}\, dt$$

$$= e^{(2m+1)i\pi/2}\, f_{-2m-1}(1) \int_{\mathbb{R}} (\sqrt{1+t^2})^{-1-\lambda}\, e^{(2m+1)\, i\, \theta(t)}\, dt$$

$$= e^{(2m+1)i\pi/2}\, F_{-2m-1}(1) \int_{\mathbb{R}} (\sqrt{1+t^2})^{-1-\lambda} \left(\tfrac{1-i\,t}{\sqrt{1+t^2}}\right)^{2m+1} dt$$

$$= e^{(2m+1)i\pi/2}\, F_{-2m-1}(1) \int_{-\pi/2}^{\pi/2} (\cos\theta)^{1+\lambda}\, e^{(2m+1)\, i\, \theta}\, \tfrac{1}{(\cos\theta)^2}\, d\,\theta$$

$$= e^{(2m+1)i\pi/2}\, F_{-2m-1}(1) \int_{-\pi/2}^{\pi/2} (\cos\theta)^{\lambda-1}\, e^{(2m+1)\, i\, \theta}\, d\,\theta$$

$$= F_{-2m-1}(1) \int_0^{\pi} (\sin\, x)^{\lambda-1}\, e^{(2m+1)\, i\, x}\, d\, x$$

$$= F_{-2m-1}(1)\, \frac{\pi\,\Gamma(\lambda)\, e^{i(2m+1)\pi/2}}{2^{\lambda-1}\,\Gamma\left(1+\frac{\lambda+2m}{2}\right)\Gamma\left(1+\frac{\lambda-2m-2}{2}\right)}$$

$$= F_{-2m-1}(1)\, \frac{\pi\,\Gamma(\lambda)\, e^{i(2m+1)\pi/2}}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}+m+1\right)\Gamma\left(\frac{\lambda}{2}-m\right)}.$$

In particular,

$$D \equiv A_\omega \cdot F_1(1) = F_1(1)\, \frac{\pi\,\Gamma(\lambda)\, e^{i(-1)\pi/2}}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)} = e^{-i\pi/2}\, \frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)}.$$

Then the operator $\frac{1}{D}A_\omega$ acts on the K-type ξ_{-2m-1} as scalar multiplication by

$$d_{-2m-1} = (-1)^{m+1}\, \frac{\Gamma\left(\frac{\lambda}{2}+1\right)}{\Gamma\left(\frac{\lambda}{2}+1+m\right)}\, \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}-m\right)}.$$

This expression is not symmetric with respect to m. We distinguish various cases:

- $d_1 = 1$
- $d_{-1} = -1$
- if $m > 0$, then $d_{-2m-1} = (-1)^{m+1} \frac{\left(\frac{\lambda}{2}-m\right)_m}{\left(\frac{\lambda}{2}+1\right)_m} = (-1)^m \frac{(\lambda-2)(\lambda-4)\cdots(\lambda-2m)}{(\lambda+2)(\lambda+4)\cdots(\lambda+2m)}$
- if $m < 0$, then $d_{-2m-1} = (-1)^{m+1} \frac{\left(\frac{\lambda}{2}+1-(-m)\right)_{-m}}{\left(\frac{\lambda}{2}\right)_{-m}} = (-1)^m \frac{(\lambda-2)(\lambda-4)\cdots(\lambda-2(-m)+2)}{(\lambda+2)(\lambda+4)\cdots(\lambda+2(-m)-2)}$

  .

Shifting indices, we find:

$$d_{2m+1} = d_{-2(-m-1)-1} = (-1)^m \frac{(\lambda-2)(\lambda-4)\cdots(\lambda-2m)}{(\lambda+2)(\lambda+4)\cdots(\lambda+2m)} = -d_{-(2m+1)}$$

for each *positive* integer m.

Conclusions. Fix λ in $\mathbb{C}$ with positive real part. If we use the compact picture for both $\pi_P(\delta^- \otimes \lambda)$ and $\pi_P(\delta^- \otimes \omega \cdot \lambda)$, we can identify domain and codomain of the intertwining operator

$$A_\omega \colon \mathcal{H}^P_{\delta^-\otimes\lambda} \longrightarrow \mathcal{H}^P_{\delta^-\otimes\omega\cdot\lambda}$$

with the Hilbert space $\mathcal{H}^P_{\delta^-} = \bigoplus_{m\in\mathbb{Z}} \xi_{2m+1}$.

The operator A_ω preserves this decomposition, and it acts on each K-type ξ_j as multiplication by a scalar (that we call b_j).

In the previous notations, $b_j = D\cdot d_j$, so we have:

$$\boxed{b_1 = -b_{-1} = D}$$

and, for each *positive* integer m,

$$\boxed{b_{2m+1} = -b_{-(2m+1)} = (-1)^m D \frac{(\lambda-2)(\lambda-4)\cdots(\lambda-2m)}{(\lambda+2)(\lambda+4)\cdots(\lambda+2m)}.}$$

We have denoted by D the value of A_ω on the fine K-type $\xi_1 = \mathbb{C}\,F_1$:

$$D = e^{-i\pi/2}\,\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)}.$$

The operator $B = iA_\omega$ acts on the K-type ξ_{2m+1} as scalar multiplication by

$$\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\,\Gamma\left(\frac{\lambda}{2}+1\right)}\,d_{2m+1}$$

which is actually a real number (in agreement to the fact that B is Hermitian).

Since $\frac{\pi\,\Gamma(\lambda)}{2^{\lambda-1}\,\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+1\right)} > 0$, the signature of B is determined by the sign of the integers $\{d_{2m+1}\}$. Because

$$d_1 = 1 \;\text{ and }\; d_{-1} = -1$$

for all values of λ, B is always indefinite.

Theorem 8. *Let P be a minimal parabolic subgroup of $SL(2,\mathbb{R})$, let δ^- be the sign* *representation of M and let $\lambda > 0$ be a strictly dominant character of A. The Langlands quotient $J_P(\delta^- \otimes \lambda)$ is never infinitesimally unitary.*

Chapter 5

Signatures of Invariant Hermitian Forms on an Admissible $(\mathfrak{g}_0^{\mathbb{C}}, K)$ module

The aim of this chapter is to discuss the signature of an invariant Hermitian form on an admissible $(\mathfrak{g}_0^{\mathbb{C}}, K)$ module with respect to a given K-type.
We start by recalling some general results about Hermitian forms on finite-dimensional vector spaces.

5.1 The finite-dimensional case

5.1.1 Signature of a Hermitian operator on a f.d. vector space

Let V be a finite-dimensional vector space with a Hermitian inner product $\langle\ ,\ \rangle_{PD}$ (we use the notation "PD" to distinguish an inner product from an arbitrary, not necessarily positive definite, Hermitian form).

Let T be a Hermitian linear operator on V, so that

$$\langle Tx,\, y\rangle_{PD} = \langle x,\, Ty\rangle_{PD}$$

for all x and y in V. By the spectral theorem, T is diagonalizable. It has real eigenvalues and orthogonal (distinct) eigenspaces.
We define the signature of T to be the triple $(n^+,\, n^-,\, n^0)$, where

n^+ = the number of positive eigenvalues of T, counting multiplicities;

n^- = the number of negative eigenvalues of T, counting multiplicities;

n^0 = dimension of the kernel of T = multiplicity of the eigenvalue zero.

Since T is diagonalizable, we obtain a decomposition of V in direct sum of orthogonal eigenspaces. Collecting all the eigenspaces with positive eigenvalues (V^+), the eigenspaces with negative eigenvalues (V^-) and then adding the zero eigenspace (V^0), we can write:

$$V = V^+ \oplus V^- \oplus V^0$$

with $\dim(V^+) = n^+,\ \dim(V^-) = n^-,\ \dim(V^0) = n^0$, and

T positive definite on V^+

T negative definite on V^-

T identically zero on V^0.

5.1.2 Signature of a Hermitian form on a f.d. vector space

Let V be a finite-dimensional vector space with a Hermitian inner product $\langle\ ,\ \rangle_{PD}$. There exists a one-one correspondence between Hermitian forms on $V \times V$ and linear operators $V \to V$ which are Hermitian with respect to $\langle\ ,\ \rangle_{PD}$. The correspondence being:

$$T \quad \longmapsto \quad \langle\ ,\ \rangle_T$$

with $\langle x,\, y\rangle_T \;=\; \langle Tx,\, y\rangle_{PD}$ for all x and y in V. The inverse mapping is

$$\langle\ ,\ \rangle \quad \longmapsto \quad T_{\langle\,,\,\rangle}$$

with $T_{\langle\,,\,\rangle}$ the Hermitian operator constructed as follows: fix an orthonormal basis $\{v_j\}$ with respect to $\langle\ ,\ \rangle_{PD}$; the matrix of $T_{\langle\,,\,\rangle}$ in this basis is given by $(T_{\langle\,,\,\rangle})_{i,j} = \langle v_j,\, v_i\rangle$.

If $\langle\ ,\ \rangle$ is a Hermitian form on V, we define the signature of $\langle\ ,\ \rangle$ to be the signature of the corresponding Hermitian operator T. This definition makes sense only if we prove that it is independent of the choice of the Hermitian inner product $\langle\ ,\ \rangle_{PD}$. So, we fix another Hermitian inner product $(\ ,\ )_{PD}$ on V, and we let S be the Hermitian operator corresponding to $\langle\ ,\ \rangle$ with respect to $(\ ,\ )_{PD}$. There exists an automorphism C of V relating the two Hermitian inner products:

$$\langle x,\, y\rangle_{PD} \;=\; (Cx,\, Cy)_{PD}$$

for all x and y in V. Since

$$\langle Tx,\, y\rangle_{PD} \;=\langle x,\, y\rangle_T \;=\; (Sx,\, y)_{PD} \;=\; \langle CSx,\, Cy\rangle_{PD}$$

for all x and y in V, we deduce that $T = C^*CS$. We need to prove that T and S have the same signature. Notice that

- since T and S are both Hermitian, the operator $CSC^{-1} \;=\; (C^{-1})^*TC^{-1}$ is Hermitian

- the Hermitian operators S and CSC^{-1} have the same signatures, because they have the same eigenvalues

- the Hermitian operators T and CSC^{-1} also have the same signature, because they give rise to equivalent quadratic forms. Indeed, since $CSC^{-1} = (C^{-1})^*TC^{-1}$, the change of variable $x \mapsto y = Cx$ gives:

$$\langle y, y\rangle_{CSC^{-1}} = \langle CSC^{-1}y, y\rangle_{PD} = \langle (C^{-1})^{*}TC^{-1}y, y\rangle_{PD} = \langle TC^{-1}y, C^{-1}y\rangle_{PD} =$$

$$= \langle Tx, x\rangle_{PD} = \langle x, x\rangle_{T}.$$

Therefore, the operators T and S have the same signature, and our definition of signature of an Hermitian form on a finite dimensional vector space makes sense.

The next step is to discuss the signature of an *invariant* Hermitian form on an admissible $(\mathfrak{g}_{\mathfrak{o}}^{\mathbb{C}}, K)$ module with respect to a K-type.

5.2 Signature of an invariant Hermitian form on an admissible $(\mathfrak{g}_{\mathfrak{o}}^{\mathbb{C}}, K)$ module w.r.t. a K-type

Let V be a $(\mathfrak{g}_{\mathfrak{o}}^{\mathbb{C}}, K)$ module. An invariant Hermitian form on V is a sesquilinear pairing $V \otimes V \to \mathbb{C}$ satisfying:

(i) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(ii) $\langle X \cdot v, w\rangle = -\langle v, X \cdot w\rangle$

(iii) $\langle k \cdot v, k \cdot w\rangle = \langle v, w\rangle$

for all v, w in V, for all X in $\mathfrak{g}_0$ and for all k in K. In other words, it is a Hermitian form on V with respect to which K acts by orthogonal operators and $\mathfrak{g}_0$ acts by skew-symmetric operators.[1]

The definition of positive/negative definite (or semi-definite) invariant Hermitian form is the standard one.

[1]Equivalently, you can ask that K acts by orthogonal operators and that $\mathfrak{g}_0^{\mathbb{C}}$ acts by skew-Hermitian operators. Indeed, if you write $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, and define conjugation accordingly, then condition (ii) is equivalent to:

$$\langle X \cdot v, w\rangle = -\langle v, \overline{X} \cdot w\rangle$$

for all X in $\mathfrak{g}_0^{\mathbb{C}}$.

Remark 2. *The Harish-Chandra module of a unitary irreducible representation of G has a positive definite invariant Hermitian form. The converse is less trivial, but also true: for G reductive, any irreducible $(\mathfrak{g}_0^{\mathbb{C}}, K)$ module admitting a positive definite invariant Hermitian form is the Harish-Chandra module of a (unique) irreducible unitary representation of G.*

We now discuss how to determine whether an invariant Hermitian form on an *admissible* $(\mathfrak{g}_0^{\mathbb{C}}, K)$ module is positive semi-definite.[2]
Suppose that (π, V) is an admissible $(\mathfrak{g}_0^{\mathbb{C}}, K)$ module with an invariant Hermitian form $\langle\ ,\ \rangle$, and that V has a Hilbert space structure. Since K is compact, we can always assume that K acts unitarily with respect to the Hermitian inner product $\langle\ ,\ \rangle_{PD}$ on V. Then, by the Peter Weyl theorem, V decomposes the orthogonal direct sum of the K isotypical components:

$$V = \bigoplus_{\mu \in K^*} V(\mu).$$

Since the representation π is assumed to be admissible, each K-type μ has finite multiplicity and each isotypic $V(\mu)$ has finite dimension (equal to $\dim(\mu) \cdot m_V(\mu)$). Moreover, the isotypics of two distinct K-types are orthogonal with respect to $\langle\ ,\ \rangle_{PD}$. We show that they are also orthogonal with respect to the invariant Hermitian form $\langle\ ,\ \rangle$:
let (μ_1, E_{μ_1}) and (μ_2, E_{μ_2}) be non equivalent irreducible representations of K that appear in V, and let $W = E_{\mu_1} \oplus E_{\mu_2}$ be their direct sum. The finite dimensional vector space W inherits by restriction the two Hermitian forms $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_{PD}$ (the latter is positive definite).

[2]The results we find have an obvious analogous for negative semi-definite forms and for positive or negative definite forms. We focus on positive definite forms, because in order to detect the unitarity of a Langlands quotient $J_P(\delta \otimes \nu)$ one needs to verify that an invariant Hermitian form on $\mathcal{H}_P(\delta \otimes \nu)$ is positive definite.

There exists a linear operator L on W, Hermitian with respect to $\langle\ ,\ \rangle_{PD}$, such that

$$\langle x,\, y\rangle \;=\; \langle Lx,\, y\rangle_{PD}$$

for all x and y in W. Since both $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_{PD}$ are invariant under K, L is a self-intertwining operator for the action of K on W:

$$\langle Lx,\, y\rangle_{PD} \;=\; \langle x,\, y\rangle \;=\; \langle k\cdot x,\, k\cdot y\rangle$$

$$=\; \langle L\,k\cdot x,\, k\cdot y\rangle_{PD} \;=\; \langle k^{-1}L\,k\cdot x,\, y\rangle_{PD}$$

for all x and y in W. It easily follows that $L = k^{-1}L\,k$ (for any k in K), because $\langle\ ,\ \rangle_{PD}$ is non degenerate.

Since μ_1 and μ_2 are irreducible and inequivalent, any self-intertwining operator on $W = E_{\mu_1} \oplus E_{\mu_2}$ must be of the form $c_1\mathbb{I}_{E_{\mu_1}} \oplus c_2\mathbb{I}_{E_{\mu_2}}$. So for all vectors $v = v_1 + v_2$ and $u = u_1 + u_2$ in W, we have:

$$\langle u,\, v\rangle \;=\; \langle L(u_1+u_2),\, (v_1+v_2)\rangle_{PD} \;=\; \langle c_1u_1 + c_2u_2,\, (v_1+v_2)\rangle_{PD}$$

$$=\; c_1\langle u_1,\, v_1\rangle_{PD} + c_2\langle u_2,\, v_2\rangle_{PD}.$$

We have used the fact that E_{μ_1} and E_{μ_2} are orthogonal with respect to $\langle\ ,\ \rangle_{PD}$. In particular, $\langle u,\, v\rangle = 0$ if u is in E_{μ_1} and v is in E_{μ_2}. This shows that any two distinct K-isotypics are also orthogonal with respect to $\langle\, ,\, \rangle$. As a consequence:

$$\boxed{\langle\, ,\, \rangle \text{ is positive semi-definite on } V \Leftrightarrow \text{ its restriction to each } K\text{-isotypic is such}}\,.$$

This reduction is very advantageous: since any K-isotypic $V(\mu)$ is finite-dimensional, we can use the signature to discriminate the sign of the form on $V(\mu)$. So if

$$n_V^+(\mu) \qquad n_V^-(\mu) \qquad n_V^0(\mu)$$

denotes the signature of the restriction of $\langle\ ,\ \rangle$ to an K-isotypic $V(\mu)$, we can say that

$$\boxed{\langle\ ,\ \rangle \text{ is positive semi-definite on } V \Leftrightarrow n_V^0(\mu) = 0 \text{ for all } \mu.}$$

The following lemma shows that the integers $\{n_V^+(\mu),\ n_V^-(\mu),\ n_V^0(\mu)\}$ are all divisible by the dimension of μ, so one only needs to know the value of the triple:

$$p_V(\mu) = \frac{n_V^+(\mu)}{\dim(\mu)} \qquad q_V(\mu) = \frac{n_V^-(\mu)}{\dim(\mu)} \qquad z_V(\mu) = \frac{n_V^0(\mu)}{\dim(\mu)}$$

which we call **the signature of V with respect to the K-type μ**.

Lemma 2. *Let (π, W) be a finite-dimensional representation of K, which is isotypic of a single K-type μ. Suppose that W has an invariant Hermitian form $\langle\ ,\ \rangle$. Then, there exists a decomposition*

$$W = W^+ \oplus W^- \oplus W^0$$

of W in K-invariant subspaces such that

$\langle\ ,\ \rangle$ *is positive definite on* W^+

$\langle\ ,\ \rangle$ *is negative definite on* W^-

$\langle\ ,\ \rangle$ *is identically zero on* W^0.

Moreover, if $W = W_1^+ \oplus W_1^- \oplus W_1^0$ is a similar decomposition, then

$m_{W^+}(\mu) = m_{W_1^+}(\mu)$

$m_{W^-}(\mu) = m_{W_1^-}(\mu)$

$W^0 = W_1^0.$

Proof. Since W is finite-dimensional and K is compact, we can find an invariant inner product $\langle\ ,\ \rangle_{PD}$ on W and a Hermitian endomorphism T on W, such that

$$\langle x, y\rangle \;=\; \langle Tx, y\rangle_{PD}$$

for all x and y in W. The Hermitian forms $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_{PD}$ are both invariant under K, so T must be a self-intertwining operator on W. It follows in particular that each eigenspace of T is stable under the action of K. Write

$$W = W^+ \oplus W^- \oplus W^0$$

for the orthogonal decomposition of W induced by the Hermitian operator T, so that W^+ is the direct sum of the positive eigenspaces of T, W^- is the direct sum of the negative eigenspaces of T, and W^0 is the kernel of T. Each of these subspaces is stable under K, hence it decomposes in a direct sum of copies of μ (which is the unique K-type contained in W).

The decomposition $W = W^+ \oplus W^- \oplus W^0$ clearly has the property that the form $\langle\ ,\ \rangle$ is positive definite on W^+, negative definite on W^- and zero on W^0. We must show that if $W = W_1^+ \oplus W_1^- \oplus W_1^0$ has the same property, then the "unicity conditions" listed above hold.
This is very easy, indeed if $\{n^+, n^-, n^0\}$ is the signature of $\langle\ ,\ \rangle$ on W, then:

- $\dim(W^+) = \dim(W_1^+) = n^+$, so

$$m_{W^+}(\mu) = \frac{\dim(W^+)}{\dim(\mu)} = \frac{\dim(W_1^+)}{\dim(\mu)} = m_{W_1^+}(\mu)$$

- $\dim(W^-) = \dim(W_1^-) = n^-$, so

$$m_{W^-}(\mu) = m_{W_1^+}(\mu) = \frac{n^-}{\dim(\mu)}$$

- $W^0 = W_1^0 = Ker(T)$.

The proof the lemma is now complete. $\square$

Definition 3. *Let (π, V) be an admissible $(\mathfrak{g}_o^{\mathbb{C}}, K)$ module. Suppose that $\langle\ ,\ \rangle$ is an invariant Hermitian form on X, and that μ is a K-type appearing in V. Define the signature of $\langle\ ,\ \rangle$ with respect to μ to be the triple:*

$$p_V(\mu) = \frac{n_V^+(\mu)}{\dim(\mu)} \qquad q_V(\mu) = \frac{n_V^-(\mu)}{\dim(\mu)} \qquad z_V(\mu) = \frac{n_V^0(\mu)}{\dim(\mu)}$$

where $\{n_V^+(\mu),\ n_V^-(\mu),\ n_V^0(\mu)\}$ is the signature of $\langle\ ,\ \rangle$ on the μ- isotypic part of V.

We notice that the triple $\{p_V(\mu),\ q_V(\mu),\ z_V(\mu)\}$ consists of non negative integers, and satisfies:

- $p_V(\mu) + q_V(\mu) + z_V(\mu) = m_V(\mu) =$ with $m_V(\mu)$ the multiplicity of the K-type μ in V

- $\langle\ ,\ \rangle$ is positive semi-definite if and only if $q_V(\mu) = 0$, for all μ.

Next, we give another interpretation for $\{p_V(\mu),\ q_V(\mu),\ z_V(\mu)\}$. There is a natural way to define a Hermitian form $\langle\ ,\ \rangle_\mu$ on the vector space $\mathrm{Hom}_K(E_\mu, V)$, which is finite-dimensional, of dimension $m_V(\mu)$. The signature of $\langle\ ,\ \rangle$ with respect to μ equals the signature of $\langle\ ,\ \rangle_\mu$. Therefore:

$p_V(\mu) =$ dim. of a maximal positive definite subspace of $\mathrm{Hom}_K(E_\mu, V)$

$q_V(\mu) =$ dim. of a maximal negative definite subspace of $\mathrm{Hom}_K(E_\mu, V)$

$z_V(\mu) =$ dim. of the radical of the induced form $\langle\ ,\ \rangle_\mu$ on $\mathrm{Hom}_K(E_\mu, V)$.

The key fact here is that $\mathrm{Hom}_K(E_\mu, V) \simeq E_\mu^* \otimes V$. We provide more details: since μ is an irreducible representation of the compact group K, the finite-dimensional vector space E_μ admits a positive definite Hermitian form $\langle\ ,\ \rangle_{E_\mu}$ which is invariant under K. The dual space E_μ^* inherits a Hermitian form (also positive definite and K-invariant) in a natural way: for each F in E_μ^*, there exists a unique element v_F of E_μ such that $F(u) = \langle u, v_F\rangle_{E_\mu}$, for all u in E_μ. Define

$$\langle F_1, F_2\rangle_{E_\mu^*} = \langle v_{F_2}, v_{F_2}\rangle_{E_\mu}$$

for any F_1, F_2 in E_μ^*. The tensor product of $\langle\ ,\ \rangle_{E_\mu^*}$ and $\langle\ ,\ \rangle$ gives a Hermitian form $(\ ,\ )$ on $E_\mu^* \otimes V$. The form $(\ ,\ )$ is defined on tensors by the formula:

$$(F_1 \otimes x_1, F_2 \otimes x_2) = \langle F_1, F_2\rangle_{E_\mu^*} \langle x_1, x_2\rangle$$

and it is extended by linearity on the first component and conjugate linearity on the second one to obtain a Hermitian form on all $E_\mu^* \otimes V$.

Since $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_{E_\mu}$ are both invariant under K, so are the forms $\langle\ ,\ \rangle_{E_\mu^*}$ on E_μ^*, and $(\ ,\ )$ on $E_\mu^* \otimes V$. Then $(\ ,\ )$ descends to a Hermitian form on the space of K-fixed vectors in $E_\mu^* \otimes V$, which is isomorphic to $\mathrm{Hom}_K(E_\mu, V)$.

We obtain a Hermitian form on $\mathrm{Hom}_K(E_\mu, V)$, which we denote by $\langle\ ,\ \rangle_\mu$. The next step is to show that $\langle\ ,\ \rangle_\mu$ has signature equal to the signature of $\langle\ ,\ \rangle$ with respect to the K-type μ. Write $V(\mu)$ for the μ-isotypic part of V, and write $V(\mu) = V(\mu)^+ \oplus V(\mu)^- \oplus V(\mu)^0$ for a decomposition of $V(\mu)$ into the direct sum of a positive definite subspace, a negative definite subspace and the radical of the restriction of $\langle\ ,\ \rangle$ to $V(\mu)$, so that

$$n_V^+(\mu) = \dim(V(\mu)^+) \qquad n_V^-(\mu) = \dim(V(\mu)^-) \qquad n_V^0(\mu) = \dim(V(\mu)^0)$$

is the signature of $\langle\ ,\ \rangle$ on $V(\mu)$. Write

$$s^+(\mu) \qquad s^-(\mu) \qquad s^0(\mu)$$

for the signature of $\langle\ ,\ \rangle_\mu$ on $\mathrm{Hom}_K(E_\mu, V)$. We must show that:

$$s^+(\mu) = \tfrac{n_V^+(\mu)}{\dim(\mu)} = p_V(\mu)$$

$$s^-(\mu) = \tfrac{n_V^+(\mu)}{\dim(\mu)} = q_V(\mu)$$

$$s^0(\mu) = \tfrac{n_V^+(\mu)}{\dim(\mu)} = z_V(\mu).$$

Because the decomposition $V(\mu) = V(\mu)^+ \oplus V(\mu)^- \oplus V(\mu)^0$ is stable under the action of K, we obtain a similar decomposition for $\mathrm{Hom}_K(E_\mu, V)$:

$$\mathrm{Hom}_K(E_\mu, V) = \mathrm{Hom}_K(E_\mu, V(\mu)) =$$

$$= \mathrm{Hom}_K(E_\mu, V(\mu)^+) \oplus \mathrm{Hom}_K(E_\mu, V(\mu)^-) \oplus \mathrm{Hom}_K(E_\mu, V(\mu)^0).$$

We notice that

- on $\mathrm{Hom}_K(E_\mu, V(\mu)^+)$, the form $\langle\ ,\ \rangle_\mu$ is the tensor product of two positive definite Hermitian forms, hence it is positive definite;
- on $\mathrm{Hom}_K(E_\mu, V(\mu)^-)$, the form $\langle\ ,\ \rangle_\mu$ is the tensor product of a positive definite Hermitian form and a negative definite Hermitian form, hence it is negative definite;
- on $\mathrm{Hom}_K(E_\mu, V(\mu)^0)$, the form $\langle\ ,\ \rangle_\mu$ is the tensor product of a positive definite Hermitian form and an identically zero Hermitian form, hence it is identically zero.

The result follows:

$$s^{+}(\mu) = \dim(\mathrm{Hom}_K(E_\mu, V(\mu)^{+})) = m_{V(\mu)^{+}}(\mu) = p_V(\mu)$$

$$s^{-}(\mu) = \dim(\mathrm{Hom}_K(E_\mu, V(\mu)^{-})) = m_{V(\mu)^{-}}(\mu) = q_V(\mu)$$

$$s^{0}(\mu) = \dim(\mathrm{Hom}_K(E_\mu, V(\mu)^{0}) = m_{V(\mu)^{0}}(\mu) = z_V(\mu).$$

Chapter 6

Weyl Group Representations and Signatures of Intertwining Operators.

In this chapter we discuss the unitarity of the Langlands quotient with parameters (P, δ, ν) for a real semi-simple Lie group G, under the assumptions that:

(i) G is split and $P = MAN$ is a minimal parabolic subgroup of G

(ii) δ is the trivial representation of M and ν is a real character of A.

As shown in chapters 3 and 5, the unitarity of $J_P(\delta \otimes \nu)$ depends on the signature of a Hermitian intertwining operator for the principal series, which can be computed separately on the various K-types. We now show how to determine some of this signature by means of Weyl group computations.

6.1 Spherical Unitary Dual for Real Split Semisimple Lie Groups

Let G be a real split group. Let $P = MAN$ be a *minimal* parabolic subgroup, so that M is a finite abelian subgroup of K. Let (δ, V^{δ}) be the trivial representation of M, and let $\mathrm{Ind}_P^G(\delta \otimes \nu)$ be a spherical principal series representation, with Langlands quotient $X(\nu)$. For simplicity, we assume that ν is real, and of course dominant:

$$\langle \nu, \alpha \rangle > 0 \quad \forall \alpha \in \Delta^+.$$

As seen in chapter 3, the irreducible representation $X(\nu)$ of G is unitary if and only if the following two conditions are satisfied:

1. $\omega_0 \cdot \nu = -\nu$, with ω_0 the long element of the Weyl group;[1]
2. the Hermitian intertwining operator

$$B = \delta(\omega_0) R(\omega_0) A(\bar{P} \colon P \colon \delta \colon \nu) \colon \mathcal{H}_P(\delta \otimes \nu) \to \mathcal{H}_P(\delta \otimes \omega_0 \cdot \nu)$$

   is positive semi-definite or negative semi-definite.

Since δ is the trivial representation of M, we let $\delta(\omega_0)$ be the identity, and we write

$$B = R(\omega_0) A(\bar{P} \colon P \colon \delta \colon \nu) = R(\omega_0) A(\omega_0 P \omega_0^{-1} \colon P \colon \delta \colon \nu) \overset{\text{not.}}{=} A_P(\omega_0, \nu).$$

It follows from the considerations in chapter 5 that the Hermitian operator B is positive (or negative) semi-definite if and only if for each K-type (μ, E_μ) that appears

[1] If $\mathfrak{g}_0^{\mathbb{C}}$ is not of type $\mathcal{D}_n$ with n odd, then ω_0 is minus the identity, and the condition $\omega_0 \cdot \nu = -\nu$ becomes trivial.

in the principal series, the induced intertwining operator

$$I_\mu(\omega_0, \nu)\colon \operatorname{Hom}_K(E_\mu, \mathcal{H}_P(\delta \otimes \nu)) \to \operatorname{Hom}_K(E_\mu, \mathcal{H}_P(\delta \otimes \omega_0 \cdot \nu))$$

is positive (or negative) semi-definite. Indeed, the signature of $I_\mu(\omega_0, \nu)$ equals (by definition) the signature of B with respect to the K-type μ, and B is positive (or negative) semi-definite if and only if it is such on each K-type.

Using Frobenius reciprocity, we can interpret $I_\mu(\omega_0, \nu)$ as an endomorphism of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu, V^\delta)$, which we denote by $R_\mu(\omega_0, \nu)$. The operator $R_\mu(\omega_0, \nu)$ is again Hermitian, and its signature determines the unitarity of the principal series.[2]

The example of $SL(2, \mathbb{R})$ (chapter 4) shows that it is fairly easy to determine the signature when the real split semi-simple Lie group has rank one. Therefore, we try to reduce computations for the general case to rank-one calculations. The method of rank-one reduction has proven to be very effective for the study of intertwining operators[3] and it can be briefly described as follows:

step 1 Find a *minimal* decomposition of ω_0 as a product of simple reflections

$$\omega_0 = s_{\alpha_r} \cdots s_{\alpha_2} s_{\alpha_1}$$

(such a decomposition is called minimal if ω_0 has length r).

step 2 Find the corresponding decomposition of $B{=}A_P(\omega_0, \nu)$:

$$A_P(\omega_0, \nu) = A_P(s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1}, \nu) =$$

---

[2]Notice that since δ is the trivial representation of M, the space $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu, V^\delta)$ is simply the space of M-invariants vectors in E_μ.

[3]For instance, it was used by Stein and Knapp in [22, 23] to gain information on the irreducibility of the principal series.

$$= A_P(s_{\alpha_r},\, (s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) A_P(s_{\alpha_{r-1}},\, (s_{\alpha_{r-2}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) \cdots$$

$$\cdots A_P(s_{\alpha_2},\, s_{\alpha_1} \cdot \nu) A_P(s_{\alpha_1},\, \nu).$$

step 3 When s_β is a simple reflection and γ is a real character of A satisfying $\langle \gamma,\, \beta \rangle \geqslant 0$, interpret the Hermitian operator $A_P(s_\beta,\, \gamma)$ as an intertwining operator for a rank-one subgroup, and compute the corresponding operator $R_\mu(s_\beta,\, \gamma)$.

step 4 Find $R_\mu(\omega_0,\, \nu) = R_\mu(s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1},\, \nu) =$

$$= R_\mu(s_{\alpha_r},\, (s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu)\, R_\mu(s_{\alpha_{r-1}},\, (s_{\alpha_{r-2}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) \cdots$$

$$\cdots R_\mu(s_{\alpha_2},\, s_{\alpha_1} \cdot \nu)\, R_\mu(s_{\alpha_1},\, \nu).$$

The key-point in this method is that to each *simple* root β we can associate a subgroup L^β of G with real rank one, so that the the operator $R_\mu(s_\beta,\, \nu)$ (for the spherical principal series G with parameter ν) can actually be regarded as an operator for a principal series of L^β and can therefore be computed using the results already known for $SL(2,\, \mathbb{R})$.

We define L^β to be the group MG^β, with G^β the analytic subgroup of G whose Lie algebra is the subalgebra of $\mathfrak{g}_0$ generated by the weight vectors of weight $\pm\beta$. The subgroup

$$K^\beta = \exp(\mathbb{R} Z_\beta) = \exp(\mathbb{R}(E_\beta + \vartheta E_\beta))$$

is a maximal compact subgroup of G^β isomorphic to $SO(2, \mathbb{R})$, and it is included in K. Let

$$E_\mu = \bigoplus_{m \in \mathbb{Z}} \varphi_m$$

be the decomposition of μ in isotypic components of irreducible representations of K^β: for each integer m, we have denoted by φ_m the isotypic component of the character $\xi_m\colon \exp(tZ_\beta) \mapsto e^{i\,m\,t}$ inside μ. The decomposition

$$E_\mu = \bigoplus_{m\in\mathbb{N}} (\varphi_m + \varphi_{-m})$$

is invariant under MK^β, which is a maximal compact subgroup of $L^\beta = MK^\beta$, and the corresponding decomposition of $\mathrm{Hom}_M(\mathrm{Res}^K_M E_\mu,\, V^\delta)$ is

$$\bigoplus_{m\in\mathbb{N}} \mathrm{Hom}_M\left(\mathrm{Res}^{MK^\beta}_M (\varphi_{2m} + \varphi_{-2m}),\, \mathbb{C}\right).$$

The operator $R_\mu(s_\beta,\, \gamma)$ preserves this decomposition, and acts on each of these subspaces as scalar multiplication. We normalize $R_\mu(s_\beta,\, \gamma)$ so that it acts trivially on $\mathrm{Hom}_M(\mathrm{Res}^{MK^\beta}_M \varphi_0,\, \mathbb{C})$. The action on $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^\beta}_M (\varphi_{2m} + \varphi_{-2m}),\, \mathbb{C}\right)$ is then given by the scalar

$$\boxed{\frac{(1-\langle\gamma,\, \check\beta\rangle)(3-\langle\gamma,\, \check\beta\rangle)\cdots(2m-1-\langle\gamma,\, \check\beta\rangle)}{(1+\langle\gamma,\, \check\beta\rangle)(3+\langle\gamma,\, \check\beta\rangle)\cdots(2m-1+\langle\gamma,\, \check\beta\rangle)}}.$$

for all non negative integers m.

Remark 3. *$R_\mu(\omega_0,\, \nu)$ can be decomposed as a product of operators corresponding to simple reflections s_β, and for each of these operators an explicit formula exists. This formula depends on the decomposition of E_μ in irreducible K^β-types.*
Although each K^β is isomorphic to $SO(2)$, the decomposition changes when β varies. It is very hard to keep track of these different decompositions when you multiply the various rank-one operators to obtain $R_\mu(\omega_0,\, \nu)$.

In this section we have intentionally omitted the details, so to convey the general idea in a more effective fashion. Please refer to the following subsections for all the details.

6.1.1 The rank-one subgroup G^β attached to a (simple) root

In this subsection we construct a subgroup G^β of G of real rank one, for each root β. This construction is general, and does not actually require the semi-simple group G be split nor β be simple.

Without loss of generality we can assume that the restricted root β is reduced, i.e. that $\frac{1}{2}\beta$ is not a root. We define G^β to be the analytic subgroup of G whose Lie algebra is the the subalgebra of $\mathfrak{g}_0$ generated by the root vectors of weights $\pm\beta$ and $\pm 2\beta$. Being ϑ-stable, the Lie algebra $\mathfrak{g}_0^\beta$ is reductive (actually semi-simple), and the restriction of ϑ to $\mathfrak{g}_0^\beta$ gives a Cartan involution on $\mathfrak{g}_0^\beta$.
If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$, we set $\mathfrak{k}_0^\beta = \mathfrak{k}_0 \cap \mathfrak{g}_0^\beta$ and $\mathfrak{p}_0^\beta = \mathfrak{p}_0 \cap \mathfrak{g}_0^\beta$, so that $\mathfrak{g}_0^\beta = \mathfrak{k}_0^\beta \oplus \mathfrak{p}_0^\beta$ is the Cartan decomposition of $\mathfrak{g}_0^\beta$. The subspace $\mathfrak{a}_0^\beta = \mathfrak{a}_0 \cap \mathfrak{g}_0^\beta = \mathbb{R}H_\beta$ is a maximal abelian subspace of $\mathfrak{p}_0^\beta$. The restricted root space decomposition of $\mathfrak{g}_0^\beta$ with respect to $\mathfrak{a}_0^\beta$ is given by:

$$\mathfrak{g}_0^\beta = \mathfrak{a}_0^\beta \oplus \mathfrak{m}_0^\beta \oplus [(\mathfrak{g}_0)_\beta \oplus (\mathfrak{g}_0)_{-\beta} \oplus (\mathfrak{g}_0)_{2\beta} \oplus (\mathfrak{g}_0)_{-2\beta}]$$

with $\mathfrak{m}_0^\beta = \mathfrak{m}_0 \cap \mathfrak{g}_0^\beta = Z_{\mathfrak{k}_0^\beta}(\mathfrak{a}_0^\beta)$. If we choose β and 2β to be positive, then $\mathfrak{n}_0^\beta = (\mathfrak{g}_0)_\beta \oplus (\mathfrak{g}_0)_{2\beta}$ and
$\bar{\mathfrak{n}}_0^\beta = \vartheta(\mathfrak{n}_0^\beta) = (\mathfrak{g}_0)_{-\beta} \oplus (\mathfrak{g}_0)_{-2\beta}$. The Iwasawa decomposition of G^β is compatible with the one of G:

$$G^\beta = K^\beta A^\beta N^\beta = (K \cap G^\beta)(A \cap G^\beta)(N \cap G^\beta).$$

It follows that the Iwasawa decomposition of an element x of $G^\beta = K^\beta A^\beta N^\beta$ is the same as the Iwasawa decomposition of x regarded as an element of $G = KAN$. In particular, the function H^β (which maps x into the log of the A^β-part of x) is just the restriction to G^β of the H function on G.

We now compare the "ρ" functions:

$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \dim((\mathfrak{g}_0)_\alpha)\,\alpha\colon \mathfrak{a}_0 \to \mathbb{R}$

$\rho^\beta = \frac{1}{2}[\dim((\mathfrak{g}_0)_\beta)\,\beta + 2\,\dim((\mathfrak{g}_0)_{2\beta})\,\beta]\colon \mathfrak{a}_0^\beta = \mathbb{R}H_\beta \to \mathbb{R}.$

In particular, we ask whether ρ^β equals the restriction of ρ to $\mathfrak{a}_0^\beta$:

$$\rho(H_\beta) \stackrel{?}{=} \rho^\beta(H_\beta) = \frac{1}{2}\,[\,\dim((\mathfrak{g}_0)_\beta) + 2\,\dim((\mathfrak{g}_0)_{2\beta})\,]\cdot\|\beta\|^2.$$

The answer to this question is in general negative. It is instead affirmative when the root β is *simple*. Indeed, if β is simple, the reflection s_β carries each multiple of β into its opposite and permutes the other positive roots. The equation

$$s_\beta(\rho) = \rho - [\,\dim((\mathfrak{g}_0)_\beta) + 2\,\dim((\mathfrak{g}_0)_{2\beta})\,]\,\beta$$

implies that $[\,\dim((\mathfrak{g}_0)_\beta) + 2\,\dim((\mathfrak{g}_0)_{2\beta})\,] = 2\frac{\langle\beta,\,\rho\rangle}{\|\beta\|^2}$. Hence

$$\rho(H_\beta) = \langle\beta,\,\rho\rangle = \frac{1}{2}[\,\dim((\mathfrak{g}_0)_\beta) + 2\,\dim((\mathfrak{g}_0)_{2\beta})\,]\cdot\|\beta\|^2.$$

This shows that $\rho\,|_{\mathfrak{a}_0^\beta} = \rho^\beta$, for each simple root β.

6.1.2 The operator $R_\mu(\omega_0,\,\nu)$

In this subsection, we give an explicit construction for the operator $R_\mu(\omega_0,\,\nu)$.

The restriction of a principal series representation $\mathrm{Ind}_{P=MAN}^G(\delta\otimes\nu)$ of G to the maximal compact subgroup K is isomorphic to the induced representation $\mathrm{Ind}_M^K(\delta)$. The linear isomorphism

$$T_1\colon \mathcal{H}_P(\delta\otimes\nu) \to \mathcal{H}_\delta,\; F \mapsto f = F|_K$$

intertwines the action of K on the two spaces and it is actually an isometry. The inverse mapping $T_2\colon \mathcal{H}_\delta \to \mathcal{H}_P(\delta \otimes \nu)$ is defined by

$$F = T_2(f)\colon G \to V^\delta,\ g = (kan) \mapsto e^{-(\rho+\nu)\log(a)} f(k)$$

for each $f\colon K \to V^\delta$ in $\mathcal{H}_\delta$.[4] Consider the composition

$$\tilde{B}\colon \mathcal{H}_\delta \xrightarrow{T_2} \mathcal{H}_P(\delta \otimes \nu) \xrightarrow{B} \mathcal{H}_P(\delta \otimes \omega_0 \cdot \nu) \xrightarrow{T_1} \mathcal{H}_\delta$$

where $B = \delta(\omega_0)R(\omega_0)A(\bar{P} = \omega_0 P \omega_0^{-1}\colon P\colon \delta\colon \nu)$ is the intertwining operator introduced in chapter 3. When δ is the trivial representation of M, we can take $\delta(\omega_0)$ to be the identity, and for each F in $\mathcal{H}_P(\delta \otimes \nu)$ we have:

$$B(F)\colon G \to V^\delta, g \mapsto \int_{\bar{N}} F(x\omega_0\bar{n})\, d\bar{n}.$$

The operator $\tilde{B} = T_1 B T_2\colon \mathcal{H}_\delta \to \mathcal{H}_\delta$ is a self-intertwining operator for the representation $\mathrm{Ind}_M^K(\delta)$ of K. It is convenient to write down $\tilde{B}$ more explicitly. If f belongs to $\mathcal{H}_\delta$, then

$$\tilde{B}f\colon K \to V^\delta,\ k \mapsto \int_{\bar{N}} T_2(f)(k\omega_0\bar{n})\, d\bar{n} = \int_{\bar{N}} e^{-(\rho+\nu)\log(a(\bar{n}))} f(k\omega_0\kappa(\bar{n}))\, d\bar{n}.^5$$

Let $(\mu,\, E_\mu)$ be a K type appearing in the principal series, and consider the homomorphism

$$\mathrm{Hom}(E_\mu, \mathcal{H}_\delta) \to \mathrm{Hom}(E_\mu, \mathcal{H}_\delta)$$

induced from $\tilde{B}$ by composition on the range. Since $\widetilde{B}$ is an intertwining operator,

[4] $T_2(f)$ is the unique extension of f to G that has the suitable transformation properties under right multiplication by elements of P.

[5] We have denoted by $\kappa(\bar{n})a(\bar{n})n(\bar{n})$ the Iwasawa deconposition of an element $\bar{n}$ of $\bar{N}$. Then $(k\omega_0\kappa(\bar{n}))a(\bar{n})n(\bar{n})$ is the corresponding Iwasawa decomposition of $k\omega_0\bar{n}$.

the subspace of K-fixed vectors is preserved[6], so we obtain a homomorphism

$$I_\mu(\omega_0,\, \nu)\colon\ \mathrm{Hom}_K(E_\mu, \mathcal{H}_\delta) \to \mathrm{Hom}_K(E_\mu, \mathcal{H}_\delta).$$

By Frobenius reciprocity, we can interpret $I_\mu(\omega_0,\, \nu)$ as an endomorphism $R_\mu(\omega_0,\, \nu)$ of $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta) = (E_\mu^M)^*$. Indeed $\mathcal{H}_\delta = \mathrm{Ind}_M^K(V_\delta)$ is the representation space of the induced representation $\mathrm{Ind}_M^K \delta$ and Frobenius reciprocity states that the space $\mathrm{Hom}_K(E_\mu,\, \mathrm{Ind}_M^K(V^\delta))$ is isomorphic to $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta)$. Let us explain how one gets this isomorphism.

Let T belong to $\mathrm{Hom}_K(E_\mu,\, \mathrm{Ind}_M^K(V^\delta))$. For each v in E_μ, $T(x)$ is a function from K to V^δ; in particular, $T(v)(1)$ lives in V^δ. Define $L_2(T)$ to be the linear mapping

$$L_2(T)\colon E_\mu \to V^\delta,\ v \mapsto T(v)(1).$$

Claim 1. *$L_2(T)$ is a fixed point for the action of M on* $\mathrm{Hom}(\mathrm{Res}_M^K E_\mu,\, V^\delta)$.

Proof. For all m in M

$$(m \cdot L(T))(v) = \delta(m)(L_2(T)(\mu(m^{-1}) \cdot v)) =$$

$$= \delta(m)(T(\mu(m^{-1}) \cdot v)(1)) = T(v)(1) = L_2(T)(v)$$

because T belongs to $\mathrm{Hom}_K(E_\mu,\, \mathrm{Ind}_M^K V^\delta)$. □

We obtain a linear mapping

$$L_2\colon\ \mathrm{Hom}_K(E_\mu,\, \mathrm{Ind}_M^K(V^\delta)) \to \mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta).$$

[6]The action of K on $\mathrm{Hom}(E_\mu, \mathcal{H}_\delta)$ is given by

$$(k \cdot T)(v) = \mathrm{Ind}_M^K(\delta)(k) \cdot (T(\mu(k^{-1} \cdot v))$$

for all v in E_μ, T in $\mathrm{Hom}(E_\mu, \mathcal{H}_\delta)$ and k in K.

Its inverse L_1 is defined as follows: let $S\colon E_\mu \to V^\delta$ be an element of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu, V^\delta)$. For each v in E_μ consider the function

$$L_1(S)(v)\colon K \to V^\delta,\ k \mapsto L_1(S)(v)(k) = S(\mu(k^{-1}) \cdot v).$$

Claim 2. $L_1(S)(v)$ *belongs to* $\mathcal{H}_\delta = \operatorname{Ind}^K_M V^\delta$*.*

Proof. For all m in M

$$L_1(S)(v)(km) = S(\mu(m^{-1})\mu(k^{-1}) \cdot v) = \delta(m^{-1})[\delta(m)S(\mu(m^{-1})\mu(k^{-1}) \cdot v)] =$$

$$= \delta(m^{-1})S(\mu(k^{-1}) \cdot v) = \delta(m^{-1})L_1(S)(v)(k).$$

because S is a fixed point for the action of M on $\operatorname{Hom}(\operatorname{Res}^K_M E_{mu}, V^\delta)$. □

The result is a well defined linear mapping

$$L_1(S)\colon E_\mu \to \operatorname{Ind}^K_M V^\delta,\ v \mapsto L_1(S)(v).$$

Claim 3. $L_1(S)$ *belongs to* $\operatorname{Hom}_K(E_\mu, \operatorname{Ind}^K_M V^\delta)$*, i.e.*

$$(\operatorname{Ind}^K_M \delta)(k)(L_1(S)(\mu(k^{-1}) \cdot v)) = L_1(S)(v)$$

for each k in K and v in E_μ.

Proof. This is easy, because for any x in K

$$(\operatorname{Ind}^K_M \delta)(k)(L_1(S)(\mu(k^{-1}) \cdot v))(x) = L_1(S)(\mu(k^{-1}) \cdot v)(k^{-1}x) =$$

$$= S(\mu(k^{-1}x)^{-1}\mu(k^{-1}) \cdot v) = S(\mu(x^{-1}) \cdot v) = L_1(S)(v)(x).$$ □

We obtain a linear mapping

$$L_1\colon \operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta) \to \operatorname{Hom}_K(E_\mu,\, \operatorname{Ind}^K_M V^\delta)$$

which can be easily checked to be the inverse of L_2.

Now that the Frobenius reciprocity is understood, we go back to the construction of the operator $R_\mu(\omega_0,\, \nu)$. We define $R_\mu(\omega_0,\, \nu)$ to be the composition

$$\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta) \xrightarrow{L_2} \operatorname{Hom}_K(E_\mu,\, \operatorname{Ind}^K_M V^\delta) \xrightarrow{I_\mu(\omega_0,\, \delta)}$$

$$\xrightarrow{I_\mu(\omega_0,\, \delta)} \operatorname{Hom}_K(E_\mu,\, \operatorname{Ind}^K_M V^\delta) \xrightarrow{L_1} \operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta).$$

More explicitly, the image of an element T of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta)$ under $R_\mu(\omega_0,\, \nu)$ is the function

$$\boxed{R_\mu(\omega_0,\, \nu)T\colon E_\mu \to V^\delta,\, v \mapsto \int_{\bar{N}=\bar{N}\cap(\omega_0 N \omega_0^{-1})} e^{-(\rho+\nu)\log(a(\bar{n}))} T((\omega_0 \kappa(\bar{n}))^{-1} \cdot v)\, d\bar{n}.}$$

6.1.3 The operator $R_\mu(s_\beta,\, \gamma)$

In this subsection, we compute the operator $R_\mu(s_\beta,\, \gamma)$ under the assumptions that the semi-simple Lie group is split and the root β is simple. These assumptions are crucial. We assume γ to be a real character of A satisfying $\langle \gamma,\, \beta\rangle \geqslant 0$.

An argument similar to the one used in the previous subsection shows that the operator $R_\mu(s_\beta,\, \gamma)$ is the endomorphism of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta)$ that carries an element T into the mapping

$$R_\mu(s_\beta,\, \gamma)T\colon E_\mu \to V^\delta,\, v \mapsto \int_{\bar{N}\cap(\sigma_\beta N \sigma_\beta^{-1})} e^{-(\rho+\gamma)\log(a(\bar{n}))} T((\sigma_\beta \kappa(\bar{n}))^{-1} \cdot v)\, d\bar{n},$$

with σ_β a representative in M' for the root reflection s_β.

Since β is simple, using results described in subsection 6.1.1 we can write:

$$(R_\mu(s_\beta,\, \gamma)T)(v) = \int_{\bar{N}^\beta} e^{-(\rho+\gamma)(H(\bar{n}))} T((\sigma_\beta \kappa(\bar{n}))^{-1} \cdot v)\, d\bar{n} =$$

$$= \int_{\bar{N}^\beta} e^{-(\rho^\beta + \gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar{n}))} T((\sigma_\beta \kappa^\beta(\bar{n}))^{-1} \cdot v)\, d\bar{n}$$

and this is exactly the value at T of the "R-operator" (with parameters μ, s_β and $\gamma \mid_{\mathfrak{a}_0^\beta}$) for the rank-one subgroup $L^\beta = MG^\beta$ of G.

Notice that, as a representation of MK^β (which is a maximal compact subgroup of MG^β), the representation μ is reducible. Let us discuss its decomposition in irreducible summands.

When G is split, each restricted root is reduced and each root space is one-dimensional. In particular, the Lie algebra of G^β

$$\mathfrak{g}_0^\beta = \mathfrak{a}_0^\beta \oplus (\mathfrak{g}_0)_\beta \oplus (\mathfrak{g}_0)_{-\beta} = \mathbb{R}H_\beta + \mathbb{R}E_\beta + \mathbb{R}\vartheta(E_\beta)$$

is isomorphic to $\mathfrak{sl}(2,\, \mathbb{R})$. We have denoted by E_β a non-zero element of $(\mathfrak{g}_0)_\beta$ that satisfies the normalizing condition $B(E_\beta,\, \vartheta(E_\beta)) = -\frac{2}{\|\beta\|^2}$, with B the Killing form. E_β is a generator for the root space of weight β and is uniquely determined up to a sign. The element $Z_\beta = E_\beta + \vartheta(E_\beta)$ is a generator for $\mathfrak{k}_0^\beta$, so

$$K^\beta = \exp(\mathbb{R}Z_\beta) \simeq SO(2,\, \mathbb{R})$$

is a maximal compact subgroup of G^β. The group MK^β is a maximal compact subgroup of $L^\beta = MG^\beta$, and its structure is described by the following lemma:

Lemma 3. *Let $\beta(m) = \pm 1$ denote the scalar by which an element m of M acts on the root vector E_β. Then* (*a*) $\beta(m) = (-\beta)(m)$

(*b*) $\mathrm{Ad}(m)Z_\beta = \beta(m)Z_\beta$

(*c*) $m\sigma_\beta m^{-1} = \sigma_\beta^{\beta(m)}$.

Proof. We first show that M acts on the vector E_β by a scalar. Since the root space $(\mathfrak{g}_0)_\beta$ is one-dimensional, it is enough to show that $\mathrm{Ad}(m)(E_\beta)$ belongs to $(\mathfrak{g}_0)_\beta$, for each m in M. This is easy, because

$$[H,\, \mathrm{Ad}(m)(E_\beta)] = \mathrm{Ad}(m)[\mathrm{Ad}(m^{-1})H,\, E_\beta] = \mathrm{Ad}(m)[H,\, E_\beta] = \beta(H)\,\mathrm{Ad}(m)(E_\beta)$$

for all H in $\mathfrak{a}_0$. For each m in M, define $\beta(m)$ by the equation $\mathrm{Ad}(m)(E_\beta) = \beta(m)E_\beta$. Since $\mathrm{Ad}(m)$ commutes with the Cartan involution[7], we easily deduce that $(-\beta)(m) = \beta(m)$, for all m:

$$\mathrm{Ad}(m)(E_{-\beta}) = \mathrm{Ad}(m)(\vartheta(E_\beta)) = \vartheta(\mathrm{Ad}(m)(E_\beta)) = \beta(m)E_{-\beta}.$$

It follows that

$$B(E_\beta,\, \vartheta(E_\beta)) = B(\mathrm{Ad}(m)(E_\beta),\, \mathrm{Ad}(m)(\vartheta(E_\beta))) = (\beta(m))^2 B(E_\beta,\, \vartheta(E_\beta))$$

and, being $B(E_\beta,\, \vartheta(E_\beta)) = -\frac{2}{\|\beta\|^2} \neq 0$ (by definition of E_β), we conclude that $(\beta(m))^2 = 1$, so the function $m \mapsto \beta(m)$ only takes the values $+1$ or -1 on M.[8] Proving that $\mathrm{Ad}(m)Z_\beta = \beta(m)Z_\beta$ is trivial, because $Z_\beta = E_\beta + E_{-\beta}$ and $(-\beta)(m) =$

[7] Let Θ be the global Cartan involution. Being Θ an involutive automorphism of G which fixes K (hence M), we have

$$mxm^{-1} = \Theta(m\Theta(x)m^{-1}) \qquad \forall\, x \in G\, \forall\, m \in M.$$

Differentiating at $x = 1$ we find that $\mathrm{Ad}(m) = \vartheta\,\mathrm{Ad}(m)\vartheta$, for all m in M. The results follows from the fact that also ϑ is an involution.

[8] Using the fact that being G split each restricted root is the restriction to $\mathfrak{a}_0$ of one (and only one) root in $\Delta(\mathfrak{g}_0^{\mathbb{C}},\, \mathfrak{a}_0^{\mathbb{C}})$, and some standard results from the representation theory of $SL(2,\mathbb{C})$, you can prove that if $m = m_\alpha = \exp(\pi Z_\alpha)$ then $\beta(m_\alpha) = (-1)^{\frac{2}{\|\alpha\|^2}\langle\alpha,\,\beta\rangle} = (-1)^{\langle\check{\alpha},\,\beta\rangle}$.

$\beta(m)$. Finally, setting $\sigma_\beta = \exp(\frac{\pi}{2} Z_\beta)$ and exponentiating, we find:

$$m\sigma_\beta m^{-1} = \sigma_\beta^{\beta(m)} = \sigma_\beta^{\pm 1}.$$

and this concludes the proof. $\square$

More generally, $m \exp(tZ_\beta)\, m^{-1} = \exp(tZ_\beta)^{\beta(m)}$, so the structure of the group MK^β is understood.

We now go back to the problem of discussing the decomposition of the (irreducible) representation μ of K into isotypic components of irreducible representations of the subgroup MK^β. First, we decompose μ under the action of $K^\beta \simeq SO(2, \mathbb{R})$, obtaining:

$$E_\mu = \bigoplus_{l \in \mathbb{Z}} \varphi_l$$

where, for each integer l, we have denoted by φ_l the isotypic component of the character ξ_l of $SO(2)$. Then we notice that for all t in $\mathbb{R}$ and all v in φ_l

$$\exp(tZ_\beta) \cdot (m \cdot v) = m \cdot [(m^{-1} \exp(tZ_\beta) m) \cdot v] = m \cdot [\exp(t\beta(m^{-1}) Z_\beta) \cdot v] =$$

$$= m \cdot [\exp(t\beta(m^{-1}) Z_\beta) \cdot v] = \xi_l(\exp(t\beta(m) Z_\beta))\, (m \cdot v) = \xi_{\beta(m)l}(\exp(tZ_\beta))\, (m \cdot v)$$

hence $m \cdot \varphi_l = \varphi_{\beta(m)l} = \varphi_{\pm l}$. The decomposition

$$E_\mu = \bigoplus_{l \in \mathbb{N}} (\varphi_l + \varphi_{-l})$$

is clearly invariant under MK^β, and it coincides with the decomposition of μ in MK^β-types.[9] Because each subspace $(\varphi_l + \varphi_{-l})$ is M-stable and δ is the trivial representation,

[9] Unless, of course, M stabilizes Z_β, so each φ_l is already stable.

we can write:

$$\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta) = \bigoplus_{l\in\mathbb{N}} \mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_l+\varphi_{-l}),\, \mathbb{C}\right)$$

When l is odd, the space $\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}(\varphi_l+\varphi_{-l}),\, \mathbb{C})$ is zero. Indeed, for each M-fixed linear map $T\colon \varphi_l \oplus \varphi_{-l} \to \mathbb{C}$, and for each v in $\varphi_l \oplus \varphi_{-l}$, we must have:

$$T(v) = (m_\beta \cdot T)(v) = T(m_\beta^{-1}\cdot v) = T(-v) = -T(v).$$

So we can write: $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta) = \bigoplus_{l\in 2\mathbb{N}} \mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_l+\varphi_{-l}),\, \mathbb{C}\right) =$

$$= \mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\, \mathbb{C}) \bigoplus_{l\in 2\mathbb{N}^*} \mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_l\oplus\varphi_{-l}),\, \mathbb{C}\right).$$

We will compute the operator $R_\mu(s_\beta,\, \gamma)$ separately on each component.

If T belongs to $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}\varphi_0,\, \mathbb{C}\right)$ and v is in φ_0, then

$(R_\mu(s_\beta,\, \gamma)T)(v) = \int_{\bar N^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))} T((\sigma_\beta\kappa^\beta(\bar n))^{-1}\cdot v)\, d\bar n =$

$= \int_{\bar N^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))} T(v)\, d\bar n =$

$= \left[\int_{\bar N^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))} d\bar n\right] T(v).$

If T belongs to $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_l\oplus\varphi_{-l}),\, \mathbb{C}\right)$ and $v=(v_+ + v_-)$ is in $(\varphi_l\oplus\varphi_{-l})$, then

$(R_\mu(s_\beta,\, \gamma)T)(v) = \int_{\bar N^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))} T((\sigma_\beta\kappa^\beta(\bar n))^{-1}\cdot v)\, d\bar n =$

$= \int_{\bar N^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))} T\left(\xi_{+l}(\sigma_\beta\kappa^\beta(\bar n))^{-1} v_+ + \xi_{-l}\left(\sigma_\beta\kappa^\beta(\bar n)\right)^{-1} v_-\right) d\bar n =$

$$= \left[\int_{\bar{N}^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar{n}))} \xi_{+l}(\sigma_\beta \kappa^\beta(\bar{n}))^{-1} d\bar{n}\right] T(v_+) +$$

$$+ \left[\int_{\bar{N}^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar{n}))} \xi_{-l}(\sigma_\beta \kappa^\beta(\bar{n}))^{-1} d\bar{n}\right] T(v_-).$$

To conclude the computations we need one final observation. Consider the Lie algebra isomorphism

$$\psi_\beta \colon \mathfrak{sl}(2,\,\mathbb{R}) \to \mathfrak{g}_0^\beta = \mathbb{R}H_\beta + \mathbb{R}E_\beta + \mathbb{R}\vartheta(E_\beta)$$

defined by:

$$\underline{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto E_\beta, \quad \underline{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -\vartheta(E_\beta), \quad \underline{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \frac{+2}{\|\beta\|^2} H_\beta.$$

When G has a complexification[10], ψ_β lifts to a group homomorphism

$$\Psi_\beta \colon SL(2,\,\mathbb{R}) \to G^\beta.$$

It follows from the formula 4.1 for the Iwasawa decomposition in $SL(2,\,\mathbb{R})$ that

[10] This is always the case if G is semi-simple. Indeed every adjoint group has a complexification: if $G = \mathrm{Ad}(\mathfrak{g}_0)$, you can take $G^{\mathbb{C}}$ to be $\mathrm{Ad}(\mathfrak{g}_0^{\mathbb{C}})$.
It also true, more generally, if the group G is real reductive and satisfies the condition

$$Z(G) \cap K = \{1\}.$$

Indeed, if $G = K\,\exp(\mathfrak{p}_0)$ is the Cartan decomposition of G, and ζ is the center of the Lie algebra of G (so that $\mathfrak{g}_0 = [\mathfrak{g}_0,\,\mathfrak{g}_0] \oplus \zeta$), then we can write

$$G = \underbrace{K\,\exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0,\,\mathfrak{g}_0])}_{G^1}\,\underbrace{\exp(\mathfrak{p}_0 \cap \zeta)}_{Z^1}. \qquad (\circledast)$$

with G^1 real reductive (of the same rank as G) and Z^1 a vector group included in the center.
Because $Z(G^1) = Z(G) \cap K = \{1\}$, the group G^1 is actually semi-simple. So ($\circledast$) is a decomposition of G as a direct product of an adjoint group and a vector group, both of which have a complexification. As a result, we obtain a complexification for G.
Finally, we notice that $Z(G)$ acts by scalars on any irreducible representation of G (this is Schur's lemma), and that $Z(G) \cap K$ acts trivially on the trivial K-type included in any irreducible spherical representation ρ of G (hence on the whole representation space E_ρ). So, when dealing with spherical representations, we can assume w.l.o.g. that the condition $Z(G) \cap K = \{1\}$ is satisfied.

$$\bar{n}=\exp(t\vartheta(E_\beta)=\exp(-t\psi_\beta(\underline{f}))=\Psi_\beta(\exp(-t\,\underline{f}))=\Psi_\beta\left(\left(\begin{array}{cc}1&0\\-t&1\end{array}\right)\right)=$$

$$=\Psi_\beta\left(\left(\begin{array}{cc}\cos(\arctan(t))&\sin(\arctan(t))\\-\sin(\arctan(t))&\cos(\arctan(t))\end{array}\right)\left(\begin{array}{cc}\sqrt{1+t^2}&0\\0&1/\sqrt{1+t^2}\end{array}\right)\left(\begin{array}{cc}1&x\\0&1\end{array}\right)\right)=$$

$$=\Psi_\beta\left(\exp(\arctan(t)\,(\underline{e}-\underline{f})\right)\;\Psi_\beta\left(\exp(\ln(\sqrt{1+t^2})\,\underline{h})\right)\;\Psi_\beta\left(\exp(x\,\underline{e})\right)=$$

$$=\exp\left(\psi_\beta(\arctan(t)\,(\underline{e}-\underline{f}))\right)\;\exp\left(\psi_\beta(\ln(\sqrt{1+t^2})\,\underline{h})\right)\;\exp\left(\psi_\beta(x\,\underline{e})\right)=$$

$$=\exp\left(\arctan(t)\,Z_\beta\right)\;\exp\left(\ln(\sqrt{1+t^2})\,\tfrac{2}{\|\beta\|^2}H_\beta\right)\;\exp\left(x\,E_\beta\right).$$

This is the Iwasawa decomposition of an arbitrary element $\bar{n}=\exp(t\vartheta(E_\alpha))$ of N^β. We notice, in particular, that $H^\beta(\bar{n})=\ln\left(\sqrt{1+t^2}\right)\,\frac{2}{\|\beta\|^2}H_\beta$ and that $\kappa^\beta(\bar{n})=\exp\left(\arctan(t)\,Z_\beta\right)$. Therefore

- $\rho^\beta(H^\beta(\bar{n}))=\frac{1}{2}\beta\left(\ln(\sqrt{1+t^2})\,\frac{2}{\|\beta\|^2}H_\beta\right)=\ln(\sqrt{1+t^2})$

- $\gamma\mid_{\mathfrak{a}_0^\beta}(H^\beta(\bar{n}))=\gamma\left(\ln(\sqrt{1+t^2})\,\frac{2}{\|\beta\|^2}H_\beta\right)=\ln(\sqrt{1+t^2})\,\langle\gamma,\,\frac{2}{\|\beta\|^2}\beta\rangle=$
  $=\ln(\sqrt{1+t^2})\,\langle\gamma,\,\check{\beta}\rangle$

- $\xi_{2l}(\sigma_\beta\kappa^\beta(\bar{n}))^{-1}=\xi_{2l}\left(\exp\left(-(\frac{\pi}{2}+\arctan(t))Z_\beta\right)\right)=(-1)^l e^{-2l\,i\,[\arctan(t)]}=$
  $=(-1)^l\left(\frac{1+i\,t}{\sqrt{1+t^2}}\right)^{-2l}$, for all l in $\mathbb{Z}$.

We can now compute the action of $R_\mu(s_\beta,\,\gamma)$ on the generic element T of $\mathrm{Hom}_M(\mathrm{Res}^{MK^\beta}_M\,\varphi_0,\,\mathbb{C})$:

$$R_\mu(s_\beta,\,\gamma)T = \left[\int_{\bar{N}^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar{n}))} d\bar{n}\right] T = \left[\int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,\check{\beta}\rangle)} dt\right] T.$$

We have already computed this integral in chapter 4, its value being the scalar by which the standard intertwining operator A_ω of $SL(2,\,\mathbb{R})$ acts on the trivial K-type of the (minimal) spherical principal series with parameter $\lambda = \langle\gamma,\,\check{\beta}\rangle$:[11]

$$\int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,\check{\beta}\rangle)} dt = \frac{\pi\,\Gamma(\langle\gamma,\,\check{\beta}\rangle)}{2^{\langle\gamma,\,\check{\beta}\rangle-1}\,\Gamma\left(\frac{\langle\gamma,\check{\beta}\rangle+1}{2}\right)\,\Gamma\left(\frac{\langle\gamma,\check{\beta}\rangle+1}{2}\right)}.$$

We will denote this constant by $C_{\langle\gamma,\,\check{\beta}\rangle}$. For each positive integer l, we now compute $(R_\mu(s_\beta,\,\gamma)T)(v_+)$, for v_+ in φ_{2l} and T in $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^\beta}_M(\varphi_{2l}\oplus\varphi_{-2l}),\,\mathbb{C}\right)$:

$$(R_\mu(s_\beta,\,\gamma)T)(v_+) = \left[\int_{\bar{N}^\beta} e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar{n}))}\,\xi_{+2l}(\sigma_\beta\kappa^\beta(\bar{n}))^{-1} d\bar{n}\right] T(v_+) \; =$$

$$= \left[(-1)^l \int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,\check{\beta}\rangle)} \left(\tfrac{1+i\,t}{\sqrt{1+t^2}}\right)^{-2l} dt\right] T(v_+).$$

After a simple change of variable $(t\mapsto -t)$, we can identify the integral

$$\left[(-1)^l \int_{\mathbb{R}} (\sqrt{1+t^2})^{-(1+\langle\gamma,\,\check{\beta}\rangle)} \left(\frac{1+i\,t}{\sqrt{1+t^2}}\right)^{-2l} dt\right]$$

with the one (already computed in chapter 4) that defines the scalar by which the standard intertwining operator A_ω of $SL(2,\,\mathbb{R})$ acts on the "$+2l$" K-type of the (minimal) spherical principal series with parameter $\lambda = \langle\gamma,\,\check{\beta}\rangle$.[12] The value of this

[11] Compare this integral with the one appearing in formula 4.2 of chapter 4, for $m = 0$.
[12] Compare this integral with the one appearing in formula 4.2 of chapter 4, for $m = -l$.

integral is

$$C_{\langle\gamma,\,\check\beta\rangle}\cdot\frac{(1-\langle\gamma,\,\check\beta\rangle)(3-\langle\gamma,\,\check\beta\rangle)\cdots(2l-1-\langle\gamma,\,\check\beta\rangle)}{(1+\langle\gamma,\,\check\beta\rangle)(3+\langle\gamma,\,\check\beta\rangle)\cdots(2l-1+\langle\gamma,\,\check\beta\rangle)}.$$

For brevity of notation, denote this quantity by $C_{\langle\gamma,\,\check\beta\rangle}d_{2l}(\langle\gamma,\,\check\beta\rangle)$.

Similarly, for v_- in φ_{-2l}, we find:

$$(R_\mu(s_\beta,\,\gamma)T)(v_-)=\left[\int_{\bar N^\beta}e^{-(\rho^\beta+\gamma|_{\mathfrak{a}_0^\beta})(H^\beta(\bar n))}\xi_{-2l}(\sigma_\beta\kappa^\beta(\bar n))^{-1}d\bar n\right]T(v_-)=$$

$$=\left[(-1)^l\int_{\mathbb{R}}(\sqrt{1+t^2})^{-(1+\langle\gamma,\,\check\beta\rangle)}\left(\tfrac{1+i\,t}{\sqrt{1+t^2}}\right)^{+2l}dt\right]T(v_-)=$$

$$=\left[C_{\langle\gamma,\,\check\beta\rangle}d_{2l}(\langle\gamma,\,\check\beta\rangle)\right]T(v_-).$$

It follows that $(R_\mu(s_\beta,\,\gamma)T)=\left[C_{\langle\gamma,\,\check\beta\rangle}d_{2l}(\langle\gamma,\,\check\beta\rangle)\right]T(v)$, for any positive integer l, for all $v=v_++\,v_-$ in $(\varphi_l\oplus\varphi_{-l})$ and all T in $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_l\oplus\varphi_{-l}),\,\mathbb{C}\right)$.

Conclusions. *For each simple root β, and for each real character γ of A such that $\langle\gamma,\,\beta\rangle\geqslant 0$, the operator*

$$R_\mu(s_\beta,\,\gamma)\colon\ \mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\,V^\delta)\to\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\,V^\delta)$$

preserves the decomposition of $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\,V^\delta)$ *in MK^β-stable subspaces:*

$$\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\,\mathbb{C})\bigoplus_{l\in\mathbb{N}^*}\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_{2l}\oplus\varphi_{-2l}),\,\mathbb{C}\right).$$

Precisely, $R_\mu(s_\beta,\,\gamma)$ acts on $\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\,\mathbb{C})$ *as scalar multiplication by*

$$\boxed{C_{\langle\gamma,\,\check\beta\rangle}=\frac{\pi\,\Gamma(\langle\gamma,\,\check\beta\rangle)}{2^{\langle\gamma,\,\check\beta\rangle-1}\,\Gamma\left(\frac{\langle\gamma,\,\check\beta\rangle+1}{2}\right)\,\Gamma\left(\frac{\langle\gamma,\,\check\beta\rangle+1}{2}\right)}}$$

and on each subspace $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_{2l}\oplus\varphi_{-2l}),\,\mathbb{C}\right)$*, with* l *in* $\mathbb{N}^*$*, as scalar multiplication by*

$$\boxed{C_{\langle\gamma,\,\check\beta\rangle}d_{2l}(\langle\gamma,\,\check\beta\rangle)=C_{\langle\gamma,\,\check\beta\rangle}\cdot\frac{(1-\langle\gamma,\,\check\beta\rangle)(3-\langle\gamma,\,\check\beta\rangle)\cdots(2l-1-\langle\gamma,\,\check\beta\rangle)}{(1+\langle\gamma,\,\check\beta\rangle)(3+\langle\gamma,\,\check\beta\rangle)\cdots(2l-1+\langle\gamma,\,\check\beta\rangle)}}.$$

6.1.4 Final considerations

By assumption γ is a real dominant character of A satisfying $\langle\gamma,\,\beta\rangle\geqslant 0$. Being $\langle\gamma,\,\check\beta\rangle$ real and positive, also the constant $C_{\langle\gamma,\,\check\beta\rangle}$ is real and positive. Write $\tilde R_\mu(s_\beta,\,\gamma)=\frac{1}{C_{\langle\gamma,\,\check\beta\rangle}}R_\mu(s_\beta,\,\gamma)$ for the normalized operator. $\tilde R_\mu(s_\beta,\,\gamma)$ acts trivially on $\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\,\mathbb{C})$, and acts on each subspace $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_{2m}\oplus\varphi_{-2m}),\,\mathbb{C}\right)$ -with $m\in\mathbb{N}^*$- as scalar multiplication by

$$\frac{\prod_{j=1}^m((2j-1)-\langle\gamma,\,\check\beta\rangle)}{\prod_{j=1}^m((2j-1)+\langle\gamma,\,\check\beta\rangle)}.$$

Collecting all the normalizing factors, we can write $R_\mu(\omega_0,\,\nu)$ as a real positive multiple of a product of $\tilde R_\mu$-operators corresponding to simple reflections, and because multiplication by a real positive constant does not affect the signature, we can as well assume that this constant is one.

The following theorem summarizes all the information we gathered so far about the spherical unitary dual of a split semi-simple Lie group.

Theorem 9. *Let* G *be a real split semi-simple Lie group, let* $P=MAN$ *be a minimal parabolic subgroup of* G *and let* ω_0 *be the long element of the Weyl group. For each real dominant character* ν *of* A *satisfying the parity condition* $\omega_0\cdot\nu=-\nu$*, denote by* $X(\nu)$ *the Langlands quotient of* G *with parameters* $(P,\,\delta,\,\nu)$*, with* δ *the trivial representation of* M*. Then,* $X(\nu)$ *is unitary if and only if for each* K*-type* μ *a certain Hermitian operator* $\tilde R_\mu(\omega_0,\,\nu)$ *of* $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\,V^\delta)$ *is positive semi-definite. The operator* $\tilde R_\mu(\omega_0,\,\nu)$ *can be constructed as follows:*

- *Find a minimal decomposition of* ω_0 *as a product of simple reflections:* $\omega_0 = s_{\alpha_r} \cdots s_{\alpha_2} s_{\alpha_1}$

- *Write* $\tilde{R}_\mu(\omega_0,\, \nu) = \tilde{R}_\mu(s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1},\, \nu) =$

  $= \tilde{R}_\mu(s_{\alpha_r},\, (s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) \tilde{R}_\mu(s_{\alpha_{r-1}},\, (s_{\alpha_{r-2}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) \cdots$

  $\cdots \tilde{R}_\mu(s_{\alpha_2},\, s_{\alpha_1} \cdot \nu) \tilde{R}_\mu(s_{\alpha_1},\, \nu)$

  and observe that each factor is an operator of the form $\tilde{R}_\mu(s_\beta,\, \gamma)$*, with* s_β *a simple reflection and* γ *a real character of* A *satisfying* $\langle \gamma,\, \beta \rangle \geqslant 0$*.*

- *For* β *and* γ *as above, define* $\tilde{R}_\mu(s_\beta,\, \gamma)$ *as follows: for each integer* l*, let* φ_l *be the isotypic component of the character* $\xi_l \colon \exp(tZ_\beta) \mapsto e^{i\,l\,t}$ *of* $K^\beta \simeq SO(2)$ *inside* μ*, so that*

  $$E_\mu = \bigoplus_{l \in \mathbb{Z}} \varphi_l$$

  is the decomposition of μ *in* K^β*-types, and*

  $$\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta) = \bigoplus_{m \in \mathbb{N}} \operatorname{Hom}_M\left(\operatorname{Res}^{MK^\beta}_M (\varphi_{2m} + \varphi_{-2m}),\, \mathbb{C}\right)$$

  is a decomposition of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_\mu,\, V^\delta)$ *in* MK^β*-stable subspaces. Define the operator* $\tilde{R}_\mu(s_\beta,\, \gamma)$ *to to act on* $\operatorname{Hom}_M\left(\operatorname{Res}^{MK^\beta}_M (\varphi_{2m} + \varphi_{-2m}),\, \mathbb{C}\right)$ *as scalar multiplication by*

  $$\frac{\prod_{j=1}^m ((2j-1) - \langle \gamma,\, \check{\beta} \rangle)}{\prod_{j=1}^m ((2j-1) + \langle \gamma,\, \check{\beta} \rangle)}.$$

  for all non-negative integers m*.*

We emphasize one more time that this construction is less trivial than it looks like, because the definition of $\tilde{R}_\mu(s_\beta,\, \gamma)$ depends on the decomposition of μ in K^β-types, and this decomposition depends on the choice of the simple root β. So to compute $\tilde{R}_\mu(\omega_0,\, \nu)$ one needs to keep track of the many different decompositions. Another difficulty consists of the fact that we are required to compute the signature of the operator $\tilde{R}_\mu(\omega_0, \nu)$ on all the K-types μ that appear in the spherical principal series, whose number is in general infinite.

In the next section we give an alternative method to compute the operator $\tilde{R}_\mu(\omega_0, \nu)$, which uses only Weyl group calculations.

6.2 Spherical representations and the Weyl group

We start by recalling the formal construction of the Weyl group.
Let V be a finite-dimensional real vector space, with inner product $\langle\, ,\rangle$ and norm squared $\| \|^2$. A root system R is a finite set of non-zero elements of V satisfying the following conditions:

(i) R spans V,

(ii) for each α in R, the reflection $s_\alpha \colon V \to V, \gamma \mapsto \gamma - \frac{2\langle\gamma,\alpha\rangle}{\|\alpha\|^2}\alpha$ is an orthogonal transformation of V that carries R into itself,

(iii) $\frac{2\langle\alpha,\beta\rangle}{\|\alpha\|^2}$ is an integer, for all α and β in R.

We say that an element α of R is a root, and that s_α is a root reflection. We define the Weyl group W of the root system (R, V) to be the subgroup of $O(V)$ generated by the the root reflections.
By construction, the (real) group algebra $\mathbb{R}[W]$ of W lies in the algebra of all endomorphisms of V. We introduce a notion of positivity in V, and we call a root simple if it is positive and cannot be expressed as a sum of two positive roots. The simple roots form a basis for the root system, and the corresponding reflections (that we call "simple reflections") generate the Weyl group.

W contains a unique element ω_0 (of order two) that carries R^+ into R^-. We call ω_0 the long Weyl group element, because a minimal decomposition of ω_0 as a product of simple reflections has length $N = \#R^+$.[13]

An element ν of V is dominant if, for any root β in R, $\langle \nu,\, \check{\beta} \rangle \equiv \langle \nu,\, \frac{2\beta}{\|\beta\|^2} \rangle \geqslant 0$.

Definition 4. *For any simple root α in R, and for any element γ in V satisfying $\langle \gamma,\, \beta \rangle \geqslant 0$, define*

$$A_{s_\alpha}(\gamma) = \frac{1}{1 + \langle \gamma,\, \check{\alpha} \rangle}\, e + \frac{\langle \gamma,\, \check{\alpha} \rangle}{1 + \langle \gamma,\, \check{\alpha} \rangle}\, s_\alpha \in \mathbb{R}[W]$$

where e is the identity of W.

Definition 5. *Let ν be a dominant element of V, and let*

$$\omega_0 = s_{\alpha_N} s_{\alpha_{N-1}} \cdots s_{\alpha_1}$$

be a minimal decomposition of the long Weyl group element ω_0 as a product of simple reflections. Define

$$A(\nu) = A_{s_{\alpha_N}}((s_{\alpha_{N-1}} s_{\alpha_{N-2}} \cdots s_{\alpha_1})(\nu)) \cdot A_{s_{\alpha_{N-1}}}((s_{\alpha_{N-2}} \cdots s_{\alpha_1})(\nu)) \cdots$$

$$\cdots A_{s_{\alpha_2}}(s_{\alpha_1}(\nu)) \cdot A_{s_{\alpha_1}}(\nu).$$

$A(\nu)$ is also an element of the group algebra $\mathbb{R}[W]$. The following lemma describes its properties:

Lemma 4. *For all dominant ν in V*

1. *$A(\nu)$ is a rational function of ν*

2. *$A(0) = e$*

[13] Each element x of W can be written as a product of simple reflections. This decomposition is called minimal if it involves the least possible number of reflections; the length of a minimal decomposition of x is defined to be the length of x. It can be shown that the length of x equals the number of positive roots that are mapped by x into negative roots. Therefore the length of x is less or equal than the cardinality of R^+, for all x in W.

3. $\lim_{\nu\to\infty} A(\nu) = \omega_0$

4. $A(\nu)$ *is invertible in* $\mathbb{R}[W]$ *provided that* $\langle \nu, \check{\alpha}\rangle \neq 1$, *for all* α *in* R^+ *(the inverse of* $A_{s_\alpha}(\xi)$ *is* $A_{s_\alpha}(-\xi) = \frac{1}{1-\langle \nu, \check{\alpha}\rangle} e - \frac{\langle \nu, \check{\alpha}\rangle}{1-\langle \nu, \check{\alpha}\rangle} s_\alpha$*)*

5. $A(\nu)$ *is self-adjoint if* $\omega_0(\nu) = -\nu$

6. $A(\nu)$ *is independent of the choice of the reduced decomposition for* ω_0.

If (ψ, E_ψ) is any representation of the Weyl group W, $\psi(x) \in Aut(E_\psi)$ for all x in W and $\psi(y) \in End(E_\psi)$ for all y in $\mathbb{R}[W]$. In particular,

$$\psi(A_{s_\alpha}(\gamma)) = \frac{\mathbb{I}d + \langle \gamma, \check{\alpha}\rangle\, \psi(s_\alpha)}{1 + \langle \gamma, \check{\alpha}\rangle}$$

is a well defined endomorphism of E_ψ, for all α and γ in V such that $\langle \gamma, \alpha\rangle \geqslant 0$, with α simple. We notice that $\psi(A_{s_\alpha}(\gamma))$ is the identity on the $(+1)$-eigenspace of $\psi(s_\alpha)$, and acts as scalar multiplication by $\frac{1-\langle \gamma, \check{\alpha}\rangle}{1+\langle \gamma, \check{\alpha}\rangle}$ on the (-1)-eigenspace of $\psi(s_\alpha)$.
Because the operator $\psi(s_\alpha)$ is unitary of order two, E_ψ is the orthogonal direct sum of the $(+1)$ and (-1) eigenspaces of $\psi(s_\alpha)$, so we can equivalently define $\psi(A_{s_\alpha}(\gamma))$ by:

$$\psi(A_{s_\alpha}(\gamma))v = \begin{cases} v & \text{for } v \text{ in the } (+1)\text{-eigenspace of } \psi(s_\alpha) \\ \frac{1-\langle \xi, \check{\alpha}\rangle}{1+\langle \xi, \check{\alpha}\rangle} v & \text{for } v \text{ in the } (-1)\text{-eigenspace of } \psi(s_\alpha). \end{cases}$$

Similarly we can consider the endomorphism $\psi(A(\nu))$, for ν real and dominant, which -by definition of $A(\nu)$- admits a decomposition

$$\psi(A(\nu)) = \psi(A_{s_{\alpha_N}}((s_{\alpha_{N-1}} s_{\alpha_{N-2}} \cdots s_{\alpha_1})(\nu))) \cdot \psi(A_{s_{\alpha_{N-1}}}((s_{\alpha_{N-2}} \cdots s_{\alpha_1})(\nu))) \cdots$$

$$\cdots \psi(A_{s_{\alpha_2}}(s_{\alpha_1}(\nu))) \cdot \psi(A_{s_{\alpha_1}}(\nu)).$$

$\psi(A(\nu))$ is well defined for each ν dominant, it is independent of the choice of the reduced decomposition for ω_0 and it is Hermitian whenever $\omega_0 \cdot \nu = -\nu$.

Theorem 10 (Barbasch-Moy [7]). *If G is a split p-adic group, the spherical irreducible representation $X(\nu)$ is unitary if and only if the following two conditions are satisfied:*

(i) $\omega_0 \cdot \nu = -\nu$*;*

(ii) for each irreducible Weyl group representation ψ, the Hermitian operator $\psi(A(\nu))$ is positive-semidefinite.

In the real case, the Hermitian operator $\psi(A(\nu))$ does not generally detect unitarity.[14] Still, it can be used to create a "non-unitarity test". To formulate this idea we need a preliminary definition.

Definition 6. *An irreducible representation $(\mu,\, E_\mu)$ of K is called petite if for every simple root β, the representation $\mu\mid_{K^\beta}$ of $K^\beta \simeq SO(2)$ does only contain the characters* $0,\, \pm 1,\, \pm 2$.

Let us look at the operator $\tilde{R}_\mu(s_\beta,\, \xi)$ when $(\mu,\, E_\mu)$ is a petite representation of K, β is a simple root, and γ is a real character of A satisfying $\langle \gamma,\, \beta \rangle \geqslant 0$.
Decomposing the domain of $\tilde{R}_\mu(s_\beta,\, \gamma)$ in MK^β-stable subspaces, we obtain:

$$\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_\mu,\, V^\delta) =$$

$$= \mathrm{Hom}_M(\mathrm{Res}^{MK^\beta}_M\, \varphi_0,\, \mathbb{C}) \oplus \mathrm{Hom}_M\left(\mathrm{Res}^{MK^\beta}_M(\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$$

(there are no other components because μ is petite). The operator $\tilde{R}_\mu(\omega_0,\, \gamma)$ acts trivially on $\mathrm{Hom}_M(\mathrm{Res}^{MK^\beta}_M\, \varphi_0,\, \mathbb{C})$ and it acts as scalar multiplication by $\frac{1-\langle \gamma,\, \check{\beta}\rangle}{1+\langle \gamma,\, \check{\beta}\rangle}$ on $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^\beta}_M(\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$.
The element $\sigma_\beta = \exp\left(\frac{\pi}{2} Z_\beta\right)$ is a representative in M' for the root reflection s_β. Clearly, σ_β acts by $(+1)$ on φ_0, and by (-1) on $(\varphi_2 \oplus \varphi_{-2})$. It easily follows that it acts

[14] Unless G is a classical group. See [5]

by $(+1)$ on $\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\, \mathbb{C})$, and by (-1) on $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$. Hence, if we regard $\mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu,\, V^\delta)$ as a representation ψ_μ of the Weyl group (which might be reducible), then we can identify $\mathrm{Hom}_M(\mathrm{Res}_M^{MK^\beta}\varphi_0,\, \mathbb{C})$ with the $(+1)$-eigenspace of $\psi_\mu(s_\beta)$, and $\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$ with the (-1)-eigenspace of $\psi_\mu(s_\beta)$. We obtain that:

$$\tilde{R}_\mu(s_\beta,\, \gamma) = \begin{cases} +1 & \text{on the } (+1)\text{-eigenspace of } \psi_\mu(s_\beta) \\ \frac{1-\langle\gamma,\, \check{\beta}\rangle}{1+\langle\gamma,\, \check{\beta}\rangle} & \text{on the } (-1)\text{-eigenspace of } \psi_\mu(s_\beta). \end{cases}$$

Therefore, $\boxed{\tilde{R}_\mu(s_\beta,\, \gamma) = \psi_\mu(A_{s_\beta}(\gamma))}$, for each simple root β, and for each real character γ of A satisfying $\langle\gamma,\, \beta\rangle \geqslant 0$.

It follows that for every dominant real character ν, the full intertwining operator $\tilde{R}_\mu(\omega_0,\, \nu)$ coincides with the p-adic operator $\psi_\mu(A(\nu))$:

$$\boxed{\tilde{R}_\mu(\omega_0,\, \nu) = \psi_\mu(A(\nu))}$$

(because of the common pattern in the factorization as a product of operators relative to simple reflections). We have proved the following theorem:

Theorem 11. *Suppose G is a real split group, and $(\mu,\, E_\mu)$ is a petite irreducible representation of K. Let ψ_μ be the representation of the Weyl group W on the space of M-fixed vectors of μ. Then $\tilde{R}_\mu(\omega_0,\, \nu) = \psi_\mu(A(\nu))$, for all real dominant characters ν of A.*

Corollary 1. *Let G be a real split group, let ν be a real dominant characters of A and let $(\mu,\, E_\mu)$ be a petite K-type of the spherical Langlands quotient $X(\nu)$. The Hermitian intertwining operator on the K-type μ can be computed by means of Weyl group calculations.*

Corollary 2. *Let G be a real split group, let ν be a real dominant characters of A and* *let $X(\nu)$ be the corresponding spherical Langlands quotient. For each petite K-type μ in $X(\nu)$, denote by ψ_μ the representation of W on the space of M-fixed vectors μ^M. If $X(\nu)$ is unitary, then the algebraic operator $\psi_\mu(A(\nu))$ is positive-semidefinte.*

This is a ***non-unitarity test***, in the sense that it can be used to show that some representations are not unitary: if the algebraic operator $\psi(A(\nu))$ fails to be positive semi-definite for some representation ψ that arises from a petite K-type μ in $X(\nu)$, then we can conclude that $X(\nu)$ is not unitary.

It is quite an amazing fact that this test also detects unitarity when the group G is a classical group. This beautiful result is due to Barbasch and can be found in [5]. It can be stated as follows:

Theorem 12. *If G is a classical (real or p-adic) split group, then the spherical Langlands quotient $X(\nu)$ is unitary if and only if the invariant Hermitian form is positive semi-definite on the petite K-types.*

Actually, Barbasch uses a subset of the set of petite K-types, that he calls "relevant" K-types. For each relevant K-type μ, the representation ψ_μ of W on μ^M is irreducible, and computing the signature of the algebraic operators $\psi_\mu(A(\nu))$ (for all relevant μ) is enough to detect unitarity. Therefore, we can study the spherical unitary dual of a split classical group by means of Weyl group computations. Even if Barbasch's result has a very simple form, it is really hard to explain. So we will not do any attempt in this direction.
Instead, in the next chapter, we will give an application of the result and show how to use the Weyl group to study the spherical unitary dual of $SL(3, \mathbb{R})$.

Chapter 7

The example of $SL(3, \mathbb{R})$

7.1 The data for $SL(3, \mathbb{R})$

In this section we briefly recall the data for the group $SL(3, \mathbb{R})$ and fix the notations that we will be using throughout the chapter.

- $G = SL(3, \mathbb{R}) = \{3 \times 3 \text{ real matrices with determinant } 1\}$
- $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R}) = \{3 \times 3 \text{ real matrices with trace } 0\}$
- $\vartheta\colon \mathfrak{g}_0 \to \mathfrak{g}_0,\ X \mapsto -X^T$
- $\mathfrak{k}_0 = \{3 \times 3 \text{ real skew-symmetric matrices with trace } 0\}$
- $\mathfrak{p}_0 = \{3 \times 3 \text{ real symmetric matrices with trace } 0\}$
- $\mathfrak{a}_0 = \{3 \times 3 \text{ real diagonal matrices with trace } 0\}$
- $\mathfrak{m}_0 = \{0\}$
- $\varepsilon_j\colon \mathfrak{a}_0 \to \mathbb{R},\ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \mapsto a_j \qquad (j = 1, 2, 3)$

- $\Delta = \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0) = \{\pm(\varepsilon_1 - \varepsilon_2),\, \pm(\varepsilon_2 - \varepsilon_3),\, \pm(\varepsilon_1 - \varepsilon_3)\}$

- $\Delta^+ = \Delta^+(\mathfrak{g}_0,\, \mathfrak{a}_0) = \{\varepsilon_1 - \varepsilon_2,\, \varepsilon_2 - \varepsilon_3,\, \varepsilon_1 - \varepsilon_3\}$

- $\Pi = \{\varepsilon_1 - \varepsilon_2,\, \varepsilon_2 - \varepsilon_3\}$ (simple roots)

- $(\mathfrak{g}_0)_{\varepsilon_i - \varepsilon_j} = \mathbb{R}E_{i,j}$ $(i, j = 1,\, 2,\, 3;\, i \neq j)$

- $K = SO(3,\, \mathbb{R}) = \{3 \times 3$ real orthogonal matrices with determinant $1\}$

- $A = \{3 \times 3$ diagonal matrices with determinant 1 and non-negative entries$\}$

- $M = \{3 \times 3$ diagonal matrices with determinant 1 and diag. entries $= \pm 1\}$

$$= \left\{ \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} : m_1 m_2 m_3 = 1,\, m_j = \pm 1 \quad \forall\, j = 1,\, 2,\, 3 \right\}$$

- $N = \{3 \times 3$ upper triangular real matrices with diagonal entries $= 1\}$

- $P = MAN = \{3 \times 3$ upper triangular real matrices$\}$

- $M' = \left\{ \begin{pmatrix} \epsilon_1\, \underline{e}_{\sigma(1)} \\ \epsilon_2\, \underline{e}_{\sigma(2)} \\ \epsilon_3\, \underline{e}_{\sigma(3)} \end{pmatrix} : \sigma \in \mathcal{S}_3,\, \epsilon_j = \pm 1 \text{ and } (\epsilon_1 \epsilon_2 \epsilon_3) sgn(\sigma) = 1 \right\}$

- $W \simeq \mathcal{S}_3$ (symmetric group on 3 letters)

- $\omega_0 = s_{\varepsilon_1 - \varepsilon_3}$ (the long element of W)

- $\widehat{M} = \{\delta_0,\, \delta_1,\, \delta_2,\, \delta_3\}$ with δ_0 the trivial representation of M, and

$$\delta_j \colon M \to \mathbb{R},\ \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \mapsto m_j,\ \forall\, j = 1,\, 2,\, 3.$$

7.2 The irreducible representations of $SO(3)$

The group $K = SO(3)$ has no irreducible representations of even dimension and, up to equivalence, a unique irreducible representation of each odd dimension.

For each integer $N \geqslant 0$, consider the space $\mathcal{H}_N$ of harmonic homogeneous polynomials of degree N in three variables, with complex coefficients. $\mathcal{H}_N$ is a complex vector space of dimension $2N+1$, and $SO(3)$ acts on it by:

$$(g \cdot F)(x,\, y,\, z) = F((x,\, y,\, z)g) \quad \forall\, g \in SO(3),\, \forall\, F \in \mathcal{H}_N.$$

This representation is irreducible, and any irreducible representation of $SO(3)$ is isomorphic to some $\mathcal{H}_N$.

7.2.1 A closer look at $\mathcal{H}_N$

The following remark is helpful to understand the structure of $\mathcal{H}_N$.

Remark 4. *Every harmonic homogeneous polynomial of degree N in three variables can be written uniquely in the form:*

$$F(x,\, y,\, z) = \sum_{k=1}^{N} \frac{x^k}{k!} F_k(y,\, z)$$

with $F_0 \dots F_N$ homogeneous polynomials in two variables y and z (of degree $deg(F_k) = N-k$) satisfying:

$$F_k(y,\, z) = -\Delta_{y,z} F_{k-2}(y,\, z) \quad \forall\, k \geqslant 2.$$

We have denoted by $\Delta_{y,z}$ the Laplacian $\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Proof. Let F be a harmonic homogeneous polynomial of degree N in x, y, z. Because

F is homogeneous, we can write

$$F(x,\, y,\, z) = \sum_{k=1}^{N} \frac{x^k}{k!} F_k(y,\, z)$$

with each F_k homogeneous of degree $N-k$. We now show that because F is harmonic, the condition

$$F_k(y,\, z) = -\Delta_{y,z} F_{k-2}(y,\, z) \quad \forall\, k \geqslant 2$$

must hold. We do it by writing down an explicit expression for the Laplacian of F:

$$\Delta_{x,y,z} F = \left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2} + \tfrac{\partial^2}{\partial z^2}\right) \left(\textstyle\sum_{k=1}^{N} \tfrac{x^k}{k!}\, F_k(y,\, z)\right) =$$

$$= \textstyle\sum_{k\geqslant 0} \tfrac{1}{k!} \left(\tfrac{\partial^2 x^k}{\partial x^2}\right) F_k(y,\, z) + \sum_{k\geqslant 0} \tfrac{x^k}{k!} \left(\tfrac{\partial^2}{\partial y^2} + \tfrac{\partial^2}{\partial z^2}\right) (F_k(y,\, z)) =$$

$$= \textstyle\sum_{k\geqslant 2} \tfrac{k(k-1)}{k!} x^{k-2} F_k(y,\, z) + \sum_{k\geqslant 0} \tfrac{x^k}{k!}\, \Delta_{y,z} F_k(y,\, z) =$$

$$= \textstyle\sum_{k\geqslant 2} \tfrac{x^{k-2}}{(k-2)!}\, [F_k(y,\, z) + \Delta_{y,z} F_{k-2}(y,\, z)] + \tfrac{x^{N-1}}{(N-1)!}\, \Delta_{y,z} F_{N-1}(y,\, z) +$$

$$+ \tfrac{x^N}{N!}\, \Delta_{y,z} F_N(y,\, z) = \qquad (\circledast)$$

$$= \textstyle\sum_{k\geqslant 2} \tfrac{x^{k-2}}{(k-2)!}\, [F_k(y,\, z) + \Delta_{y,z} F_{k-2}(y,\, z)]$$

In $(\circledast)$ we have used the fact that F_N and F_{N-1} have degree less than two, so their Laplacian is zero. It is then clear that $\Delta_{x,y,z} F = 0$ if and only if $[F_k(y,\, z) + \Delta_{y,z} F_{k-2}(y,\, z)] = 0$, for all $k \geqslant 2$. □

Corollary 3. *$\mathcal{H}_N$ has dimension $2N+1$.*

Proof. Let $F(x,\, y,\, z) = \sum_{k=1}^{N} \frac{x^k}{k!}\, F_k(y,\, z)$ be a harmonic homogeneous polynomial of degree N in $x,\, y,\, z$. F is completely determined by $F_0 = \sum_{j=0}^{N} c_j\, y^j z^{N-j}$ (which is a homogeneous polynomial of degree N in y, z) and by $F_1 = \sum_{j=0}^{N-1} d_j\, y^j z^{N-j}$ (which is a homogeneous polynomial of degree $N-1$ in y, z). So F depends on a total of $(N+1)+N = 2N+1$ complex parameters. $\square$

7.2.2 Spherical K-types

In this subsection we study restriction of $(\mu_n, \mathcal{H}_n)$ to M, and its decomposition in isotypic components of irreducible representations of M. In particular, we want to determine whether $\mathcal{H}_n$ contains the trivial M-type.

It is easy to see the action of M on $\mathcal{H}_n$ is given by

$$\mu_n \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \cdot F(x,\, y,\, z) = F(m_1 x,\, m_2 y,\, m_3 z)$$

for all $m = diag(m_1, m_2, m_3)$ in M, and all F in $\mathcal{H}_n$. We write $F = \sum_{k=0}^{N} \frac{x^k}{k!} F_k(y,\, z)$ and observe that, when N is *even*

- $$\mu_n \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot F = \begin{cases} +F & \text{if } F_1 = 0 \\ -F & \text{if } F_0 = 0 \end{cases}$$

- $$\mu_n \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot F = \begin{cases} +F & \text{if } F_0 \text{ only contains even powers of } z, \\ & \text{and } F_1 \text{ only contains odd powers of } z \\ -F & \text{if } F_0 \text{ only contains odd powers of } z, \\ & \text{and } F_1 \text{ only contains even powers of } z \end{cases}$$

- $\mu_n \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot F = \begin{cases} +F & \text{if } F_0 \text{ only contains even powers of } y, \\ & \text{and } F_1 \text{ only contains odd powers of } y \\ & \\ -F & \text{if } F_0 \text{ only contains odd powers of } z, \\ & \text{and } F_1 \text{ only contains even powers of } y. \end{cases}$

Comparing these results with the character table of M

	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
δ_0	1	1	1	1
δ_1	1	1	-1	-1
δ_2	1	-1	1	-1
δ_3	1	-1	-1	1

we conclude that

- F spans a copy of δ_0 if and only if $F_1 = 0$ and F_0 only involves even powers of both y and z. The mapping $F \mapsto F_0$ defines an isomorphism from the isotypic component of δ_0 in $\mathcal{H}_n$ to the vector space (of dimension $\frac{N}{2}+1$) of all the homogeneous polynomials of degree N in y, z that only contain even powers of both y and z:

$$V(\delta_0) \simeq \{a_0 y^N + a_2 y^{N-2} z^2 + \dots a_{N-2} y^2 z^{N-2} + a_N z^N\}.$$

- F spans a copy of δ_1 if and only if $F_1 = 0$ and F_0 only involves odd powers of both y and z. The mapping $F \mapsto F_0$ defines an isomorphism from the isotypic component of δ_1 in $\mathcal{H}_n$ to the vector space (of dimension $\frac{N}{2}$) of all the homogeneous polynomials of degree N in y, z that only contain odd powers of both y and z.

$$V(\delta_1) \simeq \{a_1 y^{N-1} z + a_3 y^{N-3} z^3 + \dots a_{N-3} y^3 z^{N-3} + a_{N-1} y z^{N-1}\}.$$

- F spans a copy of δ_2 if and only if $F_0 = 0$ and F_1 only involves even powers of y and odd powers of z. The mapping $F \mapsto F_1$ defines an isomorphism from the isotypic component of δ_2 in $\mathcal{H}_n$ to the vector space (of dimension $\frac{N}{2}$) of all the homogeneous polynomials of degree $N-1$ in y, z that only contain even powers of y and odd powers of z.

$$V(\delta_2) \simeq \{a_1 y^{N-2} z + a_3 y^{N-4} z^3 + \dots a_{N-3} y^2 z^{N-3} + a_{N-1} z^{N-1}\}.$$

- F spans a copy of δ_3 if and only if $F_0 = 0$ and F_1 only involves odd powers of y and even powers of z. The mapping $F \mapsto F_1$ defines an isomorphism from the isotypic component of δ_2 in $\mathcal{H}_n$ to the vector space (of dimension $\frac{N}{2}$) of all the homogeneous polynomials of degree $N-1$ in y, z that only contain odd powers of y and even powers of z.

$$V(\delta_3) \simeq \{a_0 y^{N-1} + a_2 y^{N-3} z^2 + \dots a_{N-3} y^3 z^{N-4} + a_{N-2} y z^{N-2}\}.$$

The situation for N *odd* is slightly different, but can be analyzed in a similar way. The results are as follows:

- The isotypic component of δ_0 in $\mathcal{H}_n$ has dimension $\frac{N-1}{2}$, and the mapping

  $$V(\delta_0) \to \{a_1 y^{N-2} z + a_3 y^{N-4} z^3 + \dots a_{N-4} y^3 z^{N-4} + a_{N-2} y z^{N-2}\}, F \mapsto F_1$$

  is an isomorphism.

- The isotypic component of δ_1 in $\mathcal{H}_n$ has dimension $\frac{N+1}{2}$, and the mapping

  $$V(\delta_1) \to \{a_0 y^{N-1} + a_2 y^{N-3} z^2 + \dots a_{N-3} y^2 z^{N-3} + a_{N-1} z^{N-1}\}, F \mapsto F_1$$

  is an isomorphism.

- The isotypic component of δ_2 in $\mathcal{H}_n$ has dimension $\frac{N+1}{2}$, and the mapping

  $$V(\delta_2) \to \{a_0 y^{N} + a_2 y^{N-2} z^2 + \dots a_{N-2} y^2 z^{N-2} + a_{N-1} y z^{N-1}\}, F \mapsto F_0$$

  is an isomorphism.

- The isotypic component of δ_3 in $\mathcal{H}_n$ has dimension $\frac{N+1}{2}$, and the mapping

  $$V(\delta_3) \to \{a_1 y^{N-1} z + a_3 y^{N-3} z^3 + \dots a_{N-2} y^2 z^{N-2} + a_N y z^{N}\}, F \mapsto F_0$$

  is an isomorphism.

We have proved the following claim:

Claim 4. *The irreducible representation $\mathcal{H}_N$ is spherical for each $N \neq 1$. Indeed, for all $m \geqslant 0$, we have*

$$\mathcal{H}_{2m} = \delta_0^{m+1} \oplus \delta_1^m \oplus \delta_2^m \oplus \delta_3^m$$

$$\mathcal{H}_{2m+1} = \delta_0^m \oplus \delta_1^{m+1} \oplus \delta_2^{m+1} \oplus \delta_3^{m+1}.$$

7.2.3 Petite K-types

In this subsection we determine the values of N for which the irreducible representation $(\mu_N, \mathcal{H}_N)$ of $K = SO(3)$ is petite.

For each simple root α, we must look at the $SO(2)$ subgroup attached to α

$$K^\alpha = \exp(\mathbb{R}Z_\alpha) = \exp(\mathbb{R}(E_\alpha + \vartheta E_\alpha))$$

and check whether the restriction of μ_N to K^α only contains the characters 0, ± 1 and ± 2 of $SO(2)$. This is equivalent to verify that for each simple root α, $d\mu_N(Z_\alpha)$ only acts with eigenvalues 0, $\pm i$ and $\pm 2i$.

We start with an explicit description of the subgroups K^α.

- For $\alpha = \varepsilon_1 - \varepsilon_2$, we take $Z_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore

$$K^\alpha = \left\{ \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

- For $\alpha' = \varepsilon_2 - \varepsilon_3$, we take $Z_{\alpha'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Therefore

$$K^{\alpha'} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} : t \in \mathbb{R} \right\}.$$

We notice that Z_α and $Z_{\alpha'}$ are conjugate (via an element of M'):

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

hence $d\mu_N(Z_\alpha)$ and $d\mu_N(Z_{\alpha'})$ have the same eigenvalues. We just need to consider one of the two simple roots. We will focus on $\alpha' = \varepsilon_2 - \varepsilon_3$, and we will assume that N is even (the odd case is similar).

An easy computation shows that

$$d\mu_N(Z_{\alpha'}) \colon \mathcal{H}_N \to \mathcal{H}_N,\ F(x, y, z) \mapsto y \left(\frac{\partial F}{\partial z} \right) - z \left(\frac{\partial F}{\partial y} \right).$$

Since $d\mu_N(Z_{\alpha'})$ does not touch the variable x, it preserves the subspaces

$$\mathcal{H}_N^{(1)} = V(\delta_0) \oplus V(\delta_1) \qquad \mathcal{H}_N^{(2)} = V(\delta_2) \oplus V(\delta_3)$$

of $\mathcal{H}_N$.[1] Identifying each element F of $\mathcal{H}_N^{(1)}$ with its F_0-part

$$a_0 z^N + a_1 y z^{N-1} + \cdots + a_{N-1} y^{n-1} z + a_N y^N$$

and writing down the explicit matrix for the action of $d\mu_N(Z_{\alpha'})$ on $\mathcal{H}_N^{(1)}$, we find that $d\mu_N(Z_{\alpha'})$ has *even* eigenvalues

$$-Ni,\ -(N-2)i, \ldots,\ -2i,\ 0,\ 2i, \ldots,\ (N-2)i,\ Ni.$$

This is compatible with the fact that

$$m_{\alpha'} = \exp(\pi Z_{\alpha'}) = \mathrm{diag}(1,\ -1,\ -1)$$

acts on $V(\delta_0) \oplus V(\delta_1)$ as scalar multiplication by $+1$. Similarly[2], we find that $d\mu_N(Z_{\alpha'})$ acts on $\mathcal{H}_N^{(2)}$ with *odd* eigenvalues

$$-(N-1)i,\ -(N-3)i, \ldots,\ -i,\ i, \ldots,\ (N-3)i,\ (N-1)i,$$

compatibly with the fact that $m_{\alpha'}$ acts by -1 on $V(\delta_2) \oplus V(\delta_3)$. We conclude that

Claim 5. *For each simple root α, the restriction of $\mathcal{H}_N$ to $K^\alpha \simeq SO(2)$ is given by*

$$\boxed{\mathcal{H}_N \mid_{K^\alpha} = \bigoplus_{l=-N}^{N} \xi_l}\,.$$

Therefore, $\mathcal{H}_N$ is petite if and only if $N = 0$, 1 or 2.

Corollary 4. *The only irreducible representations of $SO(3)$ that are both petite and*

[1] Because N is even, we can identify $\mathcal{H}_N^{(1)}$ with the subspace of $\mathcal{H}_N$ that consists of polynomials with even powers of x (and arbitrary powers of y or z) and $\mathcal{H}_N^{(2)}$ with the subspace of $\mathcal{H}_N$ that consists of polynomials with odd powers of x (and arbitrary powers of y or z).

[2] After identifying each element F of $\mathcal{H}_N^{(2)}$ with its F_1-part.

spherical are $\mathcal{H}_0$ and $\mathcal{H}_2$.

7.3 The representations of the Weyl group

We identify the Weyl group of $SO(3)$ with the symmetric group in three letters $W = \mathcal{S}_3$. There are three conjugancy classes, corresponding to the three cycle structures (1), (23) and (123), hence three irreducible inequivalent representations that we denote by U, U' and V. More precisely:

- U is the trivial representation of $\mathcal{S}_3$. Every permutation acts by $+1$.
- U' is the sign representation of $\mathcal{S}_3$. A permutation σ acts by $sgn(\sigma)$.
- W is "the other" representation of $\mathcal{S}_3$, necessarily[3] of dimension 2.

The character table of $\mathcal{S}_3$ can easily be computed using the equation

$$\sum_{\nu \text{ irred.}} \dim(\nu)\mathcal{X}_\nu(\sigma) = 0 \qquad \forall\, \sigma \in \mathcal{S}_3 - \{1\}.$$

We obtain:

$\mathcal{X}$	1	(23)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

.

We now give a more explicit realization of the two-dimensional representation V of $\mathcal{S}_3$. Let E_V be the hyperplane

$$\{\overrightarrow{v} \in \mathbb{C}^3 \colon v_1 + v_2 + v_3 = 0\}$$

[3]The sum of the square of the dimensions of the irreducible representations equals the size of the group.

and let a permutation π of $\mathcal{S}_3$ act on E_V by

$$\pi \cdot (v_1,\, v_2,\, v_3) = \big(v_{\pi^{-1}(1)},\, v_{\pi^{-1}(2)},\, v_{\pi^{-1}(3)}\big)$$

for all $\overrightarrow{v}$ in E_V. We notice that if ω is a primitive cubic root of 1, then $1+\omega+\omega^2 = 0$. So the vectors

$$\overrightarrow{x} = (1,\, \omega^2,\, \omega) \qquad \overrightarrow{y} = (1,\, \omega,\, \omega^2)$$

belong to E_V, and actually form a basis of E_V. It is easy to check that $\overrightarrow{x}$ and $\overrightarrow{y}$ are eigenvectors of τ=(123) of eigenvalue ω and ω^2 respectively, and that σ=(23) permutes $\overrightarrow{x}$ and $\overrightarrow{y}$ (hence the vectors $\overrightarrow{x}+\overrightarrow{y}$ and $\overrightarrow{x}-\overrightarrow{y}$ are eigenvectors of σ=(23) of eigenvalue $+1$ and -1 respectively).

It follows that σ=(23) acts on E_V with trace zero, and τ=(123) acts on E_V with trace $\omega+\omega^2 = -1$. Therefore E_V is a realization of the irreducible two-dimensional representation V of $\mathcal{S}_3$.

We conclude this analysis of the representations of the symmetric group $\mathcal{S}_3$ by giving an explicit formula for the decomposition of a representation $(\psi,\, E_\psi)$ of $\mathcal{S}_3$ into isotypic components of irreducible representations:

$$\boxed{\psi = U^a \oplus (U')^b \oplus V^c}$$

with

$$a = m_1(\psi(\sigma)) - m_\omega(\psi(\tau))$$

$$b = m_{-1}(\psi(\sigma)) - m_\omega(\psi(\tau))$$

$$c = m_\omega(\psi(\tau)).$$

We have denoted by $m_z(\psi(\pi))$ the algebraic multiplicity of the eigenvalue z for $\psi(\pi)$, for each permutation $\pi = \tau$ or σ, and each complex number $z = 1,\, -1$ or ω. This formula easily follows form the facts that

- σ has eigenvalues

$+1$ on each copy of U

-1 on each copy of U'

$+1, -1$ on each copy of V

- τ has eigenvalues

$+1$ on each copy of U

$+1$ on each copy of U'

ω, ω^2 on each copy of V.

7.3.1 $\psi_{\mathcal{H}_2}$: the representation of $\mathcal{S}_3$ on $(\mathcal{H}_2)^M$

In this subsection we study the representation ψ_ν of the Weyl group on the space of M invariants of a petite spherical representation μ of $SO(3)$.
If $\mu = \mathcal{H}_0$, then μ is the trivial representation of K. Clearly $\mu^M = \mathbb{C}$ and W acts trivially on it. The case $\mu = \mathcal{H}_2$ is more interesting. Recall that

$$E_{\mathcal{H}_2} = \underbrace{\{ay^2 + bz^2 - (a+b)x^2 \colon a,\, b \in \mathbb{C}\}}_{V(\delta_0)} \oplus \underbrace{\mathbb{C}\, yz}_{V(\delta_1)} \oplus \underbrace{\mathbb{C}\, xz}_{V(\delta_2)} \oplus \underbrace{\mathbb{C}\, xy}_{V(\delta_3)}$$

with K acting by:

$$(g \cdot F)(x,\, y,\, z) = F((x,\, y,\, z)g) \quad \forall\, g \in K, \forall\, F \in \mathcal{H}_2.$$

We want to determine the representation of $W = \mathcal{S}_3$ on the two-dimensional subspace

$$(E_{\mathcal{H}_2})^M = \{ay^2 + bz^2 - (a+b)x^2 \colon a,\, b \in \mathbb{C}\}.$$

It is enough to understand the action of σ=(23) and τ=(123). The first step is to pick representatives in M' for the two permutations. We choose

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \tau = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

For all $F = ay^2 + bz^2 - (a+b)x^2$ in $(E_{\mathcal{H}_2})^M$, we have

$$\psi_{\mathcal{H}_2}(\sigma) \cdot F = F\left((x,\, y\, z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right) = F(x,\, -z,\, y) = by^2 + az^2 - (a+b)x^2.$$

Therefore, the matrix associated to $\psi_{\mathcal{H}_2}(\sigma)$ with respect to the basis $\{y^2 - x^2,\, z^2 - x^2\}$

is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It has eigenvalues $+1$ and -1. Similarly, for τ, we have:

$$\psi_{\mathcal{H}_2}(\tau) \cdot F = F\left((x,\, y\, z) \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \right) = F(-y,\, -z,\, x) = -(a+b)y^2 + az^2 + bx^2.$$

We associate to $\psi_{\mathcal{H}_2}(\tau)$ the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues ω and ω^2.

We conclude that

$$a = m_1(\psi_{\mathcal{H}_2}(\sigma)) - m_\omega(\psi_{\mathcal{H}_2}(\tau)) = 0$$

$$b = m_{-1}(\psi_{\mathcal{H}_2}(\sigma)) - m_\omega(\psi_{\mathcal{H}_2}(\tau)) = 0$$

$$c = m_\omega(\psi_{\mathcal{H}_2}(\tau)) = 1.$$

Hence, $\psi_{\mathcal{H}_2} = V$.

Conclusions. *The representation of the Weyl group on the space of M-invariants in $\mathcal{H}_2$ is irreducible, and isomorphic to V.*

7.4 Signature on the K-type $\mathcal{H}_2$

In this section we compute the signature w.r.t. the K-type $\mathcal{H}_2$ of the standard intertwining operator for a principal series $I_P(\delta \otimes \nu)$ of $SL(3, \mathbb{R})$, under the assumptions that $P = MAN$ is a minimal parabolic subgroup, δ is the trivial representation of M and ν is a real dominant character of A.

Because $\mathcal{H}_2$ is a petite representation, there are two ways to do this:[4]

(1) construct the "analytic" operator $\tilde{R}_{\mathcal{H}_2}(\omega_0, \nu)$ and compute its signature

(2) construct the "algebraic" operator $\psi_{\mathcal{H}_2}(\nu)$ and compute its signature.

In both cases ν is intended to be a real dominant character of A

$$\nu = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \lambda_3 \varepsilon_3 \qquad \left(\sum_{j=1}^{3} \lambda_j = 0 \text{ and } \lambda_1 > \lambda_2 > \lambda_3 \right)$$

satisfying the "symmetry" condition $\omega_0 \cdot \nu = -\nu$. As usual, we have denoted by ω_0 the long element of the Weyl group: $\omega_0 = s_{\varepsilon_1 - \varepsilon_3}$. It is easy to check that ν must be of the form

$$\nu = \lambda \varepsilon_1 - \lambda \varepsilon_3$$

for some λ real and positive.

7.4.1 The "analytic" operator $\tilde{R}_{\mathcal{H}_2}(\omega_0, \nu)$

Recall the main steps for the construction of $\tilde{R}_{\mathcal{H}_2}(\omega_0, \nu)$:

[4] Please, refer to chapter 6 for notations.

(i) Find a minimal decomposition of ω_0 as a product of simple reflections:

$$\omega_0 = s_{\alpha_r} \cdots s_{\alpha_2} s_{\alpha_1}$$

(ii) Decompose $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ accordingly:

$$\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu) = \tilde{R}_{\mathcal{H}_2}(s_{\alpha_r} s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1},\, \nu) =$$
$$= \tilde{R}_{\mathcal{H}_2}(s_{\alpha_r},\, (s_{\alpha_{r-1}} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \nu) \cdots \tilde{R}_{\mathcal{H}_2}(s_{\alpha_2},\, s_{\alpha_1} \cdot \nu) \tilde{R}_{\mu}(s_{\alpha_1},\, \nu)$$

(iii) For each simple root β and for each real character γ of A (satisfying $\langle \gamma,\, \beta \rangle \geqslant 0$), define $\tilde{R}_{\mathcal{H}_2}(s_\beta,\, \gamma)$ as follows: for all integers l, let φ_l be the isotypic component of the character $\xi_l\colon \exp(tZ_\beta) \mapsto e^{i l t}$ of $K^\beta \simeq SO(2)$ inside $\mathcal{H}_2$, so that

$$E_{\mathcal{H}_2} = \bigoplus_{l \in \mathbb{Z}} \varphi_l$$

is the decomposition of $\mathcal{H}_2$ in K^β-types, and

$$\mathrm{Hom}_M(\mathrm{Res}_M^K E_{\mathcal{H}_2},\, V^\delta) = \bigoplus_{m \in \mathbb{N}} \underbrace{\mathrm{Hom}_M\left(\mathrm{Res}_M^{MK^\beta}(\varphi_{2m} + \varphi_{-2m}),\, \mathbb{C}\right)}_{U_m}$$

is a decomposition of $\mathrm{Hom}_M(\mathrm{Res}_M^K E_{\mathcal{H}_2},\, V^\delta)$ in MK^β-stable subspaces. Define the operator $\tilde{R}_{\mathcal{H}_2}(s_\beta,\, \gamma)$ to act on U_m as scalar multiplication by

$$\frac{\prod_{j=1}^m ((2j-1) - \langle \gamma,\, \check{\beta} \rangle)}{\prod_{j=1}^m ((2j-1) + \langle \gamma,\, \check{\beta} \rangle)}$$

for all non-negative integers m.

Steps (i) and (ii) are really easy:

$$w_0 = s_{\varepsilon_1 - \varepsilon_3} = s_{\varepsilon_1 - \varepsilon_2}\, s_{\varepsilon_2 - \varepsilon_3}\, s_{\varepsilon_1 - \varepsilon_2}$$

and the corresponding decomposition of the operator $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_3},\, \nu)$ is

$$\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, (s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2})\cdot\nu)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2-\varepsilon_3},\, s_{\varepsilon_1-\varepsilon_2}\cdot\nu)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, \nu).$$

We observe that

$$\nu = \lambda\varepsilon_1 - \lambda\varepsilon_3$$

$$s_{\varepsilon_1-\varepsilon_2}\cdot\nu = \lambda\varepsilon_2 - \lambda\varepsilon_3$$

$$(s_{\varepsilon_2-\varepsilon_3}\, s_{\varepsilon_1-\varepsilon_2})\cdot\nu = -\lambda\varepsilon_2 + \lambda\varepsilon_3$$

so can write:

$$\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu) = \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_3},\, \lambda\varepsilon_1 - \lambda\varepsilon_3) =$$

$$= \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, -\lambda\varepsilon_2 + \lambda\varepsilon_3)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2-\varepsilon_3},\, \lambda\varepsilon_2 - \lambda\varepsilon_3)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, \lambda\varepsilon_1 - \lambda\varepsilon_3).$$

To continue, we need to understand the decomposition of $(\mathcal{H}_2,\, E_{\mathcal{H}_2})$ in irreducible representations of the $SO(2)$ subgroup associated to each simple root.

We start with $\alpha = \varepsilon_1 - \varepsilon_2$. Then

$$K^\alpha = \left\{ k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

An easy computations shows that

$$k_\theta \cdot F(x,\, y,\, z) = F(x\, \cos(\theta) - y\, \sin(\theta),\, x\, \sin(\theta) + y\, \cos(\theta),\, z)$$

for all F in $E_{\mathcal{H}_2}$. In particular

$$k_\theta \cdot (yz - i\,xz) = e^{i\,\theta}(yz - i\,xz)$$

$$k_\theta \cdot (yz + i\,xz) = e^{-i\,\theta}(yz + i\,xz)$$

$$k_\theta \cdot (2z^2 - x^2 - y^2) = +(2z^2 - x^2 - y^2)$$

$$k_\theta \cdot (y^2 - x^2 - 2i\,xy) = e^{2i\,\theta}(y^2 - x^2 - 2i\,xy)$$

$$k_\theta \cdot (y^2 - x^2 + 2i\,xy) = e^{-2i\,\theta}(y^2 - x^2 + 2i\,xy).$$

The decomposition of $E_{\mathcal{H}_2}$ in K^α-types is therefore given by:

$$E_{\mathcal{H}_2} = \underbrace{\mathbb{C}(2z^2 - x^2 - y^2)}_{\varphi_0} \oplus \underbrace{\mathbb{C}(y^2 - x^2 - 2i\,xy)}_{\varphi_2} \oplus \underbrace{\mathbb{C}(y^2 - x^2 + 2i\,xy)}_{\varphi_{-2}} \oplus$$

$$\oplus \underbrace{\mathbb{C}(yz - i\,xz)}_{\varphi_{+1}} \oplus \underbrace{\mathbb{C}(yz + i\,xz)}_{\varphi_{-1}}\ .$$

The corresponding decomposition of $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^\delta)$ in MK^α-stable subspaces is

$$\mathrm{Hom}_M(\mathrm{Res}^{MK^\alpha}_M\, \varphi_0,\, \mathbb{C}) \,\oplus\, \mathrm{Hom}_M\left(\mathrm{Res}^{MK^\alpha}_M(\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$$

with

- $\mathrm{Hom}_M(\mathrm{Res}^{MK^\alpha}_M\, \varphi_0,\, \mathbb{C}) \simeq \mathbb{C}(2z^2 - x^2 - y^2)$
- $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^\alpha}_M(\varphi_2 \oplus \varphi_{-2})\right) \simeq \mathbb{C}(y^2 - x^2\right)$
- $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^\delta) \simeq \mathbb{C}(y^2 - x^2) \,\oplus\, \mathbb{C}(z^2 - x^2) \simeq (E_{\mathcal{H}_2})^M$.

Once this decomposition is understood, it is really easy to compute the operator $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1 - \varepsilon_2},\, \gamma)$, for each real character γ satisfying $\langle \gamma,\, \varepsilon_1 - \varepsilon_2 \rangle \geqslant 0$. Indeed, $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1 - \varepsilon_2},\, \gamma)$ acts trivially $\mathrm{Hom}_M(\mathrm{Res}^{MK^\alpha}_M\, \varphi_0,\, \mathbb{C})$, and it acts on $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^\alpha}_M(\varphi_2 \oplus \varphi_{-2})\right)$ as scalar multiplication by

$$\frac{1-\langle\gamma,\,\frac{2(\varepsilon_1-\varepsilon_2)}{\|\varepsilon_1-\varepsilon_2\|^2}\rangle}{1+\langle\gamma,\,\frac{2(\varepsilon_1-\varepsilon_2)}{\|\varepsilon_1-\varepsilon_2\|^2}\rangle}=\frac{1-\langle\gamma,\,\varepsilon_1-\varepsilon_2\rangle}{1+\langle\gamma,\,\varepsilon_1-\varepsilon_2\rangle}.$$

The matrix associated to $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\,\gamma)$ with respect to the basis

$$\mathcal{B}=\{(y^2-x^2),\,(z^2-x^2)\}$$

of $\operatorname{Hom}_M(\operatorname{Res}^K_M E_{\mathcal{H}_2},\,V^\delta)\simeq(E_{\mathcal{H}_2})^M$ is therefore given by

$$\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\,\gamma)\rightsquigarrow\begin{pmatrix}\frac{1-\langle\gamma,\varepsilon_1-\varepsilon_2\rangle}{1+\langle\gamma,\varepsilon_1-\varepsilon_2\rangle} & \frac{-\langle\gamma,\varepsilon_1-\varepsilon_2\rangle}{1+\langle\gamma,\varepsilon_1-\varepsilon_2\rangle}\\ 0 & 1\end{pmatrix}.$$

When $\gamma=-\lambda\varepsilon_2+\lambda\varepsilon_3$ or $\gamma=+\lambda\varepsilon_1-\lambda\varepsilon_3$, we have $\langle\gamma,\,\varepsilon_1-\varepsilon_2\rangle=+\lambda$. Hence:

$$\boxed{\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\,-\lambda\varepsilon_2+\lambda\varepsilon_3)=\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\,\lambda\varepsilon_1-\lambda\varepsilon_3)\rightsquigarrow\begin{pmatrix}\frac{1-\lambda}{1+\lambda} & \frac{-\lambda}{1+\lambda}\\ 0 & 1\end{pmatrix}}.$$

The construction for $\alpha'=\varepsilon_2-\varepsilon_3$ is very similar. The subgroup

$$K^{\alpha'}=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{pmatrix}:\theta\in\mathbb{R}\right\}$$

acts on $E_{\mathcal{H}_2}$ by

$$k_\theta\cdot F(x,\,y,\,z)=F(x,\,y\,\cos(\theta)-z\,\sin(\theta),\,y\,\sin(\theta)+z\,\cos(\theta))$$

for all F in $E_{\mathcal{H}_2}$. In particular

$$k_\theta\cdot(xy+i\,xz)=e^{i\,\theta}(xy+i\,xz)$$

$$k_\theta \cdot (xy - i\,xz) = e^{-i\,\theta}(xy - i\,xz)$$

$$k_\theta \cdot (y^2 + z^2 - 2x^2) = +(y^2 + z^2 - 2x^2)$$

$$k_\theta \cdot (y^2 - z^2 + 2i\,yz) = e^{2i\,\theta}(y^2 - z^2 + 2i\,yz)$$

$$k_\theta \cdot (y^2 - z^2 - 2i\,yz) = e^{-2i\,\theta}(y^2 - z^2 - 2i\,yz).$$

The decomposition of $E_{\mathcal{H}_2}$ in $K^{\alpha'}$-types is therefore given by:

$$E_{\mathcal{H}_2} = \underbrace{\mathbb{C}(y^2 + z^2 - 2x^2)}_{\varphi_0} \oplus \underbrace{\mathbb{C}(y^2 - z^2 + 2i\,yz)}_{\varphi_2} \oplus \underbrace{\mathbb{C}(y^2 - z^2 - 2i\,yz)}_{\varphi_{-2}} \oplus$$

$$\oplus \underbrace{\mathbb{C}(xy + i\,xz)}_{\varphi_{+1}} \oplus \underbrace{\mathbb{C}(xy - i\,xz)}_{\varphi_{-1}}.$$

The corresponding decomposition of $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^\delta)$ in $MK^{\alpha'}$-stable subspaces is

$$\mathrm{Hom}_M(\mathrm{Res}^{MK^{\alpha'}}_M\, \varphi_0,\, \mathbb{C}) \,\oplus\, \mathrm{Hom}_M\left(\mathrm{Res}^{MK^{\alpha'}}_M (\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$$

with

- $\mathrm{Hom}_M(\mathrm{Res}^{MK^{\alpha'}}_M\, \varphi_0,\, \mathbb{C}) \simeq \mathbb{C}(y^2 + z^2 - 2x^2)$
- $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^{\alpha'}}_M (\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right) \simeq \mathbb{C}(y^2 - z^2)$
- $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^\delta) \simeq \mathbb{C}(y^2 - x^2) \,\oplus\, \mathbb{C}(z^2 - x^2) \simeq (E_{\mathcal{H}_2})^M.$

For each real character γ satisfying $\langle \gamma,\, \varepsilon_2 - \varepsilon_3 \rangle \geqslant 0$, the operator $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2 - \varepsilon_3},\, \gamma)$ acts trivially $\mathrm{Hom}_M(\mathrm{Res}^{MK^{\alpha'}}_M\, \varphi_0,\, \mathbb{C})$, and it acts on $\mathrm{Hom}_M\left(\mathrm{Res}^{MK^{\alpha'}}_M (\varphi_2 \oplus \varphi_{-2}),\, \mathbb{C}\right)$ as scalar multiplication by

$$\frac{1 - \langle \gamma,\, \frac{2(\varepsilon_2 - \varepsilon_3)}{\|\varepsilon_2 - \varepsilon_3\|^2} \rangle}{1 + \langle \gamma,\, \frac{2(\varepsilon_2 - \varepsilon_3)}{\|\varepsilon_2 - \varepsilon_3\|^2} \rangle} = \frac{1 - \langle \gamma,\, \varepsilon_2 - \varepsilon_3 \rangle}{1 + \langle \gamma,\, \varepsilon_2 - \varepsilon_3 \rangle}.$$

The matrix associated to $\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2-\varepsilon_3},\, \gamma)$ with respect to the basis

$$\mathcal{B} = \{(y^2 - x^2),\, (z^2 - x^2)\}$$

of $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^\delta) \simeq (E_{\mathcal{H}_2})^M$ is therefore given by

$$\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, \gamma) \rightsquigarrow \begin{pmatrix} \frac{1}{1+\langle\gamma,\varepsilon_2-\varepsilon_3\rangle} & \frac{\langle\gamma,\varepsilon_2-\varepsilon_3\rangle}{1+\langle\gamma,\varepsilon_2-\varepsilon_3\rangle} \\ \\ \frac{\langle\gamma,\varepsilon_2-\varepsilon_3\rangle}{1+\langle\gamma,\varepsilon_2-\varepsilon_3\rangle} & \frac{1}{1+\langle\gamma,\varepsilon_2-\varepsilon_3\rangle} \end{pmatrix}.$$

When $\gamma = \lambda\varepsilon_2 - \lambda\varepsilon_3$, we have $\langle\gamma,\, \varepsilon_2 - \varepsilon_3\rangle = 2\lambda$. Hence, we obtain:

$$\boxed{\tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2-\varepsilon_3},\, \lambda\varepsilon_2 - \lambda\varepsilon_3) \rightsquigarrow \begin{pmatrix} \frac{1}{1+2\lambda} & \frac{2\lambda}{1+2\lambda} \\ \\ \frac{2\lambda}{1+2\lambda} & \frac{1}{1+2\lambda} \end{pmatrix}}.$$

Let us now consider the the full intertwining operator:

$$\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu) = \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_3},\, \lambda\varepsilon_1 - \lambda\varepsilon_3) =$$

$$= \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, -\lambda\varepsilon_2 + \lambda\varepsilon_3)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_2-\varepsilon_3},\, \lambda\varepsilon_2 - \lambda\varepsilon_3)\, \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1-\varepsilon_2},\, \lambda\varepsilon_1 - \lambda\varepsilon_3).$$

If we denote by R be the matrix associated to $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ with respect to the basis $\mathcal{B}$ of $(E_{\mathcal{H}_2})^M$, then

$$R = \frac{1}{(1+\lambda)^2}\frac{1}{(1+2\lambda)} \begin{pmatrix} 1-\lambda & -\lambda \\ 0 & 1+\lambda \end{pmatrix} \begin{pmatrix} 1 & 2\lambda \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} 1-\lambda & -\lambda \\ 0 & 1+\lambda \end{pmatrix}$$

$$= \left(\tfrac{1-\lambda}{1+\lambda}\right) \tfrac{1}{1+2\lambda} \begin{pmatrix} 1-2\lambda & 0 \\ 2\lambda & 1+2\lambda \end{pmatrix}.$$

An easy study of the eigenvalues of R

$$\eta_1 = \left(\tfrac{1-\lambda}{1+\lambda}\right)\left(\tfrac{1-2\lambda}{1+2\lambda}\right)$$

$$\eta_2 = \tfrac{1-\lambda}{1+\lambda}$$

shows that (for λ real and positive)

- R is positive definite if $0 < \lambda < \frac{1}{2}$
- R is positive semi-definite if $\lambda = \frac{1}{2}$
- R is indefinite if $\frac{1}{2} < \lambda < 1$ or $\lambda > 1$
- R is zero if $\lambda = 1$.

7.4.2 The "algebraic" operator $\psi_{\mathcal{H}_2}(A(\nu))$

These are the main steps for the construction of $\psi_{\mathcal{H}_2}(A(\nu))$:

(i) Find a minimal decomposition of ω_0 as a product of simple reflections:

$$w_0 = s_{\varepsilon_1-\varepsilon_3} = s_{\varepsilon_1-\varepsilon_2}\, s_{\varepsilon_2-\varepsilon_3}\, s_{\varepsilon_1-\varepsilon_2}$$

(ii) Decompose $A(\nu)$ accordingly:

$$A(\nu) = A_{s_{\varepsilon_1-\varepsilon_2}}((s_{\varepsilon_2-\varepsilon_3}s_{\varepsilon_1-\varepsilon_2})\cdot\nu)\, A_{s_{\varepsilon_2-\varepsilon_3}}(s_{\varepsilon_1-\varepsilon_2}\cdot\nu)\, A_{s_{\varepsilon_1-\varepsilon_2}}(\nu).$$

(iii) For each simple root β and for each real character γ of A (satisfying $\langle\gamma,\,\beta\rangle \geqslant 0$)

that appear in the above decomposition, construct the element

$$A_{s_\beta}(\gamma) = \frac{1}{1 + \langle\gamma,\, \check{\beta}\rangle}\, e + \frac{\langle\gamma,\, \check{\beta}\rangle}{1 + \langle\gamma,\, \check{\beta}\rangle}\, s_\beta$$

of $\mathbb{C}[W]$. More explicitly, set

- $A_{s_{\varepsilon_1-\varepsilon_2}}((s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}) \cdot \nu) = A_{s_{\varepsilon_1-\varepsilon_2}}(-\lambda\varepsilon_2 + \lambda\varepsilon_3) = \frac{e+\lambda\, s_{\varepsilon_1-\varepsilon_2}}{1+\lambda}$

- $A_{s_{\varepsilon_2-\varepsilon_3}}(s_{\varepsilon_1-\varepsilon_2} \cdot \nu) = A_{s_{\varepsilon_2-\varepsilon_3}}(\lambda\varepsilon_2 - \lambda\varepsilon_3) = \frac{e+2\lambda\, s_{\varepsilon_2-\varepsilon_3}}{1+2\lambda}$

- $A_{s_{\varepsilon_1-\varepsilon_2}}(\nu) = A_{s_{\varepsilon_1-\varepsilon_2}}(\lambda\varepsilon_1 - \lambda\varepsilon_3) = \frac{e+\lambda\, s_{\varepsilon_1-\varepsilon_2}}{1+\lambda}$.

(iv) With calculations in $\mathbb{C}[W]$, compute $A(\nu)$:

$$A(\nu) = A_{s_{\varepsilon_1-\varepsilon_2}}((s_{\varepsilon_2-\varepsilon_3} s_{\varepsilon_1-\varepsilon_2}) \cdot \nu)\, A_{s_{\varepsilon_2-\varepsilon_3}}(s_{\varepsilon_1-\varepsilon_2} \cdot \nu)\, A_{s_{\varepsilon_1-\varepsilon_2}}(\nu)$$

$$= \left(\tfrac{e+\lambda\, s_{\varepsilon_1-\varepsilon_2}}{1+\lambda}\right) \left(\tfrac{e+2\lambda\, s_{\varepsilon_2-\varepsilon_3}}{1+2\lambda}\right) \left(\tfrac{e+\lambda\, s_{\varepsilon_1-\varepsilon_2}}{1+\lambda}\right)$$

$$= \tfrac{(1+\lambda^2)\, e + 2\lambda^2\, (s_{1,2}\, s_{2,3} + s_{2,3}\, s_{1,2}) + 2\lambda\, (s_{1,2} + s_{2,3}) + 2\lambda^3\, s_{1,3}}{(1+\lambda)^2 (1+2\lambda)}$$

(for brevity of notations we have set $s_{\varepsilon_i-\varepsilon_j} = s_{i,j}$, for all $i,\, j = 1 \ldots 3$).

(v) Evaluate the representation $\psi_{\mathcal{H}_2}$ of W at the element $A(\nu)$ of $\mathbb{C}[W]$.

Recall that $\psi_{\mathcal{H}_2} = V$ (the two-dimensional irreducible representation of the Weyl group $\mathcal{S}_3$), and that E_V has a basis $\{\overrightarrow{x},\, \overrightarrow{y}\}$ with the following properties:

- $\sigma = (23)$ permutes $\overrightarrow{x}$ and $\overrightarrow{y}$
- $\overrightarrow{x}$ and $\overrightarrow{y}$ are eigenvectors of $\tau = (123)$, of eigenvalues ω and ω^2 respectively. Here ω a primitive cubic root of 1.

With respect to the basis $\{\overrightarrow{x}, \overrightarrow{y}\}$ we can write:

$$\psi_{\mathcal{H}_2}((12)) \rightsquigarrow \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \qquad \psi_{\mathcal{H}_2}((23)) \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\psi_{\mathcal{H}_2}((13)) \rightsquigarrow \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \qquad \psi_{\mathcal{H}_2}(e) \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

obtaining the following matrix for $\psi_{\mathcal{H}_2}(A(\nu))$:

$$\psi_{\mathcal{H}_2}(A(\nu)) \rightsquigarrow A = \frac{1}{(1+\lambda)^2(1+2\lambda)} \begin{pmatrix} (1-\lambda^2) & -2\lambda(1-\lambda^2)\omega^2 \\ -2\lambda(1-\lambda^2)\omega & (1-\lambda^2) \end{pmatrix} =$$

$$= \left(\frac{1-\lambda}{1+\lambda}\right) \frac{1}{1+2\lambda} \begin{pmatrix} 1 & -2\lambda\omega^2 \\ -2\lambda\omega & 1 \end{pmatrix}.$$

The eigenvalues of A are

$\zeta_1 = \left(\frac{1-\lambda}{1+\lambda}\right)\left(\frac{1-2\lambda}{1+2\lambda}\right)$

$\zeta_2 = \frac{1-\lambda}{1+\lambda}$.

Therefore (for λ real and positive)

- A is positive definite if $0 < \lambda < \frac{1}{2}$
- A is positive semi-definite if $\lambda = \frac{1}{2}$
- A is indefinite if $\frac{1}{2} < \lambda < 1$ or $\lambda > 1$

- A is zero if $\lambda = 1$.

7.5 Conclusions

In this section, we summarize the information gathered in the chapter about the unitarity of a spherical principal series of $SL(3, \mathbb{R})$.

Let $P = MAN$ be the minimal parabolic subgroup of $SL(3)$ consisting of real 3×3 upper triangular matrices. Let ω_0 be the long Weyl group element, and let ν be a real dominant character of A satisfying the formal symmetry condition $\omega_0 \cdot \nu = -\nu$. More explicitly, set $\omega_0 = s_{\varepsilon_1 - \varepsilon_3}$ and set $\nu = \lambda\varepsilon_1 - \lambda\varepsilon_3$, for $\lambda > 0$.

Denote by $X(\nu)$ the Langlands quotient of $SL(3)$ with parameters (P, δ_0, ν), with δ_0 the trivial representation of M. $X(\nu)$ is unitary if and only if the Hermitian operator

$$\tilde{R}_\mu(\omega_0, \nu)\colon \operatorname{Hom}_M(\operatorname{Res}^K_M E_\nu, V^{\delta_0}) \to \operatorname{Hom}_M(\operatorname{Res}^K_M E_\nu, V^{\delta_0})$$

is positive semi-definite for all spherical K-types (μ, E_μ).

By a theorem of Barbasch, it is enough to check the signature of $\tilde{R}_\mu(\omega_0, \nu)$ for the petite K-types. Apart from the trivial one, $K = SO(3)$ has only one spherical petite irreducible representation: the K-type $\mathcal{H}_2$.[5]
So it all comes down to computing the signature of the operator

$$\tilde{R}_{\mathcal{H}_2}(\omega_0, \mu) = \tilde{R}_{\mathcal{H}_2}(s_{\varepsilon_1 - \varepsilon_3}, \lambda\varepsilon_1 - \lambda\varepsilon_3).$$

[5] $(\mathcal{H}_2, E_{\mathcal{H}_2})$ is the 5-dimensional representation of $SO(3)$ on the space of homogeneous harmonic polynomials of degree 2 in three variables.

We find that

- $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ is positive definite if $0 < \lambda < \frac{1}{2}$

- $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ is positive semi-definite if $\lambda = \frac{1}{2}$

- $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ is indefinite if $\frac{1}{2} < \lambda < 1$ or $\lambda > 1$

- $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ is zero if $\lambda = 1$.

We therefore conclude that

$$\boxed{\text{the Langlands quotient } X(\nu) \text{ is unitary for } 0 < \lambda \leq \tfrac{1}{2} \text{ or } \lambda = 1}.$$

There is an alternative method to construct the operator $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$, which is purely algebraic in nature. We identify $\mathrm{Hom}_M(\mathrm{Res}^K_M\, E_{\mathcal{H}_2},\, V^{\delta_0})$ with the space of M invariants in $E_{\mathcal{H}_2}$, which carries a representation $\psi_{\mathcal{H}_2}$ of the Weyl group. Once this representation is understood, we replace the analytic operator $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$ with an algebraic operator $\psi_{\mathcal{H}_2}(A(\nu))$, that is also Hermitian and has the same signature as $\tilde{R}_{\mathcal{H}_2}(\omega_0,\, \nu)$.[6] The remarkable fact is that $\psi_{\mathcal{H}_2}(A(\nu))$ can be computed by means *only* of Weyl group calculations.

[6]The fact that $\mathcal{H}_2$ is petite is crucial for this construction.

Chapter 8

The petite spherical representations of $SO(2n, \mathbb{R})$

In this chapter we study the irreducible representations μ of $SO(2n, \mathbb{R})$ that satisfy the following requirements:

- for each restricted root α of $SL(2n, \mathbb{R})$, the restriction of μ to the $SO(2)$ subgroup associated to α only contains the characters zero, plus or minus one, plus or minus two (i.e. μ is *petite*)
- μ contains an M-fixed vector, where M is the subgroup of diagonal matrices in $SO(2n, \mathbb{R})$ (i.e. μ is *spherical*).

These are exactly the K-types that appear in a spherical principal series of $SL(2n, \mathbb{R})$ on which the intertwining operator can be computed by means of Weyl group computations.[1]

[1]Please refer to chapter 6 for details.

8.1 The petite representations of $SO(2n, \mathbb{R})$

First, we fix some notations. Let K be the group $SO(2n, \mathbb{R})$, and let H be the maximal torus

$$H = \left\{ \left(\begin{array}{cc|c|cc} \cos(\theta_1) & \sin(\theta_1) & \dots & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & \dots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & \cos(\theta_n) & \sin(\theta_n) \\ 0 & 0 & \dots & -\sin(\theta_n) & \cos(\theta_n) \end{array} \right) : \theta_1, \dots, \theta_n \in \mathbb{R} \right\}.$$

We denote by $\mathfrak{k}_0$ and $\mathfrak{h}_0$ the Lie algebras of K and H, and by $\mathfrak{k}$ and $\mathfrak{h}$ their complexification (so $\mathfrak{k} = \mathfrak{k}_0^{\mathbb{C}}$ and $\mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}$).

$\mathfrak{h}$ is a Cartan subalgebra of the complex semi-simple Lie algebra $\mathfrak{k} = \mathfrak{so}(2n, \mathbb{C})$, and it is explicitly given by

$$\mathfrak{h} = \left\{ \left(\begin{array}{cc|c|cc} 0 & \theta_1 & \dots & 0 & 0 \\ -\theta_1 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & 0 & \theta_n \\ 0 & 0 & \dots & -\theta_n & 0 \end{array} \right) : \theta_1, \theta_2, \dots, \theta_n \in \mathbb{C} \right\}.$$

For brevity of notations, it is convenient to introduce a basis of $\mathfrak{h}$. We pick the elements

$$H_j = E_{2j-1,2j} - E_{2j,2j-1} \qquad (j = 1 \dots n)$$

where as usual $E_{a,b}$ stands for an elementary matrix[2].

Since $\{H_j\}_{j=1\dots n}$ is a basis of $\mathfrak{h}$, we write $\mathfrak{h} = \bigoplus_{j=1}^n \mathbb{C} H_j$.

[2]For each a, $b = 1, \dots, 2n$, $E_{a,b}$ is a square matrix of size $2n$ with entry 1 where the ath row and the bth column meet, all other entries being 0.

For each $j = 1 \dots n$, let ψ_j be the complex linear functional on $\mathfrak{h}$ defined by

$$\psi_j(\textstyle\sum_{k=1}^n \theta_k H_k) \mapsto -i\theta_j \quad (j = 1 \dots n).$$

More explicitly,

$$\psi_j \left(\begin{array}{cc|c|cc} 0 & \theta_1 & \dots & 0 & 0 \\ -\theta_1 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & 0 & \theta_n \\ 0 & 0 & \dots & -\theta_n & 0 \end{array} \right) = -i\theta_j$$

for all $\theta_1, \dots, \theta_n$ in $\mathbb{C}$.

There exists a one-one correspondence between the set of dominant analytically integral forms λ on $\mathfrak{h}$ and the set of irreducible representations μ_λ of K, the correspondence being that λ is the highest weight of μ_λ.

The dominant analytically integral forms on $\mathfrak{h}$ are all the expressions

$$\lambda = a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1} + a_n\psi_n$$

with integer coefficients $a_1, \dots, a_{n-1}, a_n$ satisfying

$$a_1 \geqslant \cdots \geqslant a_{n-1} \geqslant \mid a_n \mid.$$

We ask which of these forms give rise to petite representations of $SO(2n, \mathbb{R})$.

The answer to this question is easy to state:

$$\mu_{a_1\psi_1+\cdots+a_n\psi_n} \text{ is petite } \Leftrightarrow a_j \in \{0, \pm 1, \pm 2\} \quad \forall j = 1, \dots, n$$

but the proof requires some preparation. Once we have a deep understanding of the restricted root space decomposition of $\mathfrak{sl}(2n, \mathbb{R})$, and an explicit construction for the $SO(2)$ subgroup K^α associated to each restricted root α, we can look at the restriction of μ to each K^α and impose that this restriction only contains the 0, ± 1 or ± 2 characters of $SO(2)$.

8.1.1 The restricted root system of $\mathfrak{sl}(2n, \mathbb{R})$

In this section we describe the various constituents of the restricted root space decomposition of $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus_{\alpha \in \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0)} (\mathfrak{g}_0)_\alpha .$$

Fix the Cartan involution $\vartheta \colon \mathfrak{g}_0 \to \mathfrak{g}_0$, $X \mapsto -X^T$. Then:

- $\mathfrak{k}_0 = \mathfrak{so}(2n, \mathbb{R})$ is the (+1)-eigenspace of ϑ
- $\mathfrak{p}_0 = \mathfrak{g}_0 \cap \{\text{real symmetric matrices}\}$ is the (-1)-eigenspace of ϑ
- $\mathfrak{a}_0 = \mathfrak{g}_0 \cap \{\text{real diagonal matrices}\}$ is a maximal abelian subspace of $\mathfrak{p}_0$

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} b_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{2n-1} & 0 \\ 0 & 0 & \dots & 0 & b_{2n} \end{pmatrix} : b_1, \dots,\, b_{2n} \in \mathbb{R} \text{ s.t. } \sum_{j=1}^{2n} b_j = 0 \right\}$$

  We pick the basis $\{H_j = E_{2j,2j} - E_{2j+1,2j+1} \colon j = 1, \dots, 2n-1\}$ in $\mathfrak{a}_0$.

- $\mathfrak{m}_0 = \{0\}$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$
- $\Delta = \Delta(\mathfrak{g}_0,\, \mathfrak{a}_0) = \{\pm(\varepsilon_i - \varepsilon_j)\}_{i,\, j=1,\dots,\, 2n,\, i<j}$ is the set of restricted roots, where for each j we have denoted by ε_j the real linear functional on $\mathfrak{a}_0$ defined by

$$\varepsilon_j \colon \mathfrak{a}_0 \to \mathbb{R},\ \mathrm{diag}(b_1, \dots,\, b_{2n}) \mapsto b_j$$

- $\Delta^+ = \{\varepsilon_i - \varepsilon_j\}_{i,\, j=1,\dots,\, 2n,\, i<j}$ is the set of positive restricted roots

- $\Pi = \{\varepsilon_i - \varepsilon_{i+1}\}_{i,\, j=1,\dots,\, 2n-1}$ is the set of simple restricted roots

- $(\mathfrak{g}_0)_{\varepsilon_i - \varepsilon_j} = \mathbb{R}E_{i,j}$ is the restricted root space associated to a root $\varepsilon_i - \varepsilon_j$.

8.1.2 The $SO(2)$ attached to a restricted root

To each restricted root α in Δ, we attach a subgroup K^α isomorphic to $SO(2, \mathbb{R})$ in the following way:

1 We pick an element E_α in $(\mathfrak{g}_0)_\alpha$ that satisfies the normalizing condition

$$B(E_\alpha,\, \vartheta(E_\alpha)) = -\tfrac{2}{\|\alpha\|^2},$$

where B is the Killing form on $\mathfrak{g}_0$ and $\|\alpha\|^2 = B(H_\alpha,\, H_\alpha)$, with H_α the unique element of $\mathfrak{a}_0$ such that

$$B(H_\alpha,\, H) = \alpha(H) \qquad \forall\, H \in \mathfrak{a}_0.$$

2 We construct the skew-symmetric matrix

$$Z_\alpha = E_\alpha + \vartheta(E_\alpha) = E_\alpha - (E_\alpha)^T,$$

which generates a one-dimensional Lie algebra $\mathfrak{k}^\alpha$ isomorphic to $\mathfrak{so}(2,\, \mathbb{R})$.

3 We define K^α to be the analytic subgroup of $SO(2n, \mathbb{R})$ with Lie algebra $\mathfrak{k}^\alpha$, so

$$K^\alpha = \exp\{\mathbb{R}Z_\alpha\}.$$

The first step is to understand the Killing form on $\mathfrak{g}_0 = \mathfrak{sl}(2n,\, \mathbb{R})$. For each X and Y in $\mathfrak{g}_0$, we have[3]:

$$B(X,\, Y) = Tr(\mathrm{ad}(X)\,\mathrm{ad}(Y)) = (2n)Tr(XY).$$

[3]See [16], Ch.III

In particular, if $H = \sum_{j=1}^{2n} x_j E_{j,j}$ and $H' = \sum_{j=1}^{2n} x'_j E_{j,j}$, then

$$B(H,\, H') = (2n) \sum_{j=1}^{2n} x_j \cdot x'_j.$$

It easily follows that

$$H_{\varepsilon_k - \varepsilon_l} = \tfrac{1}{2n}(E_{k,k} - E_{l,l})$$

for all integers $l,\, k$ $(l,\, k = 1 \ldots 2n,\, l \neq k)$. Therefore

$$\|\alpha\|^2 = B(H_\alpha,\, H_\alpha) = \tfrac{2}{2n}$$

for any root $\alpha = \varepsilon_k - \varepsilon_l$.

Since any root space is one-dimensional, the vector $E_{\varepsilon_k - \varepsilon_l}$ must be a multiple of the generator $E_{k,l}$, say $E_{\varepsilon_k - \varepsilon_l} = cE_{k,l}$, and because

$$B(E_{k,l},\, \vartheta(E_{k,l})) = -(2n)Tr(E_{k,l}(E_{k,l})^T) = -(2n)Tr(E_{k,l}E_{l,k}) = -(2n)$$

the complex number c must satisfy the equation

$$-(2n)\|c\|^2 = -\tfrac{2}{2/(2n)}$$

(whose solutions are ± 1).

For each root $\alpha = \varepsilon_k - \varepsilon_l$, we set $E_\alpha = E_{k,l}$ and $Z_\alpha = E_{k,l} + \vartheta(E_{k,l}) = E_{k,l} - E_{l,k}$. Z_α is a generator for the $\mathfrak{so}(2)$ associated to the root α, the corresponding $SO(2)$ subgroup is therefore $K^\alpha = \exp\{\mathbb{R}Z_\alpha\}$. Here is an explicit description of K^α: if

$\alpha = \pm(\varepsilon_k - \varepsilon_l)$, with $1 \leqslant k < l \leqslant 2n$, then

$$\begin{array}{cc}
\begin{pmatrix}
1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & \cos(\theta) & 0 & \dots & 0 & \sin(\theta) & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & -\sin(\theta) & 0 & \dots & 0 & \cos(\theta) & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{pmatrix}
&
\begin{matrix}
\leftarrow 1 \\ \vdots \\ \\ \leftarrow k \\ \\ \vdots \\ \\ \leftarrow l \\ \\ \vdots \\ \leftarrow 2n
\end{matrix}
\end{array}$$

with θ in $\mathbb{R}$.

8.1.3 Petite representations

Let $\lambda = a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1} + a_n\psi_n$ be an analytically integral dominant form on $\mathfrak{h}$, so that

$$a_1, \dots, a_{n-1}, a_n \in \mathbb{Z}$$

$$a_1 \geqslant \cdots \geqslant a_{n-1} \geqslant |\, a_n \,| \qquad (\circledast),$$

and let μ_λ be the irreducible representation of $SO(2n, \mathbb{R})$ with highest weight λ. Let α be a restricted root for $SL(2n, \mathbb{R})$ and let K^α be the corresponding $SO(2)$ subgroup. In this section we study the restriction of μ_λ to K^α. In particular, we ask whether this restriction consists only of the characters $\{0, \pm 1, \pm 2\}$ of $SO(2)$.

Notice that, for each α in Δ, the group K^α is isomorphic to $SO(2)$ via

$$\exp(\theta Z_\alpha) \leftrightarrow \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

so, if we want K^α to act by the characters

$$\xi_0 \colon \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \longrightarrow 1$$

$$\xi_{\pm 1} \colon \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \longrightarrow e^{\pm i\theta}$$

$$\xi_{\pm 2} \colon \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \longrightarrow e^{\pm 2i\theta}$$

then Z_α must act with eigenvalues $\{0, \pm i, \pm 2i\}$.

For each $j = 1, \ldots, n$, consider the element $H_j = E_{2j-1,2j} - E_{2j,2j-1}$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2n, \mathbb{C})$. Because $H_j = Z_{\varepsilon_{2j-1}-\varepsilon_{2j}}$, the eigenvalues of H_j must also lie in the set $\{0, \pm i, \pm 2i\}$. The list of eingenvalues of H_j in $d\mu_\lambda$ can be found by looking at the coefficient of ψ_j in all the possible weights of μ_λ, and multiplying this coefficient by $-i$.[4]

We conclude that μ_λ is petite if and only if *all* the coefficients of *all* the weights of $d\mu_\lambda$ lie in the set $\{0, \pm 1, \pm 2\}$.

In particular, the coefficients $\{a_j\}_{j=1,\ldots,n}$ of the highest weight of μ_λ must belong to this set. Taking also the condition ($\circledast$) into account, we see that μ_λ is petite only if its highest weight λ falls in this list:

[4]If y is a weight vector of weight $b_1\psi_1 + \cdots + b_n\psi_n$, then

$$d\mu_\lambda(H_j) \cdot y = (b_1\psi_1 + \cdots + b_n\psi_n)(H_j)y = b_j\varepsilon_j(H_j)y = (-ib_j)y$$

so $(-ib_j)$ is an eigenvalue of $d\mu_\lambda(H_j)$. Since the weight vectors from a basis of the representation space, it is clear that all the eigenvalues of (H_j) are obtained this way.

- $0\psi_1 + \cdots + 0\psi_n$
- $1\psi_1 + \cdots + 1\psi_k \qquad 0 < k < n$
- $(1\psi_1 + \cdots + 1\psi_{n-1}) \pm 1\psi_n$
- $2\psi_1 + \cdots + 2\psi_k \qquad 0 < k < n$
- $(2\psi_1 + \cdots + 2\psi_a) + (1\psi_{a+1} + \cdots + 1\psi_b) \qquad 0 < a, b < n$
- $(2\psi_1 + \cdots + 2\psi_k) + (1\psi_{k+1} + \cdots + 1\psi_{n-1}) \pm 1\psi_n \qquad 0 < k < n$
- $(2\psi_1 + \cdots + 2\psi_{n-1}) \pm 2\psi_n$.

The next step is to show that each representation in this list is actually petite. We need a case by case analysis.

- If $\lambda = 0\psi_1 + \cdots + 0\psi_n$, then μ_λ is the trivial representation of $SO(2n, \mathbb{R})$, and it is of course petite.

- If $\lambda = 1\psi_1 + \cdots + 1\psi_k$, with $0 < k < n$, then $\mu_\lambda = \Lambda^k\mathbb{C}^{2n}$ is the kth wedge product of the standard representation of $SO(2n, \mathbb{R})$, and it is also petite.

  We start by determining the weights of the standard representation of $SO(2n, \mathbb{R})$. Let $\{\overrightarrow{e_k}\}_{k=1,\ldots,2n}$ be the canonical basis of $\mathbb{C}^{2n}$; we notice that the vector $(\overrightarrow{e_{2j-1}} + \overrightarrow{e_{2j}})$ is a $(+1)$-eigenvector of H_j and a 0-eigenvector of H_l for $l \neq j$. Hence it is a weight vector for $\mathfrak{h}$ of weight

$$0\psi_1 + \cdots + 0\psi_{j-1} + 1\psi_j + 0\psi_{j+1} \cdots + 0\psi_n = +\psi_j.$$

  Similarly, $(\overrightarrow{e_{2j-1}} - \overrightarrow{e_{2j}})$ is a (-1)-eigenvector of H_j and a 0-eigenvector of H_l for $l \neq j$, so it is a weight vector (for $\mathfrak{h}$) of weight

$$0\psi_1 + \cdots + 0\psi_{j-1} - 1\psi_j + 0\psi_{j+1} \cdots + 0\psi_n = -\psi_j.$$

Because the vectors $\{\overrightarrow{e_{2j-1}} \pm \overrightarrow{e_{2j}}\}_{j=1,\ldots,n}$ form a basis of $\mathbb{C}^{2n}$, we deduce that the weights of the standard representation of $SO(2n, \mathbb{R})$ are $\{\pm\psi_j\}_{j=1,\ldots,n}$.

Let us now look at the weights of $\Lambda^k\mathbb{C}^{2n}$. It follows from the product rule for differentiation that the wedge product of k weight vectors of $\mathbb{C}^{2n}$ is a weight vector for $\Lambda^k\mathbb{C}^{2n}$ (relative to the sum of the weights of the k vectors). For dimensional reasons, all the weight vectors of $\Lambda^k\mathbb{C}^{2n}$ are obtained this way. We therefore conclude that each weight of $\Lambda^k\mathbb{C}^{2n}$ is of the form

$$\pm\psi_{i_1} \pm \psi_{i_2} \cdots \pm \psi_{i_r}$$

for some $r \leqslant k$ and some (distinct) indices $i_1, i_2, \ldots, i_r$ in $\{1, 2, \ldots, n\}$.

Every ψ_j appears with coefficient 0 or ± 1, hence the representation $\Lambda^k\mathbb{C}^{2n}$ is petite.

- If $\lambda = (1\psi_1 + \cdots + 1\psi_{n-1}) \pm 1\psi_n$, then μ_λ is one of the two irreducible summands of $\Lambda^n\mathbb{C}^{2n}$. In particular every weight of μ_λ is also a weight of $\Lambda^n\mathbb{C}^{2n}$, and therefore the same argument used above shows that the coefficient of ψ_j in each weight lies in the set $\{0, +1, -1\}$. We conclude that μ_λ is petite.

- If $\lambda = 2\psi_1 + \cdots + 2\psi_k$, with $0 < k < n$, then μ_λ is an irreducible summand of $(\Lambda^k\mathbb{C}^{2n}) \otimes (\Lambda^k\mathbb{C}^{2n})$. It is easy to check that the weights of $(\Lambda^k\mathbb{C}^{2n}) \otimes (\Lambda^k\mathbb{C}^{2n})$ are of the form $\nu + \nu'$, with ν and ν' (not necessarily distinct) weights of $\Lambda^k\mathbb{C}^{2n}$. So they are of the form

$$(\pm 2\psi_{i_1} \cdots \pm 2\psi_{i_r}) + (\pm\psi_{j_1} \cdots \pm \psi_{j_s})$$

for some (distinct) indices $i_1, \ldots, i_r, j_1, \ldots, j_s$ in $\{1, 2, \ldots, n\}$, with $r + s \leqslant k$. Since every weight of μ_λ is also a weight of $(\Lambda^k\mathbb{C}^{2n}) \otimes (\Lambda^k\mathbb{C}^{2n})$, the coefficient of ψ_j in a weight of μ_λ lies in the set $\{0, \pm 1, \pm 2\}$. We conclude that μ_λ is petite.

- If $\lambda = (2\psi_1 + \cdots + 2\psi_{n-1}) \pm 2\psi_n$, then μ_λ is an irreducible summand of $(\Lambda^n\mathbb{C}^{2n}) \otimes (\Lambda^n\mathbb{C}^{2n})$, and the same argument used above shows that it is petite.

- ◦ If $\lambda = (2\psi_1 + \cdots + 2\psi_a) + (1\psi_{a+1} + \cdots + 1\psi_b)$, with $0 < a,\, b < n$, then μ_λ is an irreducible summand of $(\Lambda^a\mathbb{C}^{2n}) \otimes (\Lambda^b\mathbb{C}^{2n})$ and it is again petite.
- ◦ If $\lambda = (2\psi_1 + \cdots + 2\psi_k) + (1\psi_{k+1} + \cdots + 1\psi_{n-1}) \pm 1\psi_n$, with $0 < k < n$, then μ_λ is an irreducible summand of $(\Lambda^k\mathbb{C}^{2n}) \otimes (\Lambda^n\mathbb{C}^{2n})$ and it is also petite.

This concludes the proof. We obtain:

Proposition 2. *The petite representations of $SO(2n,\, \mathbb{R})$ are exactly the ones with highest weight*

- $0\psi_1 + \cdots + 0\psi_n$
- $1\psi_1 + \cdots + 1\psi_k \qquad 0 < k < n$
- $(1\psi_1 + \cdots + 1\psi_{n-1}) \pm 1\psi_n$
- $2\psi_1 + \cdots + 2\psi_k \qquad 0 < k < n$
- $(2\psi_1 + \cdots + 2\psi_a) + (1\psi_{a+1} + \cdots + 1\psi_b) \qquad 0 < a < b < n$
- $(2\psi_1 + \cdots + 2\psi_k) + (1\psi_{k+1} + \cdots + 1\psi_{n-1}) \pm 1\psi_n \qquad 0 < k < n$
- $(2\psi_1 + \cdots + 2\psi_{n-1}) \pm 2\psi_n$.

8.2 The petite spherical representations of $SO(2n,\, \mathbb{R})$

Finding the petite *spherical* representations of $SO(2n,\, \mathbb{R})$ accounts to determining which petite representations contain an M-fixed vector.
Here M is the group of diagonal matrices of size $2n$ with entries ± 1 and determinant one. It is a finite abelian group of order 2^{2n-1} and its irreducible representations are the form

$$\delta_S\colon M \to \mathbb{C},\; m = \mathrm{diag}(m_1,\, m_2, \ldots,\, m_{2n}) \mapsto \delta_S(m) = \prod_{j \in S} m_{j_1} \qquad (\divideontimes)$$

with S a subsets of $\{1, 2, \ldots, 2n\}$ of cardinality less or equal to n.[5] In particular, $\delta_\varnothing$ is the trivial representation of M.

A petite irreducible representation of $SO(2n)$ is spherical if and only if it contains $\delta_\varnothing$. The previous proposition contains a list of the highest weights of the petite representations; for each λ in this list we study the decomposition of μ_λ in M-types.

If $\lambda = 0\psi_1 + \cdots + 0\psi_n$, then μ_λ is the trivial representation of $SO(2n, \mathbb{R})$, $\mu_\lambda \mid_M = \delta_\varnothing$ and of course μ_λ is spherical.

If $\lambda = 1\psi_1 + \cdots + 1\psi_k$, with $0 < k < n$, then $\mu_\lambda = \Lambda^k\mathbb{C}^{2n}$. We will show that restriction of μ_λ to M does not include $\delta_\varnothing$, hence μ_λ is *not* spherical.

Let $m = \text{diag}(m_1, m_2, \ldots, m_{2n})$ be an arbitrary element of M. We notice that m acts on a basis vector $\overrightarrow{e_{\underline{j}}} = \overrightarrow{e_{j_1}} \wedge \overrightarrow{e_{j_2}} \wedge \cdots \wedge \overrightarrow{e_{j_k}}$ of $\Lambda^k\mathbb{C}^{2n}$ as scalar multiplication by $(m_{j_1} \cdot m_{j_2} \cdots m_{j_k})$. It follows that, for every multi-index $\underline{j} = (j_1, j_2, \ldots, j_k)$, the one-dimensional subspace $\mathbb{C}\overrightarrow{e_{\underline{j}}}$ is a copy of the representation $\delta_{\{j_1, j_2, \ldots, j_k\}}$ of M. Therefore:

$$\mu_{1\psi_1+\cdots+1\psi_k} \mid_M = \bigoplus\nolimits_{S\subseteq\{1\ldots 2n\}:\, |S|=k} \delta_S.$$

Being $0 < k < n$, any of these summands is equivalent to $\delta_\varnothing$. So μ_λ is not spherical.

If $\lambda = (1\psi_1 + \cdots + 1\psi_{n-1}) \pm 1\psi_n$, then μ_λ is an irreducible summand of $\Lambda^n\mathbb{C}^{2n}$ and the same argument used above shows that μ_λ is *not* spherical.

If $\lambda = (2\psi_1 + \cdots + 2\psi_a) + (1\psi_{a+1} + \cdots + 1\psi_b)$, with $0 < a < b < n$, then μ_λ is an irreducible summand of $\Lambda^a\mathbb{C}^{2n} \otimes \Lambda^b\mathbb{C}^{2n}$. We will show that the restriction of

[5] It is clear that, for each $S \subseteq \{1, 2, \ldots, 2n\}$, $(\circledast)$ defines a character of M. It is also easy to check that $\delta_S \backsimeq \delta_{S'}$ if and only if S' is the complement of S (because any element of M has determinant one). This gives a list of $\frac{1}{2}2^{2n} = 2^{2n-1}$ inequivalent representations of M, basically *all* the representations of M because M has cardinality 2^{2n-1}.

$\Lambda^a\mathbb{C}^{2n} \otimes \Lambda^b\mathbb{C}^{2n}$ to M does not contain the trivial representation. Hence μ_λ is *not* spherical.

To prove that $\Lambda^a\mathbb{C}^{2n} \otimes \Lambda^b\mathbb{C}^{2n}$ does not include the M-type $\delta_\varnothing$, we notice that every basis vector

$$(\overrightarrow{e_{i_1}} \wedge \cdots \wedge \overrightarrow{e_{i_a}}) \otimes (\overrightarrow{e_{j_1}} \wedge \cdots \wedge \overrightarrow{e_{j_b}})$$

spans a copy of an irreducible representation δ_S not isomorphic to the trivial M-type. Indeed, the cardinality of S satisfies:

$$0 < b - a < \#S = \#\left(\{i_1, i_2, \ldots, i_a\} \triangle \{j_1, j_2, \ldots, j_b\}\right) < b < n$$

($\triangle$ being the symmetric difference).

If $\lambda = 2\psi_1 + \cdots + 2\psi_k$, with $0 < k < n$, then μ_λ is an irreducible summand of $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$. Showing that the representation $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ is spherical is easy, but proving that also μ_λ is spherical is quite harder.

We start with $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$. The isotypic of the trivial M-type in $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ has dimension $\binom{2n}{k}$ and is spanned by the vectors

$$(\overrightarrow{e_{i_1}} \wedge \cdots \wedge \overrightarrow{e_{i_k}}) \otimes (\overrightarrow{e_{i_1}} \wedge \cdots \wedge \overrightarrow{e_{i_k}})$$

with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant 2n$. The next step is to show that some of these copies of $\delta_\varnothing$ actually lie in μ_λ. This requires a long argument, that we explain the following lemmas.

Lemma 5. *As a representation of $SO(2n, \mathbb{R})$,*
$\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n} = \mu_{\sum_{j=1}^{k} 2\psi_j} \oplus \mu_{\sum_{j=1}^{k-1} 2\psi_j} \oplus \cdots \oplus \mu_{2\psi_1} \oplus \mu_0 \oplus$ *some non spherical types.*

Corollary 5. *The trivial M-type appears in the irreducible representation $\mu_{\sum_{j=1}^{k} 2\psi_j}$ of $SO(2n, \mathbb{R})$ with multiplicity $\binom{2n}{k}$ - $\binom{2n}{k-1}$.*

We start with the proof of lemma 5.

Proof. For brevity of notations, we set $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n} = \rho$. Recall that the weights of ρ are of the form

$$(\pm 2\psi_{i_1} \cdots \pm 2\psi_{i_r}) + (\pm\psi_{j_1} \cdots \pm \psi_{j_t})$$

for some (distinct) indices $i_1, \dots, i_r, j_1, \dots, j_t$ in $\{1, 2, \dots, n\}$, with $r + t \leqslant k$. It follows that $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ can only include K-types whose highest weight is of the form

$$\upsilon_0 = 0$$

$$\upsilon_s = 2\psi_1 + \cdots + 2\psi_s$$

$$\gamma_c = \psi_1 + \cdots + \psi_c$$

$$\gamma_{a+b} + \gamma_b = (+2\psi_1 + \cdots + 2\psi_a) + (\psi_{a+1} + \cdots + \psi_{a+b}).$$

Since the K-types μ_{γ_c} and $\mu_{\gamma_a+\gamma_{a+b}}$ are non-spherical (for all possible a, b and c), ρ can only include the spherical K-types $\mu_{\upsilon_0}, \mu_{\upsilon_1}, \dots, \mu_{\upsilon_k}$.

We want to prove that, for every $s = 0, 1, \dots, k$, the K-type μ_{υ_s} appears in the decomposition of ρ in irreducible representations of $SO(2n, \mathbb{R})$ with multiplicity one. With this in mind, we study the decomposition of ρ in irreducible $\mathfrak{so}(2n, \mathbb{C})$-modules.[6]

When $s = 0$, i.e. υ_s is the 0 weight, the result follows from the fact that $\Lambda^k\mathbb{C}^{2n}$ is self dual:[7]

$$(\Lambda^k\mathbb{C}^{2n})^* = \Lambda^k(\mathbb{C}^{2n})^* = \Lambda^k\mathbb{C}^{2n}$$

[6] Look at $\rho = \Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ as a $\mathfrak{so}(2n, \mathbb{C})$-representation; all the weights have integers coefficients. In particular, the highest weight of each irreducible summand is analytically integral. Hence, any decomposition of ρ in $\mathfrak{so}(2n, \mathbb{C})$-modules lifts to a decomposition in $SO(2n, \mathbb{R})$-representations.

[7] This, in turn, follows from the fact that the standard representation of $SO(2n, \mathbb{R})$ is self dual: let $\eta = \mathbb{C}^{2n}$, and let η^* be its dual. Then, for each x of $SO(2n, \mathbb{R})$, we have

$$\eta^*(x) = \eta(x^{-1})^T = (x^{-1})^T = x = \eta(x)$$

hence η is self dual.

and from this general formula for the decomposition of a tensor product in highest weight representations:[8]

$$\mathrm{mult}_{V^\mu \otimes V^\nu}(V^\lambda) = \sum_{\tau \in W} \mathrm{sgn}(\tau)\, \mathrm{m}_\mu(\lambda + \rho - \tau \cdot (\nu + \rho))$$

(where of course W is the Weyl group). When $V^\nu = V^\mu = \Lambda^k \mathbb{C}^{2n}$ and $\lambda = 0$, we find:

$$\mathrm{m}_\mu(\lambda + \rho - \tau \cdot (\nu + \rho)) = \mathrm{m}_\mu(\rho - \tau \cdot (\nu + \rho)) = \mathrm{m}_\mu(\tau^{-1} \cdot \rho - (\nu + \rho)) \quad \circledast$$

$$= \mathrm{m}_\mu(-\nu - \rho + \tau^{-1} \cdot \rho) = \mathrm{m}_\mu(\nu + (\rho - \tau^{-1} \cdot \rho)) \quad \circledast\circledast$$

$$= \begin{cases} 1 & \text{if } \tau = Id \\ 0 & \text{if } \tau \neq Id. \end{cases} \quad \circledast\circledast\circledast, \text{ where}$$

($\circledast$) follows from the fact that weights that are conjugate under W have the same multiplicity

($\circledast\circledast$) follows from the fact that the weights of a self-dual representation are symmetric with respect to zero[9], and finally

($\circledast\circledast\circledast$) follows from the fact that if τ is any non-trivial Weyl group element, then $(\rho - \tau^{-1} \cdot \rho)$ is a sum of positive roots so $\mu + (\rho - \tau^{-1} \cdot \rho)$ cannot be a weight. Instead, if τ is the identity of W, then $\mu + (\rho - \tau^{-1} \cdot \rho) = \mu$, and it has multiplicity one because it is the highest weight.

Therefore:

$$\mathrm{mult}_{\Lambda^k(\mathbb{C}^{2n}) \otimes \Lambda^k(\mathbb{C}^{2n})}(V^0) = 1.$$

We now discuss the case $s > 1$. We intend to show that for each $s = 1 \dots k$, $\mu_{v_s} = \mu_{\sum_{j=1}^s 2\psi_j}$ is an irreducible summand of $\Lambda^k \mathbb{C}^{2n} \otimes \Lambda^k \mathbb{C}^{2n}$.

[8] See [15], corollary 7.1.6.

[9] For any representation μ, the weights of μ^* are the negative of the weights of μ.

When $k = 1$ the result is obvious, because $\upsilon_1 = 2\psi_1$ is the highest weight of $\Lambda^1(\mathbb{C}^{2n}) \otimes \Lambda^1(\mathbb{C}^{2n}) = \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. Similarly, if $s = k$ then $\upsilon_k = \sum_{j=1}^{k} 2\psi_j$ is the highest weight of $\Lambda^k(\mathbb{C}^{2n}) \otimes \Lambda^k(\mathbb{C}^{2n})$.

So we assume $1 \leqslant s \leqslant k-1$. The trick in this case is to pass from a calculation on the Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ to a computation on $\mathfrak{gl}(s, \mathbb{C}) \oplus \mathfrak{so}(2n-2s, \mathbb{C})$, which is the Levi factor of a suitable parabolic subalgebra of $\mathfrak{so}(2n, \mathbb{C})$. Here are all the details...

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{k} = \mathfrak{so}(2n, \mathbb{C})$ already introduced at the beginning of the chapter:

$$\mathfrak{h} = \left\{ \left(\begin{array}{cc|c|cc} 0 & \theta_1 & \dots & 0 & 0 \\ -\theta_1 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & 0 & \theta_n \\ 0 & 0 & \dots & -\theta_n & 0 \end{array} \right) : \theta_1, \theta_2, \dots, \theta_n \in \mathbb{C} \right\}$$

and let $\Psi = \Psi(\mathfrak{k}, \mathfrak{h})$ be the corresponding root system:

$$\Psi = \{\pm(\psi_i \pm \psi_j)\}_{i,\, j=1,\dots,\, n,\, i<j}.$$

We denote by Ψ^+ the set of positive roots:

$$\Psi^+ = \{\psi_i \pm \psi_j\}_{i,\, j=1,\dots,\, n,\, i<j}$$

and by $\mathfrak{b}$ the corresponding Borel subalgebra: $\mathfrak{b} = \mathfrak{h} \oplus_{\alpha \in \Psi^+} \mathfrak{k}_\alpha$. The parabolic subalgebras of $\mathfrak{k}$ containing $\mathfrak{b}$ are parameterized by the class of subsets of Ψ including Ψ^+; the one corresponding to a subset Υ is of the form $\mathfrak{q} = \mathfrak{h} \oplus_{\alpha \in \Upsilon} \mathfrak{k}_\alpha$.

The subalgebras

$$\mathfrak{l} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Upsilon \cap -\Upsilon} \mathfrak{k}_\alpha \right) \qquad \mathfrak{u} = \left(\bigoplus_{\alpha \in \Upsilon :\, -\alpha \notin \Upsilon} \mathfrak{k}_\alpha \right)$$

are respectively called the Levi factor of $\mathfrak{q}$ and the nilpotent radical of $\mathfrak{q}$, and they are a key tool for the study of the finite-dimensional irreducible representations of $\mathfrak{k}$. Indeed each representation of $\mathfrak{k}$ on V gives rise to a representation of $\mathfrak{l}$ on the space of $\mathfrak{u}$ invariants

$$V^{\mathfrak{u}} = \{v \in V \colon X \cdot v = 0 \quad \forall X \in \mathfrak{u}\}$$

and the structure of the $\mathfrak{l}$-module $V^{\mathfrak{u}}$ completely determines the structure of the $\mathfrak{k}$-module V. In more details:[10]

- $\mathfrak{l}$ is reductive. It has center $\mathfrak{h}'' = \bigcap_{\alpha \in \Upsilon \cap -\Upsilon} \ker(\alpha) \subseteq \mathfrak{h}$ and semi-simple part

$$\mathfrak{l}_{ss} = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in \Upsilon \cap -\Upsilon} \mathfrak{k}_\alpha \right)$$

  with $\mathfrak{h}' = \bigoplus_{\alpha \in \Upsilon \cap -\Upsilon} \mathbb{C} H_\alpha$ is a Cartan subalgebra of $\mathfrak{l}_{ss}$

- for each irreducible finite-dimensional $\mathfrak{k}$-module V, the corresponding $\mathfrak{l}$-module $V^{\mathfrak{u}}$ is irreducible and has the same highest weight as V

- two irreducible finite-dimensional $\mathfrak{k}$-modules V_1 and V_2 are equivalent if and only if the corresponding $\mathfrak{l}$-modules $V_1^{\mathfrak{u}}$ and $V_2^{\mathfrak{u}}$ are equivalent

- for every irreducible finite-dimensional $\mathfrak{l}$-module E whose highest weight λ is algebraically integral and dominant for $\Psi^+(\mathfrak{k}, \mathfrak{h})$, there exists a finite dimensional $\mathfrak{k}$-module V, with highest weight λ, such that $E = V^{\mathfrak{u}}$.

We want to apply these remarks to the parabolic subalgebra corresponding to the set of roots

$$\Upsilon = \Psi^+ \oplus \left(\bigoplus_{1 \leqslant i < j \leqslant s} -(\psi_i - \psi_j) \right) \oplus \left(\bigoplus_{k,\, l = s+1,\, k<l}^{n} -(\psi_k \pm \psi_l) \right)$$

so that

[10] This material can be found in [20].

$$\Upsilon \cap -\Upsilon = \{\pm(\psi_i - \psi_j))\}_{1\leqslant i<j\leqslant s} \cup \{\pm(\psi_k \pm \psi_l)\}_{k,\, l=s+1,\ldots,n,\, k<l}.$$

We notice that the Levi factor

$$\mathfrak{l} = \mathfrak{h} \bigoplus_{\alpha\in\Upsilon\cap-\Upsilon} \mathfrak{k}_\alpha = \left(\left(\sum_{1\leqslant i<j\leqslant s} H_{\pm(\psi_i-\psi_j)}\right) \oplus \left(\bigoplus_{1\leqslant i<j\leqslant s} \mathfrak{k}_{\pm(\psi_i-\psi_j)}\right)\right) \oplus$$

$$\oplus \left(\left(\sum_{s+1\leqslant k<l\leqslant n} H_{\pm(\psi_k\pm\psi_l)}\right) \oplus \left(\bigoplus_{s+1\leqslant k<l\leqslant n} \mathfrak{k}_{\pm(\psi_k\pm\psi_l)}\right)\right)$$

is isomorphic to the reductive Lie algebra $\mathfrak{gl}(s,\, \mathbb{C}) \oplus \mathfrak{so}(2n-2s,\, \mathbb{C})$. [11]

Each dominant weight $a_1\psi_1 + \cdots + a_s\psi_s + a_{s+1}\psi_{s+1} + \cdots + a_n\psi_n$ of $\mathfrak{k} = \mathfrak{so}(2n,\, \mathbb{C})$ can be regarded as a weight of $\mathfrak{l} = \mathfrak{gl}(s,\, \mathbb{C}) \oplus \mathfrak{so}(2n-2s,\, \mathbb{C})$ by grouping the first s variables and the last $n-s$ variables together. We write

$$k_1\xi_1 + \cdots + k_s\xi_s \quad (k_1 \geqslant k_2 \geqslant \cdots \geqslant k_s;\ k_i - k_{i+1} \in \mathbb{Z} \quad \forall\, i)$$

for the dominant weights of $\mathfrak{gl}(s,\, \mathbb{C})$, and

$$k_1\zeta_1 + \cdots + k_{n-s}\zeta_{n-s} \quad (k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{n-s};\ 2k_i \in \mathbb{Z} \;\&\; k_i - k_j \in \mathbb{Z} \quad \forall\, i,\, j).$$

the ones of $\mathfrak{so}(2n-2s,\, \mathbb{C})$.[12] If

$$\nu = a_1\psi_1 + \cdots + a_s\psi_s + a_{s+1}\psi_{s+1} + \cdots + a_n\psi_n$$

is a weight of $\mathfrak{so}(2n,\, \mathbb{C})$, we denote by

$$\widetilde{\nu} = (a_1\xi_1 + \cdots + a_s\xi_s) + (a_{s+1}\zeta_1 + \cdots + a_n\zeta_{n-s})$$

the corresponding weight of $\mathfrak{l} = \mathfrak{gl}(s,\, \mathbb{C}) \oplus \mathfrak{so}(2n-2s,\, \mathbb{C})$.

It follows from the previous remarks that the $\mathfrak{so}(2n,\, \mathbb{C})$-representation with highest weight υ_s is an irreducible summand of $\Lambda^k(\mathbb{C}^{2n}) \otimes \Lambda^k(\mathbb{C}^{2n})$, with some multiplicity m_s, if and only if the $\mathfrak{gl}(s,\, \mathbb{C}) \oplus \mathfrak{so}(2n-2s,\, \mathbb{C})$-representation with highest weight $\widetilde{\upsilon_s}$ is an irreducible summand of $(\Lambda^k(\mathbb{C}^{2n}) \otimes \Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} = (\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} \otimes (\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}}$, with

[11] It is $\mathfrak{gl}(s,\, \mathbb{C})$ instead of $\mathfrak{sl}(s,\, \mathbb{C})$ because the ψ_j's are linear independent (here there is no relation $\sum_{j=1}^{s} \psi_j = 0$).

[12] Here $n-s \geqslant 2$, because $n > k > s$.

the same multiplicity m_s.

So it all comes to understanding the decomposition of $(\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} \otimes (\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}}$ in irreducible $\mathfrak{gl}(s, \mathbb{C}) \oplus \mathfrak{so}(2n-2s, \mathbb{C})$- modules.

Since $\Lambda^k(\mathbb{C}^{2n})$ is the irreducible representation of $\mathfrak{so}(2n, \mathbb{C})$ with highest weight $\psi_1 + \cdots + \psi_s + \psi_{s+1} + \cdots + \psi_k$, the space of $\mathfrak{u}$-invariants $(\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}}$ is the irreducible representation of $\mathfrak{gl}(s, \mathbb{C}) \oplus \mathfrak{so}(2n-2s, \mathbb{C})$ with highest weight

$$(\xi_1 + \cdots + \xi_s) + (\zeta_1 + \cdots + \zeta_{k-s}).$$

It is a standard fact that the irreducible representation of $\mathfrak{so}(2n-2s, \mathbb{C})$ with highest weight $(\zeta_1 + \cdots + \zeta_{k-s})$ is $\Lambda^{k-s}\mathbb{C}^{2n-2s}$. The following claim identifies the irreducible representation of $\mathfrak{gl}(s, \mathbb{C})$ with highest weight $(\xi_1 + \cdots + \xi_s)$.

Claim 6. *The irreducible representation of $\mathfrak{gl}(s, \mathbb{C})$ with highest weight $(\xi_1 + \cdots + \xi_s)$ is the the differential of the determinant representation of $GL(s, \mathbb{C})$.*

Proof. Let $\xi = c_1 \xi_1 + \cdots + c_{s-1} \xi_{s-1} + c_s \xi_s$ be an analytically integral form for $\mathfrak{gl}(s, \mathbb{C})$, and let Φ_ξ be the irreducible representation of $GL(s, \mathbb{C})$ with highest weight ξ. We can describe Φ_ξ as follows:

- The restriction of Φ_ξ to $SL(s, \mathbb{C})$ is the unique irreducible representation $\Phi_{\widehat{\xi}}$ of $SL(s, \mathbb{C})$ with highest weight $\widehat{\xi} = c_1 \xi_1 + \cdots + c_{s-1} \xi_{s-1} + c_s(-\xi_1 - \cdots - \xi_{s-1})$.

- The representation Φ_ξ of $GL(s, \mathbb{C})$ is the unique extension of $\Phi_{\widehat{\xi}}$ to $GL(s, \mathbb{C})$ that satisfies the condition:

$$\Phi_\xi(zI_{s \times s}) = z^{c_1 + c_2 + \cdots + c_s} \mathrm{Id} \quad \forall\, z \in \mathbb{C}^*$$

  with $I_{s \times s}$ the identity matrix of size s.

If A is an element of $GL(s, \mathbb{C})$ and z is an s^{th}-root of the determinant of A, so that the matrix $\frac{1}{z}A$ belongs to $SL(s, \mathbb{C})$, then

$$\Phi_\xi(A) = z^{c_1+c_2+\cdots+c_s}\Phi_{\widehat{\xi}}\left(\tfrac{1}{z}A\right).$$

Let $\xi = \xi_1 + \cdots + \xi_{s-1} + \xi_s$, then $\widehat{\xi} = 0$ and $\Phi_{\widehat{\xi}}$ is the trivial representation of $SL(s,\, \mathbb{C})$. For A in $GL(s,\, \mathbb{C})$, we choose z s.t. $\det(A) = z^s$, and we obtain

$$\Phi_\xi(A) = z^{\overbrace{1+1+\cdots+1}^{s}}\Phi_{\widehat{\xi}}\left(\frac{1}{z}A\right) = z^s\,\mathrm{Id} = \det(A)\mathrm{Id}.$$

This shows that $\Phi_{\xi_1+\cdots+\xi_{s-1}+\xi_s}$ is the differential of determinant representation of $GL(s,\, \mathbb{C})$, as claimed. $\square$

As a consequence, we obtain that the representation of $GL(s,\, \mathbb{C}) \times SO(2n-2s,\, \mathbb{C})$ on the space of $\mathfrak{u}$-invariants in $\Lambda^k(\mathbb{C}^{2n})$ is equal to $(det) \otimes \Lambda^{k-s}\mathbb{C}^{2n-2s}$. Hence

$$(\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} \otimes (\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} = [(det) \otimes \Lambda^{k-s}\mathbb{C}^{2n-2s}] \otimes [(det) \otimes \Lambda^{k-s}\mathbb{C}^{2n-2s}] =$$

$$= [(det)^2 \otimes (\Lambda^{k-s}\mathbb{C}^{2n-2s})^{\otimes 2}].$$

We notice that $(\Lambda^{k-s}\mathbb{C}^{2n-2s})^{\otimes 2}$ contains the trivial representation of $SO(2n-2s,\, \mathbb{C})$ with multiplicity one.[13] Therefore, the representation $(\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}} \otimes (\Lambda^k(\mathbb{C}^{2n}))^{\mathfrak{u}}$ of $GL(s,\, \mathbb{C}) \times SO(2n-2s,\, \mathbb{C})$ contains the irreducible representation $(det)^2 \otimes (trivial)$, also with multiplicity one.
Because $(det)^2 \otimes (trivial)$ has highest weight

$$2\xi_1 + \cdots + 2\xi_{s-1} + 2\xi_s + 0\zeta_1 + 0\zeta_2 + \cdots + 0\zeta_{n-s} = \widetilde{\upsilon}_s$$

it follows that the representation $\Lambda^k(\mathbb{C}^{2n}) \otimes \Lambda^k(\mathbb{C}^{2n})$ of $SO(n,\, \mathbb{C})$ contains the irreducible representation with highest weight υ_s (with multiplicity one). This finally ends the proof of the lemma. $\square$

We now give the proof of corollary 5.

[13] Apply the same argument used in the discussion of the case $s = 0$.

Proof. We must show that, for each $k = 1 \dots n$, the irreducible representation of $SO(2n, \mathbb{R})$ with highest weight $\sum_{j=1}^{k} 2\psi_j$ contains the trivial M-type $\delta_\varnothing$ with multiplicity $\binom{2n}{k} - \binom{2n}{k-1}$.

Recall that, for each $k = 1 \dots n$, the multiplicity of $\delta_\varnothing$ in $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ is equal to $\binom{2n}{k}$, and that:

$$\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n} = \mu_{\sum_{j=1}^{k} 2\psi_j} \oplus \mu_{\sum_{j=1}^{k-1} 2\psi_j} \oplus \cdots \oplus \mu_{2\psi_1} \oplus \mu_0 \oplus \Omega$$

with Ω is a sum of non spherical K-types. It follows that if we denote by

c_s: the multiplicity of $\delta_\varnothing$ in $\mu_{\sum_{j=1}^{s} 2\psi_j}$, for all $s = 1 \dots n-1$

c_0: the multiplicity of $\delta_\varnothing$ in μ_0 $(c_0 = 1)$

c^k: the multiplicity of $\delta_\varnothing$ in $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ $(c^k = \binom{2n}{k})$

then

$$\binom{2n}{k} = c^k = c_0 + c_1 + \cdots + c_k = 1 + \sum_{s=1}^{k} c_s$$

for all $k = 1 \dots n$. Solving this equations inductively we find that

$$c_s = \binom{2n}{s} - \binom{2n}{s-1}$$

for all $s = 1 \dots n$, and this ends the proof of the corollary, and shows that $\mu_{\sum_{j=1}^{s} 2\psi_j}$ is spherical.

$\square$

To conclude our discussion of the petite spherical representations of $SO(2n, \mathbb{R})$ we must still consider the petite representations with highest weight

$$(2\psi_1 + \cdots + 2\psi_{n-1}) \pm 2\psi_n.$$

An analysis similar to the one carried out for the K-type $2\psi_1 + \cdots + 2\psi_k$ will show that they are both spherical.

For brevity reasons, let us introduce some notations:

$$\upsilon_s = 2\varepsilon_1 + 2\varepsilon_2 + \cdots + 2\varepsilon_s \quad (s = 1 \dots n-1)$$

$$\upsilon_n^+ = 2\varepsilon_1 + 2\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + 2\varepsilon_n$$

$$\upsilon_n^- = 2\varepsilon_1 + 2\varepsilon_2 + \cdots + 2\varepsilon_{n-1} - 2\varepsilon_n$$

$$\gamma_s = \tfrac{1}{2}\upsilon_s = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_s \quad (s = 1 \dots n-1)$$

$$\gamma_n^+ = \tfrac{1}{2}\upsilon_n^+ = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} + \varepsilon_n$$

$$\gamma_n^- = \tfrac{1}{2}\upsilon_n^- = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1} - \varepsilon_n.$$

We notice that, for each $s = 1 \dots n-1$, μ_{γ_s} is the irreducible representation $\Lambda^s(\mathbb{C}^{2n})$ of $SO(2n, \mathbb{C})$, while $\mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-}$ are the two irreducible summands of $\Lambda^n(\mathbb{C}^{2n})$. We also recall that μ_{γ_s} is self-dual, for each $s = 1 \dots n-1$. It is interesting to observe that $\mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-}$ are not necessarily such. Indeed:

Lemma 6. *If n is odd, the dual of $\mu_{\gamma_n^+}$ is equal to $\mu_{\gamma_n^-}$. If n is even, $\mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-}$ are both self-dual.*

Proof. It is a standard fact[14] that, if V is an irreducible representation of highest weight λ and lowest weight λ', then V^* has highest weight $-\lambda'$ and lowest weight $-\lambda$. Also, $\lambda' = \omega_0 \cdot \lambda$, where ω_0 is the long Weyl group element. It follows that V_λ is self-dual if and only if its lowest and highest weights satisfy the formal symmetry condition $\lambda = -\lambda' = -\omega_0 \cdot \lambda$. Therefore, computing the dual of an irreducible representation comes down to determining its lowest weight.

When n is even, the long element of the Weyl group $W(\mathfrak{so}(2n, \mathbb{C}))$ is the negative of the identity, hence $\lambda' = \omega_0 \cdot \lambda = -\lambda$, and every irreducible representation is self-dual.

[14]Please, refer to [15], chapter 5 for details.

When n is odd, we notice that the negative of a weight ω of $\mu_{\gamma_n^+}$ is a weight of $\mu_{\gamma_n^-}$. Indeed, since the Weyl group of $\mathfrak{so}(2n, \mathbb{C})$ *only* contains the sign changes that involve an *even* number of signs, $-\omega$ is not conjugate to ω under $W(\mathfrak{so}(2n, \mathbb{C}))$. It follows that $-\omega$ is a weight of $\Lambda^n\mathbb{C}$ that does not lie in the same irreducible representation as ω.

Setting $\omega = \gamma_n^+$, we deduce that $-\gamma_n^+$ is a weight of $\mu_{\gamma_n^-}$. It is the lowest weight because, for every weight ν of $\mu_{\gamma_n^-}$, the negative $-\nu$ is a weight of $\mu_{\gamma_n^+}$, and hence we have:

$$(-\nu) \preccurlyeq \gamma_n^+ \Leftrightarrow (-\gamma_n^+) \preccurlyeq \nu.$$

Since the lowest weight of $\mu_{\gamma_n^-}$ is equal to $-\gamma_n^+$, the dual representation has highest weight $-(-\gamma_n^+) = +\gamma_n^+$. This concludes the proof. □

Corollary 6. *If n is even, both $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^-}$ contain the trivial K-type. If n is odd, none of them does.*

Proof. When n is even, the result is trivial: since $\mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-}$ are both self-dual, the representations $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^-}$ (of K) contain the trivial K-type with multiplicity one.[15] When n is odd, we compute the multiplicity of the trivial K-type in $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}$ and show that it is zero:

$$\mathrm{mult}_{\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}}(\mu_0) = \textstyle\sum_{\tau \in W(K)} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(0 + \rho - \tau \cdot (\gamma_n^+ + \rho)) =$$

$$= \textstyle\sum_{\tau \in W(K)} \mathrm{sgn}(\tau) \mathrm{m}_{\mu_{\gamma_n^+}}(\tau^{-1} \cdot \rho - (\gamma_n^+ + \rho)) =$$

$$= \textstyle\sum_{\tau \in W(K)} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(-\gamma_n^+ - (\rho - \tau^{-1} \cdot \rho)) = \qquad ⁜$$

$$= \textstyle\sum_{\tau \in W(K)} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(+\gamma_n^- + \rho - (\omega_0 \tau^{-1} \omega_0^{-1}) \cdot \rho) =$$

[15] Apply the same argument used in the proof of lemma 5 for $s = 0$.

$$= \sum_{\sigma \in W(K)} \operatorname{sgn}(\sigma)\, \mathrm{m}_{\mu_{\gamma_n^+}}(+\gamma_n^- + (\rho - \sigma \cdot \rho)) = 0 \qquad ⁎⁎$$

Just a few comments are necessary:

in (⁎), we apply the long Weyl group element ω_0 to the weight $-\gamma_n^+ - (\rho - \tau^{-1} \cdot \rho)$, obtaining a weight with the same multiplicity. Since

$$\omega_0 \cdot \gamma_n^+ = -\gamma_n^- = \text{the lowest weight of } \mu_{\gamma_n^+}$$

$$\omega_0 \cdot \rho = -\rho \qquad (\text{because } \omega_0 \text{ carries } \Delta^+ \text{ in } \Delta^-)$$

$$\omega_0 \cdot (\tau^{-1} \cdot \rho) = -(\omega_0 \tau^{-1} \omega_0^{-1}) \cdot \rho$$

we find that $\omega_0 \cdot (-\gamma_n^+ - (\rho - \tau^{-1} \cdot \rho)) = +\gamma_n^- + \rho - (\omega_0 \tau^{-1} \omega_0^{-1}) \cdot \rho$.

In (⁎⁎) we use the fact that $(+\gamma_n^- + (\rho - \sigma \cdot \rho))$ is never a weight of $\mu_{\gamma_n^+}$. Indeed, if σ is the identity element, then $(+\gamma_n^- + (\rho - \sigma \cdot \rho)) = +\gamma_n^-$ and if γ_n^- were a weight of $\mu_{\gamma_n^+}$, then the condition $\gamma_n^- \preceq \gamma_n^+$ would make $\gamma_n^+ - \gamma_n^- = +2\varepsilon_n$ a sum of positive roots. Similarly, if σ is a non trivial element of the Weyl group, then

$$\gamma_n^+ - (\gamma_n^- + (\rho - \sigma \cdot \rho)) = +2\varepsilon_n - \underbrace{(\rho - \sigma \cdot \rho)}_{\text{a sum of pos. roots}}$$

cannot be a sum of positive roots, or $2\varepsilon_n$ would be.

We have proved that $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}$ does not contain the trivial K-type. The argument for $\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^-}$ is similar, so we omit the details. $\square$

Corollary 7. *If n is odd, both $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^-}$ and $\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^+}$ contain the trivial K-type. If n is even, none of them does.*

Proof. Since $\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^-}$ and $\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^+}$ are isomorphic, the multiplicity of the trivial K-type is the same in the two representations. Let us show that this multiplicity is equal to one if n is odd, and to zero if n is even.

$$\mathrm{mult}_{\mu_{\gamma_n^+}\otimes\mu_{\gamma_n^-}}(\mu_0) = \sum_{\tau\in W} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(0+\rho-\tau\cdot(\gamma_n^- +\rho)) =$$

$$= \sum_{\tau\in W} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(\tau^{-1}\cdot\rho-(\gamma_n^- +\rho)) =$$

$$= \sum_{\tau\in W} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(-\,\gamma_n^- -(\rho-\tau^{-1}\cdot\rho)) =$$

$$= \sum_{\tau\in W} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(-\,\omega_0\cdot\gamma_n^- +\rho-(\omega_0\tau^{-1}\omega_0^{-1})\cdot\rho) =$$

$$= \sum_{\tau\in W} \mathrm{sgn}(\tau)\, \mathrm{m}_{\mu_{\gamma_n^+}}(-\,\omega_0\cdot\gamma_n^- +(\rho-\sigma\cdot\rho)) =$$

$$= \begin{cases} \sum_{\sigma\in W} \mathrm{sgn}(\sigma)\, \mathrm{m}_{\mu_{\gamma_n^+}}(+\,\gamma_n^- +(\rho-\sigma\cdot\rho)) = 0 & \text{if } n \text{ is even} \\ \\ \sum_{\sigma\in W} \mathrm{sgn}(\sigma)\, \mathrm{m}_{\mu_{\gamma_n^+}}(+\,\gamma_n^+ +(\rho-\sigma\cdot\rho)) = 1 & \text{if } n \text{ is odd.} \end{cases}$$

□

Once these results have been established, it is pretty easy to prove that:

Lemma 7. *As a representation of $SO(2n,\mathbb{R})$,*

$$\mu_{\gamma_n^+}\otimes\mu_{\gamma_n^+} = \Big(\bigoplus_{\substack{s=0\dots n-1\\ n-s\ \mathit{even}}} \mu_{\upsilon_s}\Big)\oplus\mu_{\upsilon_n^+}\oplus\ \textit{some non spherical types}$$

$$\mu_{\gamma_n^-}\otimes\mu_{\gamma_n^-} = \Big(\bigoplus_{\substack{s=0\dots n-1\\ n-s\ \mathit{even}}} \mu_{\upsilon_s}\Big)\oplus\mu_{\upsilon_n^-}\oplus\ \textit{some non spherical types}$$

$$\mu_{\gamma_n^+}\otimes\mu_{\gamma_n^-} = \Big(\bigoplus_{\substack{s=0\dots n-1\\ n-s\ \mathit{odd}}} \mu_{\upsilon_s}\Big)\oplus\ \textit{some non spherical types.}$$

We omit the proof of this lemma, because it is similar[16] to the one already given for the decomposition in irreducible summands of the tensor product $\mu_{\gamma_k} \otimes \mu_{\gamma_k} = \Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ with $k < n$. Once again, we need to use the "$\mathfrak{gl}(s) \oplus \mathfrak{so}(2n-2s)$ trick".

Corollary 8. *The irreducible representation $\mu_{\gamma_n^\pm}$ of $SO(2n,\mathbb{R})$ contains the trivial M-type with multiplicity $\binom{2n}{n} - \binom{2n}{n-1}$.*

Proof. Let us introduce some additional notations:

$c_n^\pm$: the multiplicity of $\delta_\varnothing$ in $\mu_{\upsilon_n^\pm}$

c^n: the multiplicity of $\delta_\varnothing$ in $\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n}$.

Recall that c_s denotes the multiplicity of $\delta_\varnothing$ in $\mu_{\sum_{j=1}^s 2\psi_j}$, for all $s = 1 \dots n-1$, and that $c_s = \binom{2n}{s} - \binom{2n}{s-1}$. Therefore, $\sum_{s=1}^{n-1} c_s = \binom{2n}{n-1}$.
The aim of this corollary is to compute $c_n^\pm$. In order to find an equation for $c_n^\pm$, we look at the decomposition

$$\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n} = (\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^+}) \oplus (\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^-}) \oplus (\mu_{\gamma_n^+} \otimes \mu_{\gamma_n^-}) \oplus (\mu_{\gamma_n^-} \otimes \mu_{\gamma_n^+})$$

$$= \mu_{\upsilon_n^+} \oplus \mu_{\upsilon_n^-} \oplus 2\left(\bigoplus_{s=0\dots n-1} \mu_{\upsilon_s}\right) \oplus \text{ some non spherical types}$$

which gives:

$$c^n = c_n^+ + c_n^- + 2\sum_{s=0}^{n-1} c_s = c_n^+ + c_n^- + 2\binom{2n}{n-1}. \tag{8.1}$$

We start by computing c^n. Consider all the vectors of the form

$$(\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \dots \wedge \overrightarrow{e_{i_n}}) \otimes (\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \dots \wedge \overrightarrow{e_{i_n}})$$

or

$$(\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \dots \wedge \overrightarrow{ei_{n)}}) \otimes (\overrightarrow{e_{j_1}} \wedge \overrightarrow{e_{j_2}} \wedge \dots \wedge \overrightarrow{e_{j_n}})$$

[16] The few elements of difference have been fully outlined.

with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ complementary sets in $\{1 \ldots 2n\}$. These vectors form a basis for the isotypic component of the trivial M-type in $\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n}$. Hence, $c^n = 2\binom{2n}{n}$.

The next step is to show that $c_n^+ = c_n^-$.

Let C be the orthogonal matrix $C = \operatorname{diag}(1, 1, \ldots, 1, -1)$ and let Φ be the (irreducible) representation of $O(2n, \mathbb{C})$ on $\Lambda^n\mathbb{C}^{2n}$. Conjugation by C preserves the subgroup $SO(2n, \mathbb{C})$ of $O(2n, \mathbb{C})$, therefore $\Phi(C)$ carries a $SO(2n, \mathbb{C})$-stable subspace of $\Lambda^n\mathbb{C}^{2n}$ into another $SO(2n, \mathbb{C})$-stable subspace. Conjugation by C also preserves the Cartan subalgebra $\mathfrak{h}$, sending an element of coordinates $\theta_1, \ldots, \theta_{n-1}, \theta_n$ into the element of coordinates $\theta_1, \ldots, \theta_{n-1}, -\theta_n$. So $\Phi(C)$ carries a weight vector of weight $a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1} + a_n\psi_n$ into a weight vector of weight $a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1} - a_n\psi_n$. It follows that $\Phi(C)$ defines an isomorphism from $\mu_{\upsilon_n^+}$ to $\mu_{\upsilon_n^-}$ and, given that all diagonal matrices commute, $\Phi(C)$ intertwines the actions of M on the two representations. We conclude that the restriction to M of $\mu_{\upsilon_n^+}$ and $\mu_{\upsilon_n^-}$ are isomorphic. Hence the trivial M-type appears in the two representations with the same multiplicity.

To conclude the proof, we use equation 8.1 to deduce that the multiplicity of the trivial M-type in $\mu_{\gamma_n^\pm}$ is equal to $\binom{2n}{n} - \binom{2n}{n-1}$. □

It is a consequence of this corollary that $\mu_{\gamma_n^+}$ and $\mu_{\gamma_n^-}$ are both spherical. We obtain a complete classification of the petite spherical representations of $SO(2n, \mathbb{R})$:

Proposition 3. *The petite spherical representations of $SO(2n, \mathbb{R})$ are exactly the ones with highest weight*

- $0\psi_1 + \cdots + 0\psi_n$
- $2\psi_1 + \cdots + 2\psi_k \qquad 0 < k < n$
- $2\psi_1 + \cdots + 2\psi_{n-1} \pm 2\psi_n$.

Our next goal is to describe the representation of the Weyl group $W = W(SL(2n, \mathbb{R}))$ on the space of M-fixed vectors of each spherical petite representation. Being W isomorphic to $\mathcal{S}_{2n}$, we start by recalling the representation theory of the symmetric group.

8.3 The irreducible representations of the symmetric group

Definition 7. *A* ***Young diagram*** *is a collection of boxes, arranged in left -justified rows, with a (weakly) decreasing number of boxes in each row.*

The correspondence

$$\{ \text{ Young diagrams with } m \text{ boxes } \} \leftrightarrow \{ \text{ partitions of } m \}$$

is clearly bijective.

Definition 8. *Let λ be a partition of m. A* ***Young tableau*** *is an array obtained by filling the m boxes of the Young diagram corresponding to λ with the integers $1 \dots m$ (no repetitions allowed).*

For instance:

1	3	2	6	8
4	7	5		

is a Young tableau of shape $(5, 3)$.

Definition 9. *A* ***standard Young tableau*** *is a Young tableau with a filling that is increasing across each row and also down each column.*

For instance:

1	3	4	5	8
2	6	7		

is a standard Young tableau of shape $(5, 3)$.

For any given partition λ of m, there is a natural action of the symmetric group $\mathcal{S}_m$ on the set of Young tableaux of shape λ: we define $\sigma \cdot T$ to be the filling that puts $\sigma(i)$ in the box where T puts i.

For example, applying the permutation $(1,7)$ to the Young tableau

$$\begin{array}{|c|c|c|c|c|}\hline 1&3&4&5&8\\ \hline 2&6&7 \\ \cline{1-3}\end{array} \quad \text{gives:} \quad \begin{array}{|c|c|c|c|c|}\hline 7&3&4&5&8\\ \hline 2&6&1 \\ \cline{1-3}\end{array}\,.$$

Fix a Young tableau T of shape $\lambda = (\lambda_1 \geqslant \ldots \lambda_k > 0)$. The row group $R(T)$ of T is the subgroup of $\mathcal{S}_m$ that consists of those permutations that permute the entries of each row among themselves. It is clearly the product of symmetric groups $\mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \ldots \mathcal{S}_{\lambda_k}$. Similarly, we define the column group $C(T)$ of T as the subgroup of $\mathcal{S}_m$ that stabilizes each column of T.

Definition 10. *Two tableaux T and T' (of shape λ) are called row equivalent if corresponding rows contain the same entries, i.e. if $T = \sigma \cdot T'$ for some σ in $R(T)$. The row equivalence class of T is called a **tabloid** (of shape λ) and is denoted by $\{T\}$.*

A tabloid $\{T\}$ is displayed by omitting the vertical lines between boxes, emphasizing that only the content of each row matters:

$$\begin{array}{|l|}\hline 1\ 3\ 4\ 5\ 8\\ \hline 2\ 6\ 7 \\ \hline\end{array} \quad = \quad \begin{array}{|l|}\hline 1\ 8\ 5\ 3\ 4\\ \hline 2\ 7\ 6 \\ \hline\end{array}\,.$$

The action of $\mathcal{S}_m$ on the set of all tableaux with m boxes of shape λ descends to an action on tabloids:

$$\sigma \cdot \{T\} = \{\sigma \cdot T\}$$

and gives rise to a representation of $\mathcal{S}_m$ of dimension $\frac{m!}{(\lambda_1!)(\lambda_1!)\cdots(\lambda_k!)}$, that we denote by M^λ.[17]

Definition 11. *We call M^λ the **permutation module** corresponding to λ.*

[17] M^λ is the complex vector space with basis the tabloids of shape λ.

An important remark:

$$M^\lambda = \mathrm{Ind}_{\mathcal{S}_{\lambda_1}\times\mathcal{S}_{\lambda_2}\times\cdots\times\mathcal{S}_{\lambda_k}}^{\mathcal{S}_m}(trivial)$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ are the parts of λ.

The permutation representation M^λ is (in general) reducible. We now discuss its decomposition in irreducible summands.

For each tableau T, let κ_T be the sum with sign of all the permutations that stabilize the columns of T:

$$\kappa_T = \sum_{q\in C(T)} \mathrm{sgn}(q)q.$$

κ_T is an element of the group algebra $\mathbb{C}[\mathcal{S}_m]$ and is called a Young symmetrizer.

Definition 12. *If T is a tableau, we call $\underline{e}_T = \kappa_T \cdot \{T\}$ the **polytabloid** associated to T.*

Let us illustrate this definition by an example:

if $T=\begin{array}{|c|c|c|}\hline 4&1&2\\ \hline 3&5 \\ \cline{1-2}\end{array}$ then $\kappa_T = 1-(4,3)-(1,5)+(4,3)(1,5)$ and $\underline{e}_T$ is given by:

$$\begin{array}{|c|}\hline 4\ 1\ 2\\ \hline 3\ 5\\ \hline\end{array} - \begin{array}{|c|}\hline 3\ 1\ 2\\ \hline 4\ 5\\ \hline\end{array} - \begin{array}{|c|}\hline 3\ 1\ 2\\ \hline 4\ 5\\ \hline\end{array} - \begin{array}{|c|}\hline 3\ 1\ 2\\ \hline 4\ 5\\ \hline\end{array}\ .$$

We notice that $\sigma\cdot\underline{e}_T = \underline{e}_{\sigma\cdot T}$, for each permutation σ in $\mathcal{S}_m$, and for each tableau T. Therefore, the subspace of M^λ spanned by the polytabloids is stable under the action of $\mathcal{S}_m$ and defines a submodule of M^λ.

Definition 13. *The **Specht module** S^λ corresponding to a partition λ is the submodule of M^λ spanned by the polytabloids of shape λ.*

Theorem 13. *For each partition λ of m, the Specht module S^λ is an irreducible representation of the symmetric group $\mathcal{S}_m$.*

Theorem 14. *Every irreducible representation of $\mathcal{S}_m$ is equivalent to a Specht module S^λ, for some partition λ of m.*

In particular, every irreducible summand of a permutation representation M^μ is a Specht module. If we write

$$M^\mu = \bigoplus_{\lambda \vdash m} \mathrm{m}_{\lambda_\mu} S^\lambda$$

for the decomposition of M^μ in isotypic components, then the multiplicity m_{λ_μ} of S^λ in M^μ can be explicitly described as follows:

$$\mathrm{m}_{\lambda_\mu} = \#\{\text{semi-standard tableaux of shape } \lambda \text{ and content } \nu\}\,.$$

More explicitly, m_{λ_μ} is the number of fillings of a Young diagram of shape λ with integers $\{1 \ldots r\}$, r being the number of parts of ν, so that

- the rows weakly increase;
- the columns strictly increase; and
- if $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r > 0)$, then μ_j equals the numbers of j's in the filling.

Suppose for example that $\mu = (m-k, k)$. For each partition λ of m, m_{λ_μ} counts the numbers of fillings of the Young diagram of λ with $(m-k)$ integers equal to 1 and k integers equal to 2, in such a way that the entries increase weakly across each row and strictly across each column. Let us count these fillings.

If λ has only one row, then there exists one (and only one) filling with these properties, namely:

$$\overbrace{\boxed{1}\boxed{1}\cdots\boxed{1}\boxed{1}}^{m-k}\overbrace{\boxed{2}\boxed{2}\cdots\boxed{2}\boxed{2}}^{k}$$

If λ has three or more rows, then $\mathrm{m}_{\lambda_\mu} = 0$, because to make the first column strictly increasing, its third box should be filled up with a 3 (which is not allowed).

Let us now look at partitions λ with two rows, say $\lambda = (m-s, s)$. Since all the $(m-k)$ entries equal to 1 must sit in the first row, s must be smaller or equal to k. If this condition is met, the partition $(m-s, s)$ admits one (and only one) filling of the prescribed form, namely

$$\begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\cdots\boxed{1}\boxed{1}}^{m-k}\overbrace{\boxed{2}\boxed{2}\cdots\boxed{2}\boxed{2}}^{k-1} \\ \boxed{2} \end{array} \qquad \text{if } s=1$$

$$\begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\cdots\boxed{1}\boxed{1}}^{m-k}\overbrace{\boxed{2}\boxed{2}\cdots\boxed{2}}^{k-2} \\ \boxed{2}\boxed{2} \end{array} \qquad \text{if } s=2$$

$\cdots$

$\cdots$

$$\begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\cdots\boxed{1}\boxed{1}}^{m-k}\overbrace{\boxed{2}}^{1} \\ \underbrace{\boxed{2}\cdot\cdot\boxed{2}\boxed{2}}_{k-1} \end{array} \qquad \text{if } s=k-1$$

$$\begin{array}{l} \overbrace{\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\cdots\boxed{1}\boxed{1}}^{m-k} \\ \underbrace{\boxed{2}\cdot\cdot\boxed{2}\boxed{2}\boxed{2}}_{k} \end{array} \qquad \text{if } s=k$$

We conclude that for each $s = 0 \dots k$, the Specht module $S^{(m-s,s)}$ appears in $M^{(m-k,k)}$ with multiplicity one. There are no other irreducible summands. Therefore:

$$M^{(m-k,k)} = \bigoplus_{s=0}^{k} S^{(m-s,s)}.$$

For future reference we recall a few more facts about Specht modules.

Each box of a Young diagram determines a *hook*, which consists of that box and all the boxes in its row or in its columns below the box. The *hook-length of a box* is the number of boxes in its hook. We denote by $h_{i,j}$ the hook-length of the box in the ith row and the jth column. For instance, in

$$\begin{array}{ccccccc} \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet \\ \cdot & \cdot & \bullet & \cdot & \cdot & & \\ \cdot & \cdot & \bullet & \cdot & & & \end{array}$$

the box $(1,3)$ has hook-length 7.

Theorem 15 (Hook Formula). *Let λ be a partition of m. The Specht module S^λ has dimension equal to $m!$ divided by the product of the hook-lengths of the boxes:*

$$\dim\left(S^\lambda\right) = \frac{m!}{\prod_{i,j} h_{i,j}}.$$

As an example, we compute the dimension of the Specht modules corresponding to the partitions of m in at most two rows.

Assume $k \geq 1$. Filling up each box of the Young diagram of $\lambda = (m-k, k)$ with the corresponding hook-length, we find:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \text{m-k+1} & \text{m-k} & \cdots & \text{m-2k+3} & \text{m-2k+2} & \text{m-2k} & \cdots & 2 & 1 \\ \hline \text{k} & \text{k-1} & \cdots & 2 & 1 & & & & \\ \hline \end{array}.$$

Therefore, the product of the hook-lengths equals $\frac{k!(m-k+1)!}{m-2k+1}$, and the dimension of the Specht module is:

$$\dim\left(S^{(m-k,k)}\right) = \frac{m!(m-2k+1)}{k!(m-k+1)!} = \binom{m}{k} - \binom{m}{k-1}.$$

When $\lambda = (m)$ has only one row, the computation is even easier. Indeed the hook-length of the box in position $(1, j)$ is equal to $m + 1 - j$. Hence, the product of the hook-lengths is equal to $\prod_{j=1}^{m}(m+1-j) = m!$ and the Specht module $S^{(m)}$ has dimension 1. This is not surprising, because $S^{(m)}$ is the trivial representation of the symmetric group $\mathcal{S}_m$.

Theorem 16 (A basis for S^λ). *Let S^λ be the Specht module corresponding to a* *partition λ of m. The set*

$$\{\, \underline{e}_T : T \text{ is a standard tableau of shape } \lambda\}$$

is a basis for S^{λ}. It consists of the polytabloids associated to all the standard tableaux of shape λ.

Let us describe the action of a transposition $(k, k+1)$ on the Specht module S^{λ} with respect to this basis.[18] For any standard tableau T, three possibilities can occur:

(1) The indices k and $k+1$ lie in the same column of T, then

$$(k, k+1) \cdot \underline{e}_T = -\, \underline{e}_T$$

(2) The indices k and $k+1$ lie in the same row of T, then

$$(k, k+1) \cdot \underline{e}_T = \underline{e}_T + (\text{a sum of other standard polytabloids } \underline{e}_{T'}, \text{ with } T^{'} \rhd T)$$

(3) The indices k and $k+1$ are nor in the same row neither in the same column of T, then

$$(k, k+1) \cdot \underline{e}_T = \underline{e}_{T'}$$

for some standard tableau $T^{'} \neq T$.

For instance, the polytabloid $\underline{e}_T \in S^{(m-k,k)}$ associated to the standard tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & \text{2k-3} & \text{2k-1} & \text{2k+1} & \text{2k+2} & \text{2k+3} & \cdots & \text{m-1} & \text{m-0} \\ \hline 2 & 4 & 6 & \cdots & \text{2k-2} & \text{2k} \\ \cline{1-6} \end{array}.$$

is a simultaneous (-1)-eigenvector for the transpositions $(1, 2)$, $(3, 4) \ldots (2k-3, 2k-2)$

[18]Please, see [27] for details.

and $(2k-1, 2k)$. It is also a simultaneous $(+1)$-eigenvector for the transpositions $(2k+1, 2k+2)$, $(2k+3, 2k+4)\ldots$.

It is clear from this example and from the considerations above that a polytabloid of shape $(m, m-k)$ can be a simultaneous (-1)-eigenvector of at most k disjoint transpositions.

We conclude this section on the representations of the symmetric group $\mathcal{S}_m$ by mentioning the branching rule, which describes what happens when a representation S^λ of $\mathcal{S}_m$ is restricted to a subgroup isomorphic to $\mathcal{S}_{m-1}$.

Definition 14. *If λ is a Young diagram with m boxes, we define an* ***inner corner*** *of λ to be a box (i,j) of λ whose removal makes λ into a Young diagram of a partition of $m-1$. Any partition obtained by such a removal will be denoted by λ^-.*

The inner corners of λ are exactly those boxes that sit simultaneously at the end of a row and at the end of a column. For instance, if we mark with a bullet the inner corners of the partition $(4,4,1)$ of $m=9$ we obtain:

$$\begin{array}{|c|c|c|c|}\hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \bullet \\ \hline \bullet \\ \cline{1-1}\end{array}$$

so λ^- can be $(4,3,1)$ or $(4,4)$.

Theorem 17 (Branching Rule). *For each partition λ of m, the decomposition of the Specht module S^λ in irreducible $\mathcal{S}_{m-1}$ representations is given by:*

$$S^\lambda \downarrow_{\mathcal{S}_{m-1}} = \bigoplus_{\lambda^-} S^{\lambda^-}.$$

For instance, $S^{(4,4,1)} \downarrow_{\mathcal{S}_8} = S^{(4,3,1)} \oplus S^{(4,4)}$.

As an application of the branching rule we notice that, for every $m \geq 3$, the representations of $\mathcal{S}_m$ whose restriction to $\mathcal{S}_3$ does *not* contain the sign representation $(1,1,1)$ correspond to partitions of m with *at most* two rows.

8.4 The "petite" Weyl group representations

Let (E_μ, μ) be a spherical representation of K, and let c be the dimension of the isotypical component of the trivial M-type in μ. The Weyl group W of $SL(2n, \mathbb{R})$ acts on the space of M-fixed vectors E_μ^M, giving rise to a c-dimensional representation that we denote by ψ_μ. In this section we describe the Weyl group representation ψ_μ corresponding to each petite spherical K-type (E_μ, μ).

We identify the Weyl group of $SL(2n, \mathbb{R})$ with the symmetric group $\mathcal{S}_{2n}$ and the set of inequivalent irreducible representations of W with the set of partitions of $2n$, as illustrated in the previous section.

If μ has highest weight $0\psi_1 + \cdots + 0\psi_n$, then $\mu = \mu_0$ is the trivial representation of $SO(2n, \mathbb{R})$. It is clear that the corresponding Weyl group representation ψ_μ is the trivial representation of $\mathcal{S}_{2n}$, that we identify with the partition $(2n)$.

If μ has highest weight $2\psi_1 + \cdots + 2\psi_k$, with $0 < k < n$, then $\mu = \mu_{\upsilon_k}$ is an irreducible summand of $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$. The isotypic of the trivial M-type in $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$ has dimension $\binom{2n}{k}$ and is spanned by the vectors

$$(\overrightarrow{e_{i_1}} \wedge \cdots \wedge \overrightarrow{e_{i_k}}) \otimes (\overrightarrow{e_{i_1}} \wedge \cdots \wedge \overrightarrow{e_{i_k}})$$

with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant 2n$. Denote by $\widetilde{\psi}_k$ the representation of Weyl group on the space of M-fixed vectors of $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n}$.

Let H_k be the subgroup of $\mathcal{S}_{2n}$ consisting of all the permutations of the indices $\{1, \ldots, k\}$, and let L_{2n-k} be the subgroup consisting of permutations of $\{k+1, \ldots, 2n\}$. Clearly $H_k \backsimeq \mathcal{S}_k$ and $L_{2n-k} \backsimeq \mathcal{S}_{2n-k}$, so we write $\mathcal{S}_{2n-k} \times \mathcal{S}_k$ for the direct product $L_{2n-k} \times H_k$.[19] The vector

$$(\overrightarrow{e_1} \wedge \overrightarrow{e_2} \wedge \overrightarrow{e_3} \wedge \cdots \wedge \overrightarrow{e_k}) \otimes (\overrightarrow{e_1} \wedge \overrightarrow{e_2} \wedge \overrightarrow{e_3} \wedge \cdots \wedge \overrightarrow{e_k})$$

[19] $\mathcal{S}_{2n-k} \times \mathcal{S}_k$ consists of permutations of $\{1, \ldots, 2n\}$ whose cycle decomposition only involves cycles that lie either in $\mathcal{S}_{2n-k}$ or in $\mathcal{S}_k$.

spans a copy of the trivial representation of the subgroup $\mathcal{S}_{2n-k} \times \mathcal{S}_k$ of $\mathcal{S}_{2n}$, so $\widetilde{\psi}_k$ contains the trivial representation of $\mathcal{S}_{2n-k} \times \mathcal{S}_k$.

It follows by Frobenius reciprocity that $\widetilde{\psi}_k$ is included in the permutation representation

$$M^{(2n-k,k)} = \mathrm{Ind}_{\mathcal{S}_{2n-k}\times\mathcal{S}_k}^{\mathcal{S}_{2n}}(trivial).$$

By dimensional reasons, $\widetilde{\psi}_k \equiv M^{(2n-k,k)}$.

Putting together all the information we have gathered so far, we obtain:

- $M^{(2n-k,k)} = \bigoplus_{s=0\ldots k} S^{(2n-s,s)} = S^{(2n)} \oplus \left(\bigoplus_{s=1\ldots k} S^{(2n-s,s)}\right)$
- $\widetilde{\psi}_k = \psi_{\mu_0} \oplus \left(\bigoplus_{s=1\ldots k} \psi_{\mu_{\upsilon_s}}\right)$

  (because $\Lambda^k\mathbb{C}^{2n} \otimes \Lambda^k\mathbb{C}^{2n} = \mu_0 \oplus (\bigoplus_{s=1\ldots k} \mu_{\upsilon_s}) \oplus$ non spherical K-types)
- $S^{(2n)} = \psi_{\mu_0} =$ the trivial repr. of W
- $\dim\left(S^{(2n-s,s)}\right) = \binom{2n}{s} - \binom{2n}{s-1} = \dim(\psi_{\mu_s})$

for any $0 < K < n$.

It easily follows that for each $s = 1\ldots k$, the Weyl group representation $\psi_{\mu_{\upsilon_s}}$ (on the space of M fixed vectors of μ_{υ_s}) is equal to the Specht module $S^{(2n-s,s)}$ corresponding to the partition $(2n-s,s)$ of $2n$. This result holds for every k, hence:

$$\psi_{\mu_{\upsilon_s}} = S^{(2n-s,s)} \quad \forall\, s = 1,\ldots,n-1.$$

Finally we discuss the case in which $\mu = \mu_{\upsilon_n^+}$ or $\mu = \mu_{\upsilon_n^-}$, so μ has highest weight $\upsilon_n^\pm = 2\psi_1 + \cdots + 2\psi_{n-1} \pm 2\psi_n$ and sits inside $\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n}$.

We have already observed that the space of M-fixed vectors in $\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n}$ has dimension $2 \cdot \binom{2n}{k}$, and it is spanned by all the vectors of the form

$$(\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \cdots \wedge \overrightarrow{e_{i_n}}) \otimes (\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \cdots \wedge \overrightarrow{e_{i_n}})$$

or

$$(\overrightarrow{e_{i_1}} \wedge \overrightarrow{e_{i_2}} \wedge \cdots \wedge \overrightarrow{ei_{n)}}) \otimes (\overrightarrow{e_{j_1}} \wedge \overrightarrow{e_{j_2}} \wedge \cdots \wedge \overrightarrow{e_{j_n}})$$

with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ complementary sets in $\{1 \ldots 2n\}$.
Denote by $\widetilde{\psi}_k$ the corresponding Weyl group representation.
We define the subgroups $L_n \backsimeq \mathcal{S}_n$ and $H_n \backsimeq \mathcal{S}_n$ as above, and we write $\mathcal{S}_n \times \mathcal{S}_n$ for their direct product. The vector

$$(\overrightarrow{e_1} \wedge \overrightarrow{e_2} \wedge \overrightarrow{e_3} \wedge \cdots \wedge \overrightarrow{e_n}) \otimes (\overrightarrow{e_1} \wedge \overrightarrow{e_2} \wedge \overrightarrow{e_3} \wedge \cdots \wedge \overrightarrow{e_n})$$

spans a copy of the trivial representation of $\mathcal{S}_n \times \mathcal{S}_n$, and so does the vector

$$(\overrightarrow{e_{n+1}} \wedge \overrightarrow{e_{n+2}} \wedge \overrightarrow{e_{n+3}} \wedge \cdots \wedge \overrightarrow{e_{2n}}) \otimes (\overrightarrow{e_{n+1}} \wedge \overrightarrow{e_{n+2}} \wedge \overrightarrow{e_{n+3}} \wedge \cdots \wedge \overrightarrow{e_{2n}}).$$

So $\widetilde{\psi}_n$ contains two copies of the trivial representation of $\mathcal{S}_n \times \mathcal{S}_n$. It follows[20] that

$$\widetilde{\psi}_n = M^{(n,n)} \oplus M^{(n,n)}.$$

We notice that

- $M^{(n,n)} = \bigoplus_{s=0\ldots n} S^{(2n-s,s)} = \left(\bigoplus_{s=0\ldots n-1} S^{(2n-s,s)}\right) \oplus S^{(n,n)}$
- $\widetilde{\psi}_n = 2\left(\bigoplus_{s=1\ldots n-1} \psi_{\mu_{\upsilon_s}}\right) \oplus \psi_{\mu_{\upsilon_n^+}} \oplus \psi_{\mu_{\upsilon_n^-}}$, because

$$\Lambda^n\mathbb{C}^{2n} \otimes \Lambda^n\mathbb{C}^{2n} = \mu_{\upsilon_n^+} \oplus \mu_{\upsilon_n^-} \oplus 2\left(\sum_{s=0\ldots n-1} \mu_{\upsilon_s}\right) \oplus \text{ some non spherical types}$$

[20]Use Frobenius reciprocity and the fact that $\dim(\widetilde{\psi}_n) = 2\,\dim(\mathrm{Ind}_{\mathcal{S}_n \times \mathcal{S}_n}^{\mathcal{S}_{2n}}(\mathit{trivial}))$.

- $\psi_{\mu_{v_s}} = S^{(2n-s,s)}$ for all $s = 0 \dots n-1$, and

- $\dim(\psi_{\mu_{v_n^\pm}}) = \binom{2n}{n} - \binom{2n}{n-1} = \dim\left(S^{(n,n)}.\right)$

Therefore, $\psi_{\mu_{v_n^+}} = \psi_{\mu_{v_n^-}} = S^{(n,n)}$. In other words, the Weyl group representation $\psi_{\mu_{v_n^\pm}}$ (on the space of M fixed vectors of $\mu_{v_n^\pm}$) is equal to the Specht module $S^{(n,n)}$.

8.5 Conclusions

The following chart describes the set of petite spherical representations of $SL(2n, \mathbb{R})$ and summarizes the results of the chapter.

highest weight	**Weyl group repr.**	**dim.**
$0 = 0\psi_1 + \cdots + 0\psi_n$	$S^{(2m)} = trivial$	1
$2\psi_1 + \cdots + 2\psi_k$ $0 < k < n$	$S^{(2n-k,k)}$	$\binom{2n}{k} - \binom{2n}{k-1}$
$2\psi_1 + \cdots + 2\psi_{n-1} \pm 2\psi_n$	$S^{(n,n)}$	$\binom{2n}{n} - \binom{2n}{n-1}$

Conclusions. *For each petite spherical representation μ of $SL(2n, \mathbb{R})$, denote by ψ_μ the representation of the Weyl group $W \simeq \mathcal{S}_{2n}$ on the space of M-fixed vectors in μ. If we identify the irreducible representations of W with partitions of $2n$, the set*

$$\{\, \psi_\mu : \ \mu \text{ petite and spherical}\}$$

consists of the partitions of $2n$ in at most two parts.

Chapter 9

Constructing petite K-types

For each representation (ρ, F_ρ) of the Weyl group of $SL(n, \mathbb{R})$ whose restriction to any $W(SL(3))$ does not contain the sign representation, we construct a finite-dimensional representation (μ_ρ, E_{μ_ρ}) of $SO(n, \mathbb{R})$ with the following properties:

- for each root α of $SL(n)$, the subgroup $K^\alpha \simeq SO(2)$ only acts with characters zero, plus or minus one, plus or minus two (i.e. μ_ρ is *petite*)
- μ_ρ contains a M-fixed vector (i.e. μ_ρ is *spherical*)
- ρ is a submodule of the representation of $W(SL(n))$ on the space of M-fixed vectors of μ_ρ .

When ρ varies in the set of Weyl group representations that do not contain any copy of the sign representation of $W(SL(3))$, this construction produces *all* the petite spherical representations of $SO(n)$.

9.1 Assumptions on ρ

In this section we motivate the assumptions on the Weyl group representation ρ.

Let μ be a petite representation of $K = SO(n, \mathbb{R})$, and let H be a subgroup of K isomorphic to $SO(3)$. The restriction of μ to L must be petite, hence it can only contain the irreducible representations $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_2$ of $SO(3)$.[1]

We look at the restriction of μ to $M' \cap SO(3)$, and ask whether it contains the sign representation of $\mathcal{S}_3$.[2] The answer is negative, because the restriction of $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_2$ to $M' \cap SO(3)$ does not contain the sign representation.[3]

Hence, if we want a Weyl group representation ρ to extend to a petite (and of course spherical) representation of K we *must* assume that the restriction of ρ to any $W(SO(3))$ does not contain any copy of the sign representation.

Next, we interpret these assumptions on ρ in terms of partitions. We identify the Weyl group of $SL(n, \mathbb{R})$ with the symmetric group of n letters, and each irreducible representation of W with a Specht module S^λ, for some partition λ of n.[4] It follows from the branching rule that the representation S^λ of $\mathcal{S}_n$ contains the sign representation $(1,1,1)$ of $\mathcal{S}_3$ if and only if the partition λ has at least three rows.[5] Hence, our assumptions on ρ can be rephrased by saying that

$$\boxed{\rho \simeq S^\lambda, \text{ for some } \lambda \vdash n \text{ with at most two rows}}\,.$$

[1]For all $N \geq 0$, the representation $\mathscr{H}_N$ of $SO(3)$ contains the character N of $SO(2)$. So $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_2$ are the only petite representations of $SO(3)$. Please, refer to chapter 7 for details and notations.

[2]We can of course regard any representation of $W(SO(3)) = \mathcal{S}_3$ as a representation of $M' \cap SO(3)$.

[3]$M' \cap SO(3)$ has a total of 5 inequivalent irreducible representations: $U = trivial$, $U' = sign$ and V, which are the three irreducible representations of the Weyl group $\mathcal{S}_3$, and two additional representations, both 3-dimensional, that we will denote by ν_1 and ν_2. They can be realized as the two irreducible summands of the (six-dimensional) reducible representation $\nu = Ind_{M \cap SO(3)}^{M' \cap SO(3)} \delta_1$, with

$$\delta_1 \colon M \to \{\pm 1\},\, m = diag(m_1, m_2, m_3) \mapsto m_1.$$

In particular, we denote by ν_1 the irreducible summand of ν on which the element $\sigma_{1,2} = exp(\frac{\pi}{2} Z_{\varepsilon_1 - \varepsilon_2})$ acts with eigenvalues $\{1, \pm i\}$ and by ν_2 the irreducible summand of ν on which $\sigma_{1,2}$ acts with eigenvalues $\{-1, \pm i\}$. Then

- $\mathscr{H}_0 \downarrow_{M' \cap SO(3)} = U$
- $\mathscr{H}_1 \downarrow_{M' \cap SO(3)} = \nu_1$
- $\mathscr{H}_2 \downarrow_{M' \cap SO(3)} = V \oplus \nu_2$.

The sign representation U^1 makes its first appearance in $\mathscr{H}_3$.

[4]Please, refer to chapter 8 for a quick description of S^λ, and to [13] or [27] for more detailed analysis of the representation theory of the symmetric group.

[5]Again, refer to chapter 8 for details.

Equivalently, $\rho \simeq S^{(n-k,k)}$, for some $k = 0, \dots, \left[\frac{n}{2}\right]$.

Finally, we give some notations.

Notations. *We denote by Δ the set of restricted roots of $SL(n\,\mathbb{R})$, and by Δ^+ a choice of positive roots. For each $\alpha \in \Delta^+$, we let E_α be an element of $(\mathfrak{g}_0)_\alpha$ satisfying the normalizing condition*

$$B(E_\alpha, \vartheta(E_\alpha)) = \frac{-2}{\|\alpha\|^2}.$$

We also take

$$Z_\alpha = E_\alpha + \vartheta(E_\alpha)$$

$$\sigma_\alpha = \exp\left(\tfrac{\pi}{2} Z_\alpha\right)$$

$$m_\alpha = \exp\left(\pi Z_\alpha\right) = \sigma_\alpha^2.$$

We recall that Z_α is an element of $\mathfrak{k}_0 = \mathrm{Lie}(K)$, m_α is an element of M and σ_α is a representative in M' for the root reflection s_α.

9.2 Sketch of the construction

As anticipated in the introduction, the main result of this chapter is an algorithm that extends a certain class of representations of the Weyl group $W(SL(n))$ to petite spherical representations of $K = SO(n)$. This algorithm actually produces *all* the petite spherical representations of K.

Because the construction is somehow complicated (although very natural), we provide a sketch that the reader can keep in mind while reading through the details.

- **Input:** A representation ρ of $W = \mathcal{S}_n$ that satisfies a very natural assumption, i.e. partition of n in at most two parts.

- **Output:** A "not too large" representation $\tilde{\rho'}$ of K that extends ρ.

  More precisely, a petite spherical representation $(\tilde{\rho'}, F_{\tilde{\rho'}})$ of K such that the representation of W on $(F_{\tilde{\rho'}})^M$ contains ρ.

- **Value:** On the K-type $\tilde{\rho'}$, the intertwining operator for a spherical principal series can be computed using only the Weyl group representation ρ.

- **Basic idea:** As a Lie algebra representation, the differential of a petite spherical representation of K is generated by the M-fixed vectors (through an iterated application of the Z_αs).

- **Main Steps.** They can be summarized as follows:

  1. We regard (ρ, F_ρ) as a representation of M', and enlarge it by adding to F_ρ the linear span of all the formal strings

     $$S = Z_{\alpha_1} \dots Z_{\alpha_r} v$$

     where $\alpha_1, \dots, \alpha_r \in \Delta^+$ are *mutually orthogonal* positive roots and $v \in F_\rho$ is a simultaneous (-1)-eigenvector for the root reflections $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_r}$. We define an action of M' on the space

     $$F_{\rho'} \equiv F_\rho \oplus \mathrm{Span}(\,\textit{strings}\,)$$

     by:

     $$\sigma \cdot (Z_{\alpha_1} \dots Z_{\alpha_r} v) = (\mathrm{Ad}(\sigma)(Z_{\alpha_1})) \cdots (\mathrm{Ad}(\sigma)(Z_{\alpha_r}))\,(\sigma \cdot v).$$

     The result of this first extension is a much larger[6] representation of M', that we denote by ρ'.

[6]Still finite-dimensional.

Remark: ρ' might contain the sign representation of $M' \cap SO(3)$, so it cannot yet be extended to a petite representation of K.

2. We take a quotient of $F_{\rho'}$ modulo some suitable equivalence relations, so to remove any copy of the sign representation of $M' \cap SO(3)$ from $F_{\rho'}$. The result is a representation $\left(\tilde{\rho}', F_{\tilde{\rho}'} = \frac{F_{\rho'}}{R}\right)$ of M', that can be extended to a petite K-type.

3. We define a representation of Lie(K) on space $F_{\tilde{\rho}'}$ such that:

   ◇ as a Lie algebra representation, $\tilde{\rho}'$ is generated by F_ρ. In particular, we require that

$$[\,Z_\beta Z_{\alpha_1} \dots Z_{\alpha_r} v] = Z_\beta \cdot [Z_{\alpha_1} \dots Z_{\alpha_r} v] = \dots = (Z_\beta Z_{\alpha_1} \dots Z_{\alpha_r}) \cdot [v\,]$$

   ◇ the representation of Lie(K) on $\tilde{\rho}'$ lifts to a petite representation of K. In particular, we require that for each positive root β, the eigenvalues of $\tilde{\rho}'(Z_\beta)$ lie in the set $\{0, \pm i, \pm 2i\}$. To be more precise, we want Z_β to act by:

   $\boxed{0}$ on the $(+1)$-eigenspace of σ_β

   $\boxed{+i}$ on the $(+i)$-eigenspace of σ_β

   $\boxed{-i}$ on the $(-i)$-eigenspace of σ_β

   $\boxed{\pm 2i}$ on the (-1)-eigenspace of σ_β.

Remark These assumptions basically determine the entire action of Z_β on the quotient space $\tilde{F}'(\,\rho) = F_\rho \oplus \frac{\text{Span}\{\text{strings}\}}{R}$.

4. We lift the representation $\tilde{\rho}'$ of Lie(K) to a representation of K, and conclude the construction.

9.3 Step 1: the representation $F_{\rho'}$ of M'

Let (ρ, F_ρ) be an irreducible representation of $W(SL(n))$ whose restriction to any $W(SL(3))$ does *not* contain any copy of the sign representation of $W(SL(3))$.[7] We can of course regard ρ as a representation of M'.

The first step is to extend F_ρ to a much larger representation of M', that we will denote by $F_{\rho'}$. This extension is obtained by *first* adding to F_ρ the formal linear span of all the strings of the form

$$(a_1 Z_{\alpha_1})(a_2 Z_{\alpha_2}) \ldots (a_k Z_{\alpha_k}) v$$

with $k \geqslant 1$ and

$a_1, a_2, \ldots, a_k$: complex numbers

$\alpha_1, \alpha_2, \ldots, \alpha_k$: mutually orthogonal positive roots

$v \in F_\rho$: a simultaneous (-1)-eigenvector for the root reflections $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots, \sigma_{\alpha_k}$,

and *then* imposing on the vector space

$$F_\rho \oplus \text{Span}\{(a_1 Z_{\alpha_1})(a_2 Z_{\alpha_2}) \ldots (a_k Z_{\alpha_k}) v\}$$

the following "linearity and commutativity equivalence relations":

- $(a_1 Z_{\alpha_1})(a_2 Z_{\alpha_2}) \ldots (a_k Z_{\alpha_k}) v = (a_1 a_2 \ldots a_k) Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_k} v$
- $Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_k}(av + a'v') = a\,(Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_k} v) + a'\,(Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_k} v')$
- $Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_k} v = Z_{\alpha_{\tau(1)}} Z_{\alpha_{\tau(2)}} \ldots Z_{\alpha_{\tau(k)}} v$

[7] At this stage, our assumption on ρ is not critical.

for each choice of $k \geqslant 1,\ \tau \in \mathcal{S}_k$, and $a,\, a',\, a_1,\, a_2, \ldots,\, a_k \in \mathbb{C}$,

for all $\alpha_1,\, \alpha_2, \ldots,\, \alpha_k \in \Delta^+$, mutually orthogonal,

and for every $v,\, v' \in F_\rho$, simultaneous (-1)-eigenvectors of $\sigma_{\alpha_1},\, \sigma_{\alpha_2}, \ldots,\, \sigma_{\alpha_k}$.

We denote the quotient space by $F_{\rho'}$:

$$\boxed{F_{\rho'} = \frac{F_\rho \oplus \operatorname{Span}\{(a_1 Z_{\alpha_1})(a_2 Z_{\alpha_2}) \ldots (a_k Z_{\alpha_k})v\}}{\text{linearity relations \& commutativity relations}}}\,.$$

Because the relations are only imposed on strings of length $k \geq 1$, they do not involve elements of the original space. Therefore, we can regard the vector space $F_{\rho'}$ as an extension of F_ρ. The next step is to define a representation ρ' of M' on $F_{\rho'}$, which extends the original representation ρ (of M' on F_ρ).

For each σ in M', we let σ act on elements of F_ρ via ρ, and on strings (of orthogonal roots) by:

$$\boxed{\rho'(\sigma) \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) \equiv (\operatorname{Ad}(\sigma) Z_{\alpha_1})(\operatorname{Ad}(\sigma) Z_{\alpha_2}) \cdots (\operatorname{Ad}(\sigma) Z_{\alpha_k})(\sigma \cdot v)}\,.$$

To show that this action is well defined we must check that

(i) for all strings, and all σ in M', $\rho'(\sigma) \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v)$ is again a well defined string;

(ii) the linearity and commutativity equivalence relations are preserved

(iii) for all strings, and all $\sigma_1,\, \sigma_2$ in M'

$$(\,\rho'(\sigma_1\, \sigma_2)) \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) = \rho'(\sigma_1) \cdot (\,\rho'(\sigma_2) \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v))\,.$$

Proof. (i) Fix an element σ of M'. For all $j = 1 \ldots k$, there exists a scalar $c_j = \pm 1$ such that $\operatorname{Ad}(\sigma)(Z_{\alpha_j}) = c_j Z_{\beta_j}$, and since

$$(\mathrm{Ad}(\sigma)Z_{\alpha_1})(\mathrm{Ad}(\sigma)Z_{\alpha_2})\cdots(\mathrm{Ad}(\sigma)Z_{\alpha_k})(\sigma\cdot v) = (c_1 Z_{\beta_1})(c_2 Z_{\beta_2})\cdots(c_k Z_{\beta_k})(\sigma\cdot v) =$$

$$= (c_1\, c_2 \ldots c_k) Z_{\beta_1} Z_{\beta_2}\cdots Z_{\beta_k}(\sigma\cdot v)$$

we just need to check that the roots β_j s are mutually orthogonal and that the vector $(\sigma\cdot v)$ is a simultaneous (-1)-eigenvector for $\sigma_{\beta_1}, \sigma_{\beta_2}, \ldots, \sigma_{\beta_k}$. We denote by w the projection of σ in the Weyl group, then

$$\langle \beta_i,\, \beta_j\rangle = \langle w(\alpha_i),\, w(\alpha_j)\rangle = \langle \alpha_i,\, \alpha_j\rangle = 0.$$

and,

$$\sigma_{\beta_j}\cdot(\sigma\cdot v) = \sigma\cdot(\sigma^{-1}\sigma_{\beta_j}\sigma)\cdot v = \sigma\cdot \underbrace{(\sigma_{\alpha_j}^{c_j}\cdot v)}_{-v} = -\sigma\cdot v.$$

Hence both conditions are satisfied.

(ii) It is easy to show that

- $\sigma\cdot((a_1 Z_{\alpha_1})(a_2 Z_{\alpha_2})\cdots(a_k Z_{\alpha_k})v) = (a_1\, a_2 \ldots a_k)\,\sigma\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k} v)$
- $\sigma\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k}(cv+dw)) = c\,(\sigma\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k} v)) + d\,(\sigma\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k} w))$
- $\sigma\cdot\left(Z_{\alpha_{\tau(1)}} Z_{\alpha_{\tau(2)}}\cdots Z_{\alpha_{\tau(k)}} v\right) = \sigma\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k} v)$

for every string, for every permutation τ and for all σ in M'. We omit the details.

(iii) Because Ad is a representation of M', we can write:

$$(\,\rho'(\sigma_1\,\sigma_2))\cdot(Z_{\alpha_1} Z_{\alpha_2}\cdots Z_{\alpha_k} v) =$$

$$= (\mathrm{Ad}(\sigma_1\,\sigma_2)(Z_{\alpha_1}))\,(\mathrm{Ad}(\sigma_1\,\sigma_2)(Z_{\alpha_2}))\cdots(\mathrm{Ad}(\sigma_1\,\sigma_2)(Z_{\alpha_k}))\,(\sigma_1\,\sigma_2\cdot v) =$$

$$= (\mathrm{Ad}(\sigma_1)\,(\mathrm{Ad}(\sigma_2)(Z_{\alpha_1})))\,(\mathrm{Ad}(\sigma_1)\,(\mathrm{Ad}(\sigma_2)(Z_{\alpha_2})))\cdots$$

$$\cdots(\mathrm{Ad}(\sigma_1)\,(\mathrm{Ad}(\sigma_2)(Z_{\alpha_1})))\,(\sigma_1\,\sigma_2\cdot v) =$$

$$= \rho'(\sigma_1)\cdot((\mathrm{Ad}(\sigma_2)(Z_{\alpha_1}))\,(\mathrm{Ad}(\sigma_2)(Z_{\alpha_2}))\cdots(\mathrm{Ad}(\sigma_2)(Z_{\alpha_k}))\,(\sigma_1\cdot(\sigma_2\cdot v))) =$$

$$= \rho'(\sigma_1)\cdot(\,\rho'(\sigma_2)\cdot(Z_{\alpha_1}Z_{\alpha_2}\cdots Z_{\alpha_k}v))$$

for all σ_1, σ_2 in M', and all strings. The proof is now complete. □

We conclude that:

Proposition 4. *For every irreducible Weyl group representation* $(\,\rho, F_\rho)$*, the representation* $(\,\rho', F_{\rho'})$ *of* M' *is well defined.*

Remark 5. $F_{\rho'}$ *is finite-dimensional.*

Proof. Because $F_{\rho'}$ is a quotient of $F_\rho \oplus \mathrm{Span}\{(a_1Z_{\alpha_1})(a_2Z_{\alpha_2})\dots(a_kZ_{\alpha_k})v\}$, it is enough to prove that there are only finitely many strings. This is obvious: there are finitely many strings of each given length and, since the roots in each string are required to be mutually orthogonal, the length of a string cannot exceed the maximum number of orthogonal roots in $\mathfrak{sl}(n)$, which is equal to $\left[\frac{n}{2}\right]$. □

When ρ is a partition of n in at most two parts, it is possible to get a better estimate of the maximum length of a string appearing in $F_{\rho'}$.

Remark 6. *If* $\rho = (n)$*, then* $F_{\rho'} = F_\rho$ *and there are no strings at all.*
If $\rho = (n-s, s)$*, for some* $1 \le s \le \left[\frac{n}{2}\right]$*, then* $F_{\rho'}$ *only contains strings of length less than or equal to* s*.*

Proof. The strings allowed in $F_{\rho'}$ are of the form $Z_{\alpha_1}Z_{\alpha_2}\cdots Z_{\alpha_k}v$, with $\alpha_1, \alpha_2, \dots, \alpha_k$ mutually orthogonal positive roots, and $v \in F_\rho$ a simultaneous (-1)-eigenvector for the root reflections $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \dots, \sigma_{\alpha_k}$.

Let us look at the possible (-1)-eigenvectors of a transposition $(j, j+1)$ in F_ρ. Pick a basis $\{\underline{e}_T\}$ of F_ρ that consists of standard polytabloids.[8] Then

- If j and $j+1$ lie in the same column of T, then $(j, j+1) \cdot \underline{e}_T = -\underline{e}_T$.
- If j and $j+1$ do not lie in the same column nor in the same row of T, then $(j, j+1) \cdot \underline{e}_T$ is still a standard polytabloid. We notice that j and $j+1$ do not lie in the same column nor in the same row of $(j, j+1) \cdot \underline{e}_T$, and that the difference $\underline{e}_T - (j, j+1) \cdot \underline{e}_T$ is a (-1)-eigenvector of $(j, j+1)$.
- If j and $j+1$ lie in the same row of T, then $(j, j+1) \cdot \underline{e}_T$ is not a multiple of a standard polytabloid. The difference $\underline{e}_T - (j, j+1) \cdot \underline{e}_T$ is a standard polytabloid $\underline{e}_S$ and a (-1)-eigenvector of $(j, j+1)$. We notice that j and $j+1$ lie in the same column of S.

Therefore, if we order the basis vectors by putting *first* the polytabloids in which j and $j+1$ lie in the same column, *then* the ones in which j and $j+1$ are not in the same row, nor in the same column, and *finally* the ones in which j and $j+1$ belong to the same row, we obtain for $\rho(j, j+1)$ a matrix of the form:

$$\rho(j, j+1) \rightsquigarrow \left(\begin{array}{c|c|c} -I & O & A \\ \hline O & B & O \\ \hline O & O & I \end{array}\right)$$

in which of course I and O are respectively the identity and the zero matrix, and B

[8]Please, refer to section 8.3 for terminology.

is given by

$$B = \left(\begin{array}{cc|cc|c|cc} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{array}\right).$$

It follows that the (-1)-eigenvectors for $\rho(j, j+1)$ are either of the form $\underline{e}_T$ with j and $j+1$ in the same column of T, or of the form $\underline{e}_T - \underline{e}_{T'}$, with $\underline{e}_{T'} = (j, j+1) \cdot \underline{e}_T$ and j, $j+1$ not in the same column nor in the same row of either T or T'.[9] This argument shows that *any* (-1)-eigenvector of $(j, j+1)$ can be expressed as linear combination of standard polytabloids in which j and $j+1$ sit in different rows.

We now go back to the problem of determining the simultaneous (-1)-eigenvectors in F_ρ for k orthogonal root reflections $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \dots, \sigma_{\alpha_k}$. Without loss of generality, we can assume that σ_{α_t} is the transposition $(2t-1, 2t)$, for all $t = 1 \dots k$.[10]
Any simultaneous (-1)-eigenvector for $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \dots, \sigma_{\alpha_k}$ lives in the subspace of F_ρ generated by the polytabloids in which, for all $t = 1 \dots k$, $2t-1$ and $2t$ sit in different rows. We notice that

- $\diamond$ because $\rho = (2n-s, s)$, every polytabloid has exactly two rows
- $\diamond$ if a polytabloid with two rows satisfies the above condition, then each of the two rows must contain exactly one of the two elements $2t-1$, $2t$ (for all $t = 1 \dots k$).

[9] This can be easily shown by computing the kernel of

$$\rho(j, j+1) + \mathbb{I}d \rightsquigarrow \left(\begin{array}{c|c|c} O & O & A \\ \hline O & B+I & O \\ \hline O & O & 2I \end{array}\right).$$

[10] Indeed, the k orthogonal root-reflections $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \dots, \sigma_{\alpha_k}$ are conjugate in the Weyl group to the k disjoint transpositions $(1, 2)$, $(3, 4)$, $\dots$, $(2k-1, 2k)$.

Hence the polytabloid must have at least k columns (of length two)

- ◇ because $\rho = (2n-s, s)$, every polytabloid has exactly s columns (of length two).

We deduce that if $k > s$ then there is no set of k orthogonal roots admitting a simultaneous (-1)-eigenvector. As a consequence, $F'_{(2n-s,s)}$ does not contain *any* string of length strictly bigger than s. □

9.4 Step 2: A quotient of $F_{\rho'}$

So far we have constructed an extension of the Weyl group representation ρ to a representation ρ' of M'. The aim of the this section is to get rid of the copies of the sign representation in $\rho' \mid_{M' \cap SO(3)}$. This step is necessary, if we want to be able to extend ρ' to a *petite* representation of K.
For clarity, we divide the argument in several parts:

1. We study the restriction to $M' \cap SO(3)$ of an arbitrary representation of M'
2. We study the restriction to $M' \cap SO(3)$ of $F_{\rho'}$
3. We get rid of all the copies of the sign representation in $\rho' \mid_{M' \cap SO(3)}$, by taking the quotient of $F_{\rho'}$ with some suitable equivalence relations.

9.4.1 The restriction to $M' \cap SO(3)$ of an arbitrary representation of M'

Let α, β, γ be three positive restricted roots of $SL(n)$ such that

$$\{\pm\alpha, \pm\beta, \pm\gamma\}$$

is a root system of type $\mathcal{A}_2$.[11] By construction, the elements Z_α, Z_β and Z_γ of $\mathfrak{so}(n)$ generate an $\mathfrak{so}(3)$. We denote the subgroup $M' \cap SO(3)$ of M' by $M'_{\alpha,\beta,\gamma}$.
In this subsection we study the restriction to $M'_{\alpha,\beta,\gamma}$ of an arbitrary representation of M'. In particular, for each (possibly reducible) representation (η, F_η) of M' and every element x of F_η, we describe the smallest $M'_{\alpha,\beta,\gamma}$-invariant subspace V_x of F_η containing x, and the action of $M'_{\alpha,\beta,\gamma}$ on this subspace.

Without loss of generality, we can assume that one of the following possibilities holds:

1. $\sigma_\alpha \cdot x = \sigma_\beta \cdot x = +x$
2. $\sigma_\alpha \cdot x = \sigma_\beta \cdot x = -x$
3. $\sigma_\alpha \cdot x = +x, \quad \sigma_\beta \cdot x \neq +x, \quad \sigma_\beta^2 \cdot x = +x$
4. $\sigma_\alpha \cdot x = -x, \quad \sigma_\beta \cdot x \neq -x, \quad \sigma_\beta^2 \cdot x = +x$
5. $\sigma_\alpha^2 \cdot x = \sigma_\beta^2 \cdot x = +x, \quad x$ not a (± 1)-eigenvector of σ_α, σ_β, or σ_γ
6. $\sigma_\alpha \cdot x = +x, \quad \sigma_\beta^2 \cdot x = -x$
7. $\sigma_\alpha \cdot x = -x, \quad \sigma_\beta^2 \cdot x = -x$
8. $\sigma_\alpha^2 \cdot x = +x, \quad \sigma_\beta^2 \cdot x = -x, \quad x$ not a (± 1)-eigenvector of σ_α
9. none of the above, i.e. x not stable under the action of $M_{\alpha,\beta,\gamma}$.

We discuss each case separately.

[11] This condition basically means that α, β and γ are mutually not orthogonal, and that γ is equal to $\pm(\alpha+\beta)$ or $\pm(\alpha-\beta)$. In other words, if we write

$$\alpha = \varepsilon_{i_1} - \varepsilon_{j_1} \qquad \beta = \varepsilon_{i_2} - \varepsilon_{j_2} \qquad \gamma = \varepsilon_{i_3} - \varepsilon_{j_3}$$

then the set $\{i_1, i_2, i_3, j_1, j_2, j_3\}$ has cardinality 3.

The first two cases are trivial: $V_x = \mathbb{C}\,x$ and $M'_{\alpha,\beta,\gamma}$ acts on it as the trivial representation in case (1) and the *sign* representation in case (2).

In case (3), $V_x = < x,\, \sigma_\beta \cdot x,\, \sigma_\gamma \cdot x >_{\mathbb{C}}$, and it is either two or three-dimensional. We need to distinguish between these two possibilities.
If $\dim(V_x) = 2$, then x and $\sigma_\beta \cdot x$ form a basis of V_x.[12] Because $\sigma_\beta^2 \cdot x = +x$, σ_β acts on V_x with eigenvalues ± 1. We write $\sigma_\gamma \cdot x = a\,x + b\,\sigma_\beta \cdot x$, and observe that w.r.t. the basis $\{x,\, \sigma_\beta \cdot x\}$ of V_x, the root reflections σ_α and σ_γ act by:

$$\sigma_\alpha \rightsquigarrow \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \qquad \sigma_\gamma \rightsquigarrow \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

Since σ_α, σ_β and σ_γ are all conjugate, they must act with the same eigenvalues. We deduce that $a = b = -1$, and the relation

$$\sigma_\alpha \cdot x \,+\, \sigma_\beta \cdot x \,+\, \sigma_\gamma \cdot x = 0$$

holds. Therefore, V_x is a copy of the standard representation of $M'_{\alpha,\beta,\gamma}$.
If $\dim(V_x) = 3$, then x, $\sigma_\beta \cdot x$ and $\sigma_\gamma \cdot x$ are linearly independent. We can decompose V_x as the sum of two irreducible $M'_{\alpha,\beta,\gamma}$-invariant subspaces:

$$V_x = \mathbb{C}(x + \sigma_\beta \cdot x + \sigma_\gamma \cdot x) \oplus < x - \sigma_\beta \cdot x,\, x - \sigma_\gamma \cdot x >_{\mathbb{C}}$$

that are respectively a copy of the trivial representation and a copy of the standard representation of $M'_{\alpha,\beta,\gamma}$.

Case (4) is similar: V_x is again equal to $< x,\, \sigma_\beta \cdot x,\, \sigma_\gamma \cdot x >_{\mathbb{C}}$, and it is either two

[12] Notice that $\sigma_\beta \cdot x \neq x$ by assumption, and that $\sigma_\beta \cdot x \neq -x$ or $\mathbb{C}\,x$ would be a one-dimensional representation of $M'_{\alpha,\beta,\gamma}$ different from both the trivial and the sign representation. Because $\sigma_\beta^2 \cdot x = +x$, it cannot be any other multiple of x.

or three-dimensional.

If $\dim(V_x) = 2$, we choose the basis $\{x,\, \sigma_\beta \cdot x\}$ of V_x, and we write $\sigma_\gamma \cdot x = a\,x + b\,\sigma_\beta \cdot x$. It is easy to see that the root reflections σ_α, σ_β and σ_γ act by:

$$\sigma_\alpha \rightsquigarrow \begin{pmatrix} -1 & -a \\ 0 & -b \end{pmatrix} \qquad \sigma_\beta \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_\gamma \rightsquigarrow \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix}.$$

Since σ_α, σ_β and σ_γ are all conjugate, they must act with the same eigenvalues. We deduce that $a = 1$ and $b = -1$. Again, the relation

$$\sigma_\alpha \cdot x \,+\, \sigma_\beta \cdot x \,+\, \sigma_\gamma \cdot x = 0$$

holds, and V_x is a copy of the standard representation of $M'_{\alpha,\beta,\gamma}$.

If $\dim(V_x) = 3$, then x, $\sigma_\beta \cdot x$ and $\sigma_\gamma \cdot x$ are linearly independent. We can decompose V_x as the sum of two irreducible $M'_{\alpha,\beta,\gamma}$-invariant subspaces:

$$V_x = \mathbb{C}(x - \sigma_\beta \cdot x - \sigma_\gamma \cdot x) \oplus < x + \sigma_\beta \cdot x,\, x + \sigma_\gamma \cdot x >_{\mathbb{C}}$$

that are respectively a copy of the *sign* representation and a copy of the standard representation of $M'_{\alpha,\beta,\gamma}$.

In case (5), $V_x = \{x,\, \sigma_\alpha \cdot x,\, \sigma_\beta \cdot x,\, \sigma_\gamma \cdot x,\, (\sigma_\alpha \sigma_\beta) \cdot x,\, (\sigma_\alpha \sigma_\gamma) \cdot x\}$. We set $y = x + \sigma_\alpha \cdot x$ and $z = x - \sigma_\alpha \cdot x$, and we notice that V_x decomposes as the direct sum of two $M'_{\alpha,\beta,\gamma}$-invariant subspaces:

$$V_x = \underbrace{< y,\, \sigma_\beta \cdot y,\, \sigma_\gamma \cdot y >_{\mathbb{C}}}_{V_x^+} \oplus \underbrace{< z,\, \sigma_\beta \cdot z,\, \sigma_\gamma \cdot z >_{\mathbb{C}}}_{V_x^-}.$$

By the previous arguments, V_x^+ is either a copy of the trivial representation, or a copy of the standard representation or a sum of the two. Similarly, V_x^- is either a copy of the *sign* representation, or a copy of the standard representation or a sum of the two.

Cases (6) and (7) are similar. In both cases $V_x = < x,\, \sigma_\beta \cdot x,\, \sigma_\gamma \cdot x >_{\mathbb{C}}$. This space is always three-dimensional, and it is an irreducible representation of $M'_{\alpha,\beta,\gamma}$ isomorphic to ν_1 in case (6), and to ν_2 in case (7).[13]

In case (8), $V_x = \{x,\, \sigma_\alpha \cdot x,\, \sigma_\beta \cdot x,\, \sigma_\gamma \cdot x,\, (\sigma_\alpha\sigma_\beta)\cdot x,\, (\sigma_\alpha\sigma_\gamma)\cdot x\}$. We set $y = x + \sigma_\alpha \cdot x$ and $z = x - \sigma_\alpha \cdot x$, and we notice that V_x decomposes as the direct sum of two $M'_{\alpha,\beta,\gamma}$-invariant subspaces:

$$V_x = \underbrace{< y,\, \sigma_\beta \cdot y,\, \sigma_\gamma \cdot y >_{\mathbb{C}}}_{V_x^+} \oplus \underbrace{< z,\, \sigma_\beta \cdot z,\, \sigma_\gamma \cdot z >_{\mathbb{C}}}_{V_x^-} .$$

By the previous arguments, V_x^+ is always a copy of the three-dimensional representation ν_1, and V_x^- is always a copy of the three-dimensional representation ν_2.

Case (9) can be reduced to the previous cases on the basis of the following observations: the elements σ_α^2, σ_β^2, σ_γ^2 of $M_{\alpha,\beta,\gamma}$ commute, each of them has order two and their product equals the identity. Therefore, for any representation η of M' we have

$$\eta(\sigma_\alpha^2) = \eta(\sigma_\beta^2)\,\eta(\sigma_\gamma^2) = \eta(\sigma_\gamma^2)\,\eta(\sigma_\beta^2) = \eta(\sigma_\alpha^2)^{-1}$$

$$\eta(\sigma_\beta^2) = \eta(\sigma_\alpha^2)\,\eta(\sigma_\gamma^2) = \eta(\sigma_\gamma^2)\,\eta(\sigma_\alpha^2) = \eta(\sigma_\beta^2)^{-1}$$

$$\eta(\sigma_\gamma^2) = \eta(\sigma_\beta^2)\,\eta(\sigma_\alpha^2) = \eta(\sigma_\alpha^2)\,\eta(\sigma_\beta^2) = \eta(\sigma_\gamma^2)^{-1}.$$

It follows that each of the vectors

[13] Recall that ν_1 and ν_2 are the two irreducible summands of $Ind_M^{M'}\delta_1$. A root reflection acts on ν_1 with eigenvalues 1, $\pm i$ and on ν_1 with eigenvalues -1, $\pm i$.

$$x_0 = x + \sigma_\alpha^2 \cdot x + \sigma_\beta^2 \cdot x + \sigma_\gamma^2 \cdot x$$

$$x_1 = x + \sigma_\alpha^2 \cdot x - \sigma_\beta^2 \cdot x - \sigma_\gamma^2 \cdot x$$

$$x_2 = x - \sigma_\alpha^2 \cdot x + \sigma_\beta^2 \cdot x - \sigma_\gamma^2 \cdot x$$

$$x_3 = x - \sigma_\alpha^2 \cdot x - \sigma_\beta^2 \cdot x + \sigma_\gamma^2 \cdot x$$

spans a one-dimensional representation of $M_{\alpha,\beta,\gamma}$.[14] Hence, we can study the space V_{x_j} by means of the same techniques used in (1)-(8).

This concludes our analysis of the restriction of an arbitrary representation (η, F_η) of M' to $M'_{\alpha,\beta,\gamma}$. In the next subsection, we set $F_\eta = F_\rho \oplus \mathrm{Span}\{strings\}$.

9.4.2 The restriction of $F_{\rho'}$ to $M'_{\alpha,\beta,\gamma}$

With abuse of notations we identify a string with its equivalence class modulo the commutativity and linearity relations. In particular, we think of $F_{\rho'}$ as the vector space generated by F_ρ and by all the strings of the form

$$Z_{\alpha_1} \dots Z_{\alpha_r} v$$

with $\alpha_1 \dots \alpha_r$ mutually orthogonal positive roots , and v an element of F_ρ satisfying $\sigma_{\alpha_1} \cdot v = \cdots = \sigma_{\alpha_r} \cdot v = -v$.

The purpose of this section is the study of the action of $M'_{\alpha,\beta,\gamma}$ on $F_{\rho'}$. Precisely, we ask the following questions:

1. When is $\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ stable under the action of $M_{\alpha,\beta,\gamma}$?

2. When is $\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ a copy of the trivial representation of $M'_{\alpha,\beta,\gamma}$?

[14] The representations $\{\mathbb{C}\, x_j\}_{j=0\dots3}$ are isomorphic to δ_0, δ_1, δ_2, δ_3.

3. When is $\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ a copy of the sign representation of $M'_{\alpha,\beta,\gamma}$?

The answer to this question is the same for all the split groups with one root lenght, so we formulate it in general terms:

Proposition 5. *Let G be a semi-simple split group whose root system admits only one root-length. Fix a Weyl group representation ρ that does not contain any copy of the sign representation of $W(SL(3))$, and form strings of orthogonal roots $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$, just like in the case of $SL(n)$. Define an action of M' on the linear span of the strings by*

$$\sigma \cdot (Z_{\alpha_1} \dots Z_{\alpha_r} v) = (\mathrm{Ad}(\sigma) Z_{\alpha_1}) \cdots (\mathrm{Ad}(\sigma) Z_{\alpha_r})(\sigma \cdot v).$$

For each triple of positive roots α, β, γ forming an $\mathcal{A}_2$, consider the subgroup $M'_{\alpha,\beta,\gamma}$ of M' generated by σ_α, σ_β and σ_γ. Then

1. *$\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ is always stable under the action of $M_{\alpha,\beta,\gamma}$, hence it defines a one-dimensional irreducible representation of $M_{\alpha,\beta,\gamma}$.*

2. *$\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ can never be a copy of the sign representation of $M'_{\alpha,\beta,\gamma}$*

3. *$\mathbb{C}(Z_{\alpha_1} \dots Z_{\alpha_r} v)$ is a copy of the trivial representation of $M'_{\alpha,\beta,\gamma}$ if and only if the roots α, β, γ are all orthogonal to $\alpha_1 \dots \alpha_r$, and the following condition holds:*

$$\sigma_{\alpha_1} \cdot v = \cdots = \sigma_{\alpha_r} \cdot v = +v.$$

These results follow easily from some properties of the split groups whose root systems have one root-length. We describe such properties in the next few lemmas.

Lemma 8. *Let G be a split group whose root system Δ admits only one root-length.*

Then for all α, β in Δ, we have:

$$\mathrm{Ad}(\sigma_\beta^2)(Z_\alpha) = \begin{cases} +Z_\alpha & \text{if } \alpha = \beta \ \textit{ or } \ \alpha \perp \beta \\ -Z_\alpha & \text{otherwise.} \end{cases}$$

Proof. For any split group, and all roots α, β in Δ, the element $m_\alpha = \sigma_\alpha^2$ acts on Z_β by the scalar ± 1. Precisely[15]

$$\mathrm{Ad}(\sigma_\beta^2)(Z_\alpha) = (-1)^{\langle \alpha, \check{\beta} \rangle} Z_\alpha.$$

Let us focus on the case in which the root system has one root-length. The Cartan integer $\langle \alpha, \check{\beta} \rangle$ takes the value:

$$\begin{array}{ll} \pm 2 & \text{if } \alpha = \pm \beta \\ 0 & \text{if } \alpha \perp \beta \\ \pm 1 & \text{otherwise.} \end{array}$$

This makes $\mathrm{Ad}(\sigma_\beta^2)(Z_\alpha) = \begin{cases} +Z_\alpha & \text{if } \alpha = \beta \text{ or } \alpha \perp \beta \\ -Z_\alpha & \text{otherwise.} \end{cases}$ □

Corollary 9. *For every string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ and for every positive root β*

$$\sigma_\beta^2 \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) = (-1)^{\#\{j \colon [Z_\beta, Z_{\alpha_j}] \neq 0\}} (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v).$$

Proof. It follows from the previous lemma and from the fact that $\sigma_\beta^2 \cdot v = v$, because ρ is a Weyl group representation. □

Remark 7. *This result does not extend to the case in which ρ is a representation of M', but not a Weyl group representation.*

Remark 8. *Corollary 9 shows part (1) of proposition 5.*

[15] Please, refer to [30], lemma 4.3.19 for a proof. In this lemma the result is stated for G quasi-split (i.e. with $\mathfrak{m}_0$ abelian) and β real, but of course both assumption are satisfied when G is split.

Lemma 9. *Let Φ be a root system with one root-length.*[16] *For all α, β in Φ, $s_\beta(\alpha)$ cannot be orthogonal to α.*

Proof. Set $\gamma = s_\beta(\alpha) = \alpha - 2\,\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\,\beta$, and suppose by contradiction that γ is orthogonal to α. Then

$$\|\gamma\|^2 = \langle\, \gamma,\, \alpha - 2\,\tfrac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\,\beta\rangle = -2\,\tfrac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\,\langle\gamma,\,\beta\rangle \quad \Leftrightarrow$$

$$\Leftrightarrow \quad 2 = \left(-2\,\tfrac{\langle\beta,\alpha\rangle}{\|\beta\|^2}\right)\left(2\,\tfrac{\langle\gamma,\beta\rangle}{\|\gamma\|^2}\right) \Leftrightarrow$$

$$\Leftrightarrow \quad 2 = \left(-2\,\tfrac{\langle\beta,\alpha\rangle}{\|\beta\|^2}\right)\left(-2\,\tfrac{\langle\beta,\alpha\rangle}{\|\beta\|^2}\right) \Leftrightarrow \qquad (\circledast)$$

$$\Leftrightarrow \quad 2 = \left(2\,\tfrac{\langle\beta,\alpha\rangle}{\|\beta\|^2}\right)^2.$$

We reach a contradiction, because the quantity $\left(2\,\frac{\langle\beta,\alpha\rangle}{\|\beta\|^2}\right)$ is an integer.
In $(\circledast)$ we used the two equalities

$$\|\beta\|^2 = \|\gamma\|^2 (= \|\alpha\|^2) \qquad \langle\gamma,\,\beta\rangle = -\langle s_\beta(\alpha),\, s_\beta(\beta)\rangle = -\langle\alpha,\,\beta\rangle.$$

$\square$

Remark 9. *This result does not extend to the case in which Φ has two root-lengths. For instance, in type $\mathcal{B}_2$*

$$s_{\varepsilon_1}(\varepsilon_1 - \varepsilon_2) = -(\varepsilon_1 + \varepsilon_2) \perp (\varepsilon_1 - \varepsilon_2)$$

and in type $\mathcal{C}_2$

$$s_{2\varepsilon_1}(\varepsilon_1 - \varepsilon_2) = -(\varepsilon_1 + \varepsilon_2) \perp (\varepsilon_1 - \varepsilon_2).$$

[16] *Φ can be of type $\mathcal{A}_n$, $\mathcal{D}_n$, E_6, E_7, E_8.*

If Φ has two root-lengths, then we can conclude that $s_\beta(\alpha)$ is not orthogonal to α only when α and β have the same length.

Corollary 10. *Let $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ be a well defined string, i.e. let $\alpha_1, \dots, \alpha_r$ be mutually orthogonal positive roots and let v be an element of F_ρ satisfying*

$$\sigma_{\alpha_1} \cdot v = \cdots = \sigma_{\alpha_r} \cdot v = -\,v.$$

Let ν be any positive root. Then

(i) $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ *is a $(+1)$-eigenvector of σ_ν if and only if the following conditions are satisfied:*

- $\sigma_\nu \cdot v = +\,v$
- $\nu \perp \{\alpha_1, \dots, \alpha_r\}$.

(ii) $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ *is a (-1)-eigenvector of σ_ν if and only of the following conditions are satisfied:*

- $\sigma_\nu \cdot v = -\,v$
- *ν belongs to the set $\{\alpha_1, \dots, \alpha_r\}$, or it is orthogonal to it.*

Proof. *(i)*. Suppose that $\sigma_\nu \cdot (Z_{\alpha_1} \dots Z_{\alpha_r} v) = +(Z_{\alpha_1} \dots Z_{\alpha_r} v)$, i.e.

$$(\mathrm{Ad}(\sigma_\nu)(Z_{\alpha_1}))(\mathrm{Ad}(\sigma_\nu)(Z_{\alpha_2})) \cdots (\mathrm{Ad}(\sigma_\nu)(Z_{\alpha_r}))(\sigma_\nu \cdot v) = +\,Z_{\alpha_1} \dots Z_{\alpha_r} v.$$

We notice that for all $j = 1 \dots r$, $\mathrm{Ad}(\sigma_\nu)(Z_{\alpha_j}) = \pm Z_{s_\nu(\alpha_j)}$. So if $s_\nu(\alpha_j) \neq \pm\,\alpha_j$, then $s_\nu(\alpha_j)$ must belong to the set $\{\alpha_1, \dots, \alpha_r\} - \{\alpha_j\}$ and hence must be orthogonal to α_j. We reach a contradiction. Hence, either ν belongs to the set $\{\alpha_1, \dots, \alpha_r\}$, or it is orthogonal to it.

We exclude the first possibility, because it would make $Z_{\alpha_1} \dots Z_{\alpha_r} v$ a (-1)-eigenvector of σ_ν. Then ν must be orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$ (and v must necessarily be a (+1)-eigenvector of σ_ν). This concludes the proof of (i). The proof of (ii) is similar.

□

As a consequence, we immediately obtain that

Corollary 11. *For any string* $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$

(a) $\mathbb{C}(Z_{\alpha_1} \ldots Z_{\alpha_r} v)$ *can never be a copy of the sign representation of* $M'_{\alpha,\beta,\gamma}$.[17]

(b) $\mathbb{C}(Z_{\alpha_1} \ldots Z_{\alpha_r} v)$ *is a copy of the trivial representation of* $M'_{\alpha,\beta,\gamma}$ *if and only if the roots* α, β, γ *are all orthogonal to* $\alpha_1 \ldots \alpha_r$, *and the following condition holds:*

$$\sigma_{\alpha_1} \cdot v = \cdots = \sigma_{\alpha_r} \cdot v = +v.$$

Remark 10. *This corollary concludes the proof of proposition 5.*

9.4.3 Eliminating the sign from $F_{\rho'}$

A combined application of the results of the past subsections shows that the sign representation of $M'_{\alpha,\beta,\gamma}$ appears in $F_{\rho'}$ in one of the following forms:

(a) $\mathbb{C}(S - \sigma_\beta \cdot S - \sigma_\gamma \cdot S)$, for a string S satisfying $\sigma_\alpha \cdot S = -S$ and $\sigma_\beta^2 \cdot S = S$

(b) $\mathbb{C}(S - \sigma_\alpha \cdot S)$, for a string S satisfying $\sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = S$ and $\sigma_\beta \cdot (S - \sigma_\alpha \cdot S) = -(S - \sigma_\alpha \cdot S)$

(c) $\mathbb{C}((S - \sigma_\alpha \cdot S) - \sigma_\beta \cdot (S - \sigma_\alpha \cdot S) - \sigma_\gamma \cdot (S - \sigma_\alpha \cdot S))$, for a string S satisfying $\sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = S$ and $\sigma_\beta \cdot (S - \sigma_\alpha \cdot S) \neq -(S - \sigma_\alpha \cdot S)$.

[17] *Or* $\mathbb{C}v$ *would be a copy of sign in* F_ρ.

We look for equivalence relations to be imposed on $F_{\rho'}$ in order to annihilate each of these possible copies of the sign representation. The first step is to set

$$\boxed{S = \sigma_\beta \cdot S + \sigma_\gamma \cdot S} \quad (\star)$$

for all strings S satisfying $\sigma_\alpha \cdot S = -S$ and $\sigma_\beta^2 \cdot S = S$.
It is a good choice, because the $(\star)$-relations annihilate all the sign representations of type (a), and they span a subspace of $F_{\rho'}$ which is invariant under M'.[18]
A natural question arises:

Question. *What happens to the sign representations of type* (b) *and* (c) *when we impose the* $(\star)$*-relations?*

Surprisingly enough, they are also annihilated.

Proof. Suppose that the string S satisfies $\sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = S$ and $\sigma_\beta \cdot (S - \sigma_\alpha \cdot S) = -(S - \sigma_\alpha \cdot S)$, so that $(S - \sigma_\alpha \cdot S)$ is a sign representation of type (b). Let us show that the equivalence class of $(S - \sigma_\alpha \cdot S)$ modulo the $(\star)$-relations is zero.
We start by observing that, if we write $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$, then α cannot be orthogonal to $\alpha_1 \dots \alpha_r$. Indeed, if α were orthogonal to $\alpha_1 \dots \alpha_r$, then the difference $S - \sigma_\alpha \cdot S = Z_{\alpha_1} \dots Z_{\alpha_r}(v - \sigma_\alpha \cdot v)$ would be a string[19] on which both σ_α and σ_β act by (-1).

[18]For all positive roots ν, α, β, γ such that $\{\pm\alpha, \pm\beta, \pm\gamma\} = \mathcal{A}_2$ and for all strings S such that $\sigma_\alpha \cdot S = -S$ and $\sigma_\beta^2 \cdot S = S$, we have:

$$\sigma_\nu \cdot S = \sigma_\nu \cdot (\sigma_\beta \cdot S) + \sigma_\nu \cdot (\sigma_\gamma \cdot S)\,.$$

This relation is equivalent to

$$(\sigma_\nu \cdot S) = \sigma_{s_\nu(\beta)} \cdot (\sigma_\nu \cdot S) + \sigma_{s_\nu(\gamma)} \cdot (\sigma_\nu \cdot S)\,,$$

which is again a relation of type $(\star)$, for the roots $s_\nu(\alpha)$, $s_\nu(\beta)$, $s_\nu(\gamma)$ (that form an $\mathcal{A}_2$) and the string $(\sigma_\nu \cdot S)$ (that satisfies $s_\nu(\alpha) \cdot (\sigma_\nu \cdot S) = -(\sigma_\nu \cdot S)$ and $s_\nu(\beta)^2 \cdot (\sigma_\nu \cdot S) = +(\sigma_\nu \cdot S)$).

[19]If α is orthogonal to $\alpha_1 \dots \alpha_r$, then $\sigma_\alpha \cdot v$ is a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1} \dots \sigma_{\alpha_r}$. Therefore, it follows from the linearity conditions that

$$S - \sigma_\alpha \cdot S = Z_{\alpha_1} \dots Z_{\alpha_r} v - Z_{\alpha_1} \dots Z_{\alpha_r}(\sigma_\alpha \cdot v) = Z_{\alpha_1} \dots Z_{\alpha_r}(v - \sigma_\alpha \cdot v).$$

As a consequence, the element $v - \sigma_\alpha \cdot v$ of F_ρ would generate a copy of the sign representation of $M'_{\alpha,\beta,\gamma}$. We reach a contradiction.

Therefore, we can assume that α is not orthogonal to α_1. Notice that α_1 cannot be equal to α, or σ_α would act by (-1)-on the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$. So we can find a positive root α'_1 such that the set $\{\pm\,\alpha, \pm\,\alpha_1, \pm\,\alpha'_1\}$ forms a root system of type $\mathcal{A}_2$. Because $\sigma_{\alpha_1} \cdot S = -S$ and $\sigma^2_\alpha \cdot S = S$, also $\sigma^2_{\alpha'_1} \cdot S = S$ and we have a relation:

$$S = \sigma_\alpha \cdot S + \sigma_{\alpha'_1} \cdot S.$$

Let us look at $T = \sigma_{\alpha'_1} \cdot S$. By construction, the string T includes the root α, so $\sigma_\alpha \cdot T = -T$, and $\sigma_\beta \cdot T \neq \pm\, T$. Since $\sigma^2_\beta \cdot T = \sigma^2_\beta \cdot (\sigma_{\alpha'_1} \cdot S) = +T$, we have a relation:

$$T = \sigma_\beta \cdot T + \sigma_\gamma \cdot T.$$

In the quotient space of $F_{\rho'}$ modulo the $(\star)$-relations, we obtain:

$$[S - \sigma_\alpha \cdot S] = [T] = \sigma_\beta \cdot [T] + \sigma_\gamma \cdot [T] = \sigma_\beta \cdot [S - \sigma_\alpha \cdot S] + \sigma_\gamma \cdot [S - \sigma_\alpha \cdot S] =$$

$$= [\sigma_\beta \cdot (S - \sigma_\alpha \cdot S)] + [\sigma_\gamma \cdot (S - \sigma_\alpha \cdot S)] = -2\,[S - \sigma_\alpha \cdot S]$$

hence $[S - \sigma_\alpha \cdot S] = 0$.

This shows that passing to the quotient modulo the $(\star)$-relations eliminates all the sign representations of type (b). We must prove that also the sign representations of type (c) are annihilated.

Suppose that the string S satisfies $\sigma^2_\alpha \cdot S = \sigma^2_\beta \cdot S = S$ and $\sigma_\beta \cdot (S - \sigma_\alpha \cdot S) \neq -(S - \sigma_\alpha \cdot S)$, so that

$$\mathbb{C}((S - \sigma_\alpha \cdot S) - \sigma_\beta \cdot (S - \sigma_\alpha \cdot S) - \sigma_\gamma \cdot (S - \sigma_\alpha \cdot S))$$

is a sign representation of type (c). The same argument used above shows that $S - \sigma_\alpha \cdot S$ is again a string.[20] We call such a string T. Because $\sigma_\alpha \cdot T = -T$ and $\sigma_\beta^2 \cdot T = T$, we have a relation $T = \sigma_\beta \cdot T + \sigma_\gamma \cdot T$. In the quotient space:

$$[(S - \sigma_\alpha \cdot S) - \sigma_\beta \cdot (S - \sigma_\alpha \cdot S) - \sigma_\gamma \cdot (S - \sigma_\alpha \cdot S)] =$$

$$= [S - \sigma_\alpha \cdot S] - \sigma_\beta \cdot [S - \sigma_\alpha \cdot S] - \sigma_\gamma \cdot [S - \sigma_\alpha \cdot S] =$$

$= [T] - \sigma_\beta \cdot [T] - \sigma_\gamma \cdot [T] = [T - \sigma_\beta \cdot T - \sigma_\gamma \cdot T] = [\,0\,]$. The answer to our question is now complete. $\square$

Corollary 12. *The $(\star)$-relations are the correct set of relations that one needs to impose in order to remove all the copies of the sign representation from $F_{\rho'}$.*

Definition 15. *Let R be the subspace of $F_{\rho'}$ generated by all the relations:*

$$\boxed{\sigma_\alpha \cdot S + \sigma_\beta \cdot S + \sigma_\gamma \cdot S = 0 \quad \Leftrightarrow \quad S = \sigma_\beta \cdot S + \sigma_\gamma \cdot S} \qquad (\star)$$

where S is a string satisfying the following two conditions:

1. *S is a (-1)-eigenvector of the root reflections σ_α*
2. *$\sigma_\beta^2 \cdot S = +S$.*

If we write $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$, the first condition means that $\sigma_\alpha \cdot v = -v$ and that the root α either belongs to the set $\{\alpha_1, \dots, \alpha_r\}$, or it is orthogonal to it. The second condition means that β fails to be orthogonal to an *even* number of α_is. Notice that γ automatically satisfies the condition $\sigma_\gamma^2 \cdot S = +S$, hence $\mathbb{C}S$ is the trivial representation of $M_{\alpha,\beta,\gamma}$.

[20] This time α is allowed to be orthogonal to $\alpha_1 \dots \alpha_r$. If this is the case, then $S - \sigma_\alpha \cdot S = Z_{\alpha_1} \dots Z_{\alpha_r}(v - \sigma_\alpha \cdot v)$. Therefore $S - \sigma_\alpha \cdot S$ is a string containing only roots orthogonal to α.

Definition 16. *Let $F_{\tilde{\rho'}}$ be the quotient space:* $\boxed{F_{\tilde{\rho'}} = \dfrac{F_{\rho'}}{R}}$.
Because R is stable under M', the action of M' on $F_{\rho'}$ descends to an action on the quotient space. The result is a well defined representation of M' on $F_{\tilde{\rho'}}$, that we denote by $\tilde{\rho'}$.

Remark 11. *$\tilde{\rho'}$ does not contain any copy of the sign representation of $M' \cap SO(3)$.*

Also notice that, for any positive root ν, all the (-1)-eigenvectors of σ_ν on $F_{\tilde{\rho'}}$ are of the form

$$[Z_{\alpha_1} \dots Z_{\alpha_r} v]$$

where $v \in F_\rho$ is a simultaneous (-1)-eigenvector of $s_\nu,\ s_{\alpha_1}, \dots,\ s_{\alpha_r}$, and $I = \{\alpha_1,\ \dots,\ \alpha_r\}$ is a set of mutually orthogonal positive roots such that ν either belongs to I or it orthogonal to it.

9.5 Step 3: The action of $\mathrm{Lie}(K)$ on $F_{\tilde{\rho'}}$

The purpose of this section is to define an action of $\mathrm{Lie}(K)$ on $F_{\tilde{\rho'}}$ that satisfies the following properties:[21]

(1) as a Lie algebra representation, it is generated by F_ρ

(2) for each positive root α, the eigenvalues of $Z_\alpha \in \mathrm{Lie}(K)$ on $F_{\tilde{\rho'}}$ belong to the set $\{0,\ \pm i,\ \pm 2i\}$

(3) the representation of $\mathrm{Lie}(K)$ on $F_{\tilde{\rho'}}$ lifts to a representation μ of K such that

- ◇ μ is petite
- ◇ μ is spherical
- ◇ the restriction of μ to M' coincides with $\tilde{\rho'}$.

[21] These properties are necessary for the existence of a lifting of the action of $\mathrm{Lie}(K)$ to a petite spherical representation of K that extends ρ.

In particular, the action of Lie(K) must satisfy the condition:

$$\exp\left(\left(\frac{\pi}{2}Z_\beta\right)\cdot[\,x\,]\right)=\left(\exp\left(\frac{\pi}{2}Z_\beta\right)\right)\cdot[\,x\,]=\rho'(\sigma_\beta)\cdot[\,x\,]$$

for all $[\,x\,]$ in $F_{\tilde{\rho}'}$ (i.e. for all x in $F_{\rho'}$), and for all positive roots β. Adding the "petite-ness" requirement, we obtain the the following set of necessary equivalences:

E_1: $\boxed{\{(0)\text{-}eigenspace\, of\; Z_\beta\}\equiv\{(+1)\text{-}eigenspace\, of\; \sigma_\beta\}}$

E_2: $\boxed{\{(i)\text{-}eigenspace\, of\; Z_\beta\}\equiv\{(i)\text{-}eigenspace\, of\; \sigma_\beta\}}$

E_3: $\boxed{\{(-i)\text{-}eigenspace\, of\; Z_\beta\}\equiv\{(-i)\text{-}eigenspace\, of\; \sigma_\beta\}}$

E_4: $\boxed{\{(2i)\text{-}eigenspace\, of\; Z_\beta\}\oplus\{(-2i)\text{-}eigenspace\, of\; Z_\beta\}\equiv\{(-1)\text{-}eigenspace\, of\; \sigma_\beta\}}$.

Let us analyze each of these equivalences in more details.

- E_1 implies that for any $x\in F_{\rho'}$ fixed by σ_β, we must define $Z_\beta\cdot[\,x\,]=0$.[22]
- E_2 and E_3 imply that each $(\pm i)$-eigenvector of σ_β in $F_{\tilde{\rho}'}$ is also an eigenvector of Z_β (relative the same eigenvalue).

  Hence, for all vectors x in $F_{\rho'}$ satisfying $\sigma_\beta^2\cdot x=-\,x$, we must have:

$$\begin{cases} Z_\beta\cdot[\,\sigma_\beta\cdot x-i\,x\,]=-i[\,\sigma_\beta\cdot x-i\,x\,]\\ Z_\beta\cdot[\,\sigma_\beta\cdot x+i\,x\,]=+i[\,\sigma_\beta\cdot x+i\,x\,]\end{cases}$$

  As a result, we need to set: $Z_\beta\cdot[\,x\,]=\sigma_\beta\cdot[\,x\,]$, for all $x\in F_{\rho'}$ s.t. $\sigma_\beta^2\cdot x=-\,x$.

[22]Recall that any string fixed by σ_β is of the form

$$Z_{\alpha_1}Z_{\alpha_2}\cdots Z_{\alpha_r}v$$

with β is orthogonal to $\alpha_1\ldots\alpha_r$ and $\sigma_\beta(v)=+v$.

- E_4 is slightly more subtle. Because we want the Lie algebra representation to be generated by F_ρ, we must define $Z_\beta \cdot [\,v\,] = [\,Z_\beta v\,]$ for all v in F_ρ for which the string $Z_\beta v$ makes sense, i.e. for all (-1)-eigenvectors of σ_β in F_ρ.

  Similarly, if $Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v$ is a string such that β is orthogonal to $\alpha_1 \dots \alpha_r$ and $\sigma_\beta(v) = -v$ (so that $Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v$ is a well defined string), we expect

$$Z_\beta \cdot [Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v] = [\,Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,].$$

  The two dimensional subspace

$$\mathrm{Span}\,(\,[\,Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,],\,[\,Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v]\,)$$

  is included in the (-1)-eigenspace of σ_β, so we want Z_β to act on it with eigenvalues $\pm 2\,i$. This can be achieved by defining[23]

$$Z_\beta \cdot [\,Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,] = -4[\,Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,].$$

These remarks basically determine the entire action of Z_β on the space

$$F_{\tilde{\rho'}} = F_\rho \oplus \frac{\mathrm{Span}\{\text{strings}\}}{R}.$$

The action of Z_β on F_ρ

The root reflection σ_β acts semi-simply on F_ρ, with eigenvalues $+1$ and -1. Therefore we can decompose F_ρ as the direct sum of the $(+1)-$ and the (-1)-eigenspace of σ_β. We define the action of Z_β separately on the two eigenspaces.

[23]With respect to the basis $\{[\,Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,],\,[\,Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v]\}$, we will have:

$$Z_\beta \rightsquigarrow \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.$$

- If $v \in F_\rho$ is a $(+1)$-eigenvector of σ_β, we define $Z_\beta \cdot [\, v\,] = 0$.

- If $v \in F_\rho$ is a (-1)-eigenvector of σ_β, we define $Z_\beta \cdot [\, v\,] = [\, Z_\beta v\,]$.

The action of Z_β on a string $[\, S\,] = [\, Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,]$

We distinguish various cases:

- If $\boxed{\sigma_\beta^2 \cdot [\, S\,] = -[\, S\,]}$, then we set $Z_\beta \cdot [\, S\,] = [\, \sigma_\beta \cdot S\,]$.

- If $\sigma_\beta^2 \cdot [\, S\,] = +[\, S\,]$, then we distinguish three cases:

  1. $\boxed{\sigma_\beta \cdot [\, S\,] = +[\, S\,]}$, then we define $Z_\beta \cdot [\, S\,] = [\, 0\,]$.

  2. $\boxed{\sigma_\beta \cdot [\, S\,] = -[\, S\,]}$. Then there are two possibilities:

     (i) $[\, S\,] = [\, Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,]$, with β orthogonal to $\alpha_1 \ldots \alpha_r$ and $\sigma_\beta(v) = -v$. In this case, we define

     $$Z_\beta \cdot [\, S\,] = Z_\beta \cdot [\, Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,] = [\, Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,]$$

     (ii) $= [\, Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,]$, with $\beta = \alpha_1$ and $\sigma_\beta(v) = -v$. We define

     $$Z_\beta \cdot [\, S\,] = Z_\beta \cdot [\, Z_\beta Z_{\alpha_2} \cdots Z_{\alpha_r} v\,] = -4[Z_{\alpha_2} Z_{\alpha_3} \cdots Z_{\alpha_r} v\,].$$

  3. If $\boxed{\sigma_\beta^2 \cdot [\, S\,] = +[\, S\,] \text{ but } \sigma_\beta \cdot x \neq \pm x}$, then the vector $[S + (\sigma_\beta \cdot S)\,]$ is a $(+1)$-eigenvector for σ_β. Therefore, $Z_\beta \cdot [\, S\,] = -Z_\beta \cdot [\, \sigma_\beta \cdot S\,]$, and we can write:

     $$Z_\beta \cdot [\, S\,] = \frac{1}{2}\left(Z_\beta \cdot [\, S\,] - Z_\beta \cdot [\, \sigma_\beta \cdot S\,]\right) = \frac{1}{2}\, Z_\beta \cdot \, [S - (\sigma_\beta \cdot S)\,].$$

     Let us look at $[S - (\sigma_\beta \cdot S)\,]$, which is a (-1)-eigenvector for σ_β in $F_{\tilde{\rho}'}$. There are two possibilities:

(i) β is orthogonal to $\alpha_1 \ldots \alpha_r$ and of course $\sigma_\beta(v - \sigma_\beta \cdot v) = -(v - \sigma_\beta \cdot v)$. Then

$$[S - (\sigma_\beta \cdot S)] = [Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r}(v - \sigma_\beta \cdot v)]$$

and, by $2(i)$, we must define:

$$Z_\beta \cdot [S] = \frac{1}{2} Z_\beta \cdot [S - (\sigma_\beta \cdot S)] = \frac{1}{2} [Z_\beta Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r}(v - \sigma_\beta \cdot v)].$$

(ii) There exists at least one root α_{i_1} to which β is not orthogonal. Because $\beta \neq \alpha_{i_1}$[24], and $(-1)^{\#\{j:\ [Z_\beta, Z_{\alpha_j}] \neq 0\}} = +1$[25], there exists one (and only one) other root α_{i_2} that is not orthogonal to β, and different from β. Without loss of generality we can assume that $\alpha_{i_1} = 1$ and $\alpha_{i_2} = 2$, so that:

$$\beta \not\perp \alpha_1,\ \beta \not\perp \alpha_2 \qquad \beta \neq \alpha_1,\ \beta \neq \alpha_2 \qquad \beta \perp \alpha_3,\ \alpha_4 \ldots \alpha_r.$$

For $i = 1, 2$, let β_i be the positive root such that $\{\alpha_i, \beta, \beta_i\}$ is a root system of type $\mathcal{A}_2$, and let $\epsilon_i = \pm 1$ be such that $[Z_{\beta_i}, Z_{\alpha_i}] = \epsilon_i Z_\beta$. We notice that for both $i = 1, 2$ we have

$$\begin{cases} \sigma_{\alpha_i} \cdot S = -S \\ \sigma_\beta^2 \cdot S = +S \end{cases}$$

so, when we pass to the quotient, we must impose the relations

$$[S] = \sigma_\beta \cdot [S] + \sigma_{\beta_1} \cdot [S]$$

[24] Or $Z_\beta \cdot [Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v] = -[Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v]$.

[25] Indeed

$$\sigma_\beta^2 \cdot (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) = (-1)^{\#\{j:\ [Z_\beta, Z_{\alpha_j}] \neq 0\}} (Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v) = +(Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_k} v)$$

by assumption.

$$[\,S\,] = \sigma_\beta \cdot [\,S\,] + \,\sigma_{\beta_2} \cdot [\,S\,].$$

Therefore, we should define:

$$Z_\beta \cdot [\,S\,] \equiv \frac{1}{2}\,Z_\beta \cdot [\,S - (\sigma_\beta \cdot S)\,] = \frac{1}{2}\,Z_\beta \cdot [\sigma_{\beta_1} \cdot S] = \frac{1}{2}\,Z_\beta \cdot [\sigma_{\beta_2} \cdot S]. \quad (\circledast)$$

We notice that:

$$[\,\sigma_{\beta_1} \cdot S\,] = \sigma_{\beta_1} \cdot [\,Z_{\alpha_1} Z_{\alpha_2} \cdots Z_{\alpha_r} v\,] =$$

$$= [\,(\mathrm{Ad}(\sigma_{\beta_1}) Z_{\alpha_1})\,(\mathrm{Ad}(\sigma_{\beta_1}) Z_{\alpha_2}) \cdots (\mathrm{Ad}(\sigma_{\beta_1}) Z_{\alpha_r})\,(\sigma_{\beta_1} \cdot v)\,] =$$

$$= [\,(\epsilon_1 Z_\beta)\,(\mathrm{Ad}(\sigma_{\beta_1}) Z_{\alpha_2}) \cdots (\mathrm{Ad}(\sigma_{\beta_1}) Z_{\alpha_r})\,(\sigma_{\beta_1} \cdot v)\,].$$

Therefore:

$$\frac{1}{2}\,Z_\beta \cdot [\,\sigma_{\beta_1} \cdot S\,] = -2\,\epsilon_1\,\sigma_{\beta_1} \cdot [\,\underbrace{Z_{\alpha_1}}_{omitted}\,Z_{\alpha_2} Z_{\alpha_3} \cdots Z_{\alpha_r} v\,].$$

Similarly, $\frac{1}{2}\,Z_\beta \cdot [\,\sigma_{\beta_2} \cdot S\,] = -2\,\epsilon_2\,\sigma_{\beta_2} \cdot [\,Z_{\alpha_1}\,\underbrace{Z_{\alpha_2}}_{omitted}\,Z_{\alpha_3} \cdots Z_{\alpha_r} v\,]$.
So the definition in ($\circledast$) actually makes sense only if we show that

$$\epsilon_1 \sigma_{\beta_1} \cdot [\,Z_{\alpha_2} Z_{\alpha_3} \cdots Z_{\alpha_r} v\,] = \epsilon_2 \sigma_{\beta_2} \cdot [\,Z_{\alpha_1} Z_{\alpha_3} \cdots Z_{\alpha_r} v\,].$$

Since both β_1 and β_2 are orthogonal to $\alpha_3 \ldots \alpha_r$, this is equivalent to proving that

$$[\epsilon_1 \sigma_{\beta_1} \cdot (Z_{\alpha_2} v)] = [\epsilon_2 \sigma_{\beta_2} \cdot (Z_{\alpha_1} v)].$$

This is not hard to check. Comparing the equations[26]

$$[\,Z_{\alpha_1}Z_{\alpha_2}v\,] = \sigma_\beta \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] + \sigma_{\beta_1} \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] \qquad (I)$$

$$[\,Z_{\alpha_1}Z_{\alpha_2}v\,] = \sigma_\beta \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] + \sigma_{\beta_2} \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] \qquad (II)$$

we find:

$$\sigma_{\beta_1} \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] = \sigma_{\beta_2} \cdot [\,Z_{\alpha_1}Z_{\alpha_2}v\,] \Leftrightarrow$$

$$\Leftrightarrow [\epsilon_1 Z_\beta(\sigma_{\beta_1}(\cdot Z_{\alpha_2}v))] = [\epsilon_2 Z_\beta(\sigma_{\beta_2} \cdot (Z_{\alpha_1}v))].$$

Now we just have to apply Z_β to obtain the result:

$$-2\epsilon_1[\sigma_{\beta_1} \cdot (Z_{\alpha_2}v)] = -2\epsilon_2[\sigma_{\beta_2} \cdot (Z_{\alpha_1}v)].\checkmark$$

This concludes the case-by-case analysis of the definition of the action of $Z_(\beta)$ on $F_{\tilde{\rho'}}$. Because the elements $\{Z_\beta\}_{\beta\in\Delta^+}$ generate the Lie algebra of K, the entire action of Lie(K) is determined.

A picture for action of Z_β on $F_{\tilde{\rho'}}$

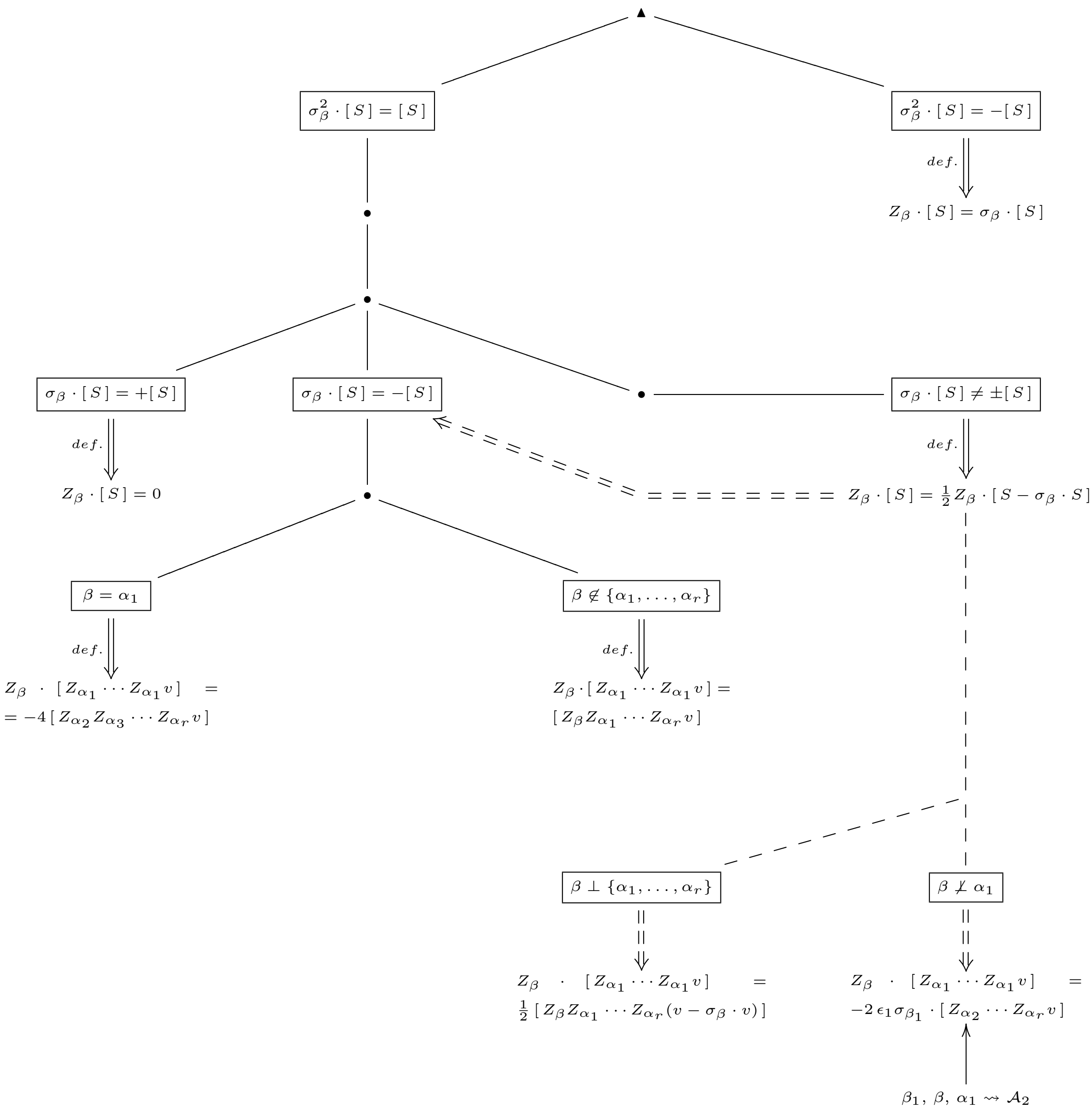
The following diagram shows how to define the action of Z_β on a string

$$[\,S\,] = [\,Z_{\alpha_1}\ldots Z_{\alpha_r}v\,].$$

[26] The first equation is the equivalence relation $\star$ for the triple $\{\alpha_1, \beta, \beta_1\}$ (which of course generates an $\mathfrak{so}(3)$), and the string $S = Z_{\alpha_1}Z_{\alpha_2}v$. It holds because:

$$\sigma_\alpha \cdot S = -S \quad \sigma_\beta^2 \cdot S = \sigma_\gamma^2 \cdot S = S \quad \sigma_\beta \cdot S \neq \pm S \quad \sigma_\gamma \cdot S \neq \pm S.$$

The second equation is the corresponding equation for the triple $\{\alpha_2, \beta, \beta_2\}$.



Some remarks

The action of Lie(K) on $F_{\tilde{\rho}'}$ is now understood. Of course, we still have to show that this definition gives rise to as well defined Lie algebra representation, which is endowed of all the desired properties.

In particular, we must verify that:

- ◇ The action of Lie(K) preserves the ($\star$)-relations.
- ◇ The action of Lie(K) satisfies the bracket relations.
- ◇ The following "eigenvalue condition" holds: for all positive roots β, the element Z_β of $Lie(K)$ acts on $F_{\tilde{\rho}'}$ with eigenvalues in the set $\{0, \pm i, \pm 2i\}$.
- ◇ The action of Lie(K) lifts to a representation of K.
- ◇ The lifting is petite.
- ◇ The restriction of the lifting to M' coincides with $\tilde{\rho}'$.

We will prove these properties in the following subsections.

9.5.1 The ($\star$)-relations are preserved

For any triple of positive roots α, β, γ forming an $\mathcal{A}_2$ and for every string S such that $\sigma_\alpha \cdot S = -S$ and $\sigma_\beta^2 \cdot S = +S$, we have a ($\star$)-relation

$$[\,S\,] = [\,\sigma_\beta \cdot S\,] + [\,\sigma_\gamma \cdot S\,].$$

In this subsection, we show that this relation is preserved by the action of Z_ν on $F_{\tilde{\rho}'}$, for every positive root ν. Equivalently

$$Z_\nu \cdot [\,S\,] = Z_\nu \cdot [\,\sigma_\beta \cdot S\,] + Z_\nu \cdot [\,\sigma_\gamma \cdot S\,]. \qquad (⋇)$$

We distinguish a few cases:

1. $\sigma_\nu^2 \cdot S = -S$ and $\nu \perp \{\alpha, \beta, \gamma\}$
2. $\sigma_\nu^2 \cdot S = -S$ and $\nu \not\perp \{\alpha, \beta, \gamma\}$

3. $\sigma_\nu^2 \cdot S = +S$ and $\nu \perp \{\alpha, \beta, \gamma\}$

4. $\sigma_\nu^2 \cdot S = +S$ and $\nu \not\perp \{\alpha, \beta, \gamma\}$

and we discuss each such case separately.

1. Because the roots β and γ are both orthogonal to ν, we have:

$$\sigma_\nu^2 \cdot (\sigma_\beta \cdot S) = \sigma_\beta \cdot (\sigma_\nu^2 \cdot S) = -(\sigma_\beta \cdot S)$$

$$\sigma_\nu^2 \cdot (\sigma_\gamma \cdot S) = \sigma_\gamma \cdot (\sigma_\nu^2 \cdot S) = -(\sigma_\gamma \cdot S).$$

The condition ($\circledast$) is therefore equivalent to

$$[\,\sigma_\nu \cdot S\,] = [\,\sigma_\nu \cdot (\sigma_\beta \cdot S)\,] + [\,\sigma_\nu \cdot (\sigma_\gamma \cdot S)\,] \Leftrightarrow \sigma_\nu \cdot [\,S\,] = \sigma_\nu \cdot [\,\sigma_\beta \cdot S\,] + \sigma_\nu \cdot [\,\sigma_\gamma \cdot S\,],$$

and it is certainly satisfied because σ_ν preserves the ($\star$)-relations. ✓

2. We apply the same argument used in (1): we suppose that β is not orthogonal to ν, and show that σ_ν^2 still acts by (-1) on $(\sigma_\beta \cdot S)$.

$$\sigma_\nu^2 \cdot (\sigma_\beta \cdot S) = \sigma_\beta \cdot ((\sigma_\beta^{-1}\sigma_\nu^2\sigma_\beta) \cdot S) = \sigma_\beta \cdot \underbrace{(\sigma_{s_\nu(\beta)}^{\pm 2} \cdot S)}_{-S} = -(\sigma_\beta \cdot S)$$

because β, ν, $\sigma_{s_\nu(\beta)}$ give rise to an $\mathcal{A}_2$, hence the commuting product $\sigma_{s_\nu(\beta)}^2\sigma_\beta^2\sigma_\nu^2$ must act by $+1$ on each representation.
Similarly, $\sigma_\nu^2 \cdot (\sigma_\gamma \cdot S) = -(\sigma_\gamma \cdot S)$.[27] ✓

3. It follows easily from the definition of the action of Lie(K) on $F_{\tilde{\rho}'}$ that

$$\sigma \cdot (Z_\nu \cdot S) = (\mathrm{Ad}(\sigma)(Z_\nu)) \cdot (\sigma \cdot S)$$

[27] γ is not necessarily orthogonal to α, but we have shown the result in both cases.

for all σ in M', all positive roots ν and every string S. Therefore, when β and γ are both orthogonal to ν we can re-write the condition ($\circledast$) as

$$[\, Z_\nu \cdot S\,] = [\,(\mathrm{Ad}(\sigma_\beta)(Z_\nu)) \cdot (\sigma_\beta \cdot S)\,] + [\,((\mathrm{Ad}(\sigma_\gamma)(Z_\nu)) \cdot (\sigma_\gamma \cdot S)\,] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad [\, Z_\nu \cdot S\,] = [\,\sigma_\beta \cdot (Z_\nu \cdot S)\,] + [\,\sigma_\gamma \cdot (Z_\nu \cdot S)\,].$$

$$\Leftrightarrow \quad [\, Z_\nu \cdot S\,] = \sigma_\beta \cdot [\, Z_\nu \cdot S\,] + \sigma_\gamma \cdot [\, Z_\nu \cdot S\,].$$

We deduce that ($\circledast$) is simply the ($\star$)-relation for the triple α, β, γ and the string $Z_\nu \cdot S$. This relation holds because, since α, β, γ are orthogonal to ν, we have

$$\sigma_\alpha \cdot (Z_\nu \cdot S) = Z_\nu \cdot (\sigma_\alpha \cdot S) = -(Z_\nu \cdot S)$$

$$\sigma_\beta^2 \cdot (Z_\nu \cdot S) = Z_\nu \cdot (\sigma_\beta^2 \cdot S) = +(Z_\nu \cdot S). \quad \checkmark$$

4. We notice that the roots $\{\nu, \alpha, \beta, \gamma\}$ give rise to an $\mathcal{A}_3$. Because σ_α^2, σ_β^2, σ_γ^2 and σ_ν^2 all act by $+1$ on S, the string S belongs to a Weyl group representation of $\mathcal{S}_4$. So it is enough to show that the action of $\mathrm{Lie}(K)$ on $F_{\tilde{\rho}'}$ is well defined for every representation ρ of $\mathcal{S}_4$. We will do it in the next chapter, in the section dedicated to the examples.

9.5.2 Checking bracket relations-PART 1

Let α and β be **not orthogonal** positive roots, and let γ be a positive root in δ such that $\{\pm\,\alpha, \pm\,\beta, \pm\,\gamma\}$ is a root system of type $\mathcal{A}_2$. Without loss of generality we can assume that the following bracket relations hold:[28]

$$[Z_\beta,\, Z_\alpha] = Z_\gamma \qquad [Z_\alpha,\, Z_\gamma] = Z_\beta \qquad [Z_\beta,\, Z_\gamma] = -Z_\alpha\,.$$

We want to verify that

$$\boxed{Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x}$$

for all x in $F_{\tilde{\rho}'}$. It is of course enough to consider the case in which x is either an element of the original space F_ρ or the equivalence class of a string:

$$x = [\,S\,] = [\,Z_{\alpha_1} \dots Z_{\alpha_r} v\,]$$

for some mutually orthogonal positive roots $\alpha_1, \dots, \alpha_r$ and a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_r}$.

When x belongs to F_ρ, we decompose F_ρ in isotypic components of irreducible representations of the $\mathcal{S}_(3)$ corresponding to α, β, γ:

$$\rho = (\text{trivial})^{m_t} \,\oplus\, (\text{standard})^{m_s} = U^{m_t} \,\oplus\, V^{m_s}$$

and we write $x = x_t + x_s$ for the corresponding decomposition of x. We only need to check that the bracket relations hold for x_t and x_s, and this problem is equivalent to showing that our construction produces a well defined Lie algebra representation when ρ is either the trivial or the standard representation of $\mathcal{S}_3$. We will do this in

[28] As a consequence, we have:

$$\sigma_\beta \sigma_\alpha \sigma_\beta^{-1} = \sigma_\alpha^{-1} \sigma_\beta \sigma_\alpha = \sigma_\gamma$$
$$\sigma_\alpha \sigma_\gamma \sigma_\alpha^{-1} = \sigma_\gamma^{-1} \sigma_\alpha \sigma_\gamma = \sigma_\beta$$
$$\sigma_\gamma \sigma_\beta \sigma_\gamma^{-1} = \sigma_\beta^{-1} \sigma_\gamma \sigma_\beta = \sigma_\alpha.$$

the next chapter, in the section dedicated to the examples.

Now we discuss the case in which $x = [\,S\,] = [\,Z_{\alpha_1} \ldots Z_{\alpha_r} v\,]$ is the equivalence class of a string. Define $\epsilon_\alpha,\, \epsilon_\beta,\, \epsilon_\gamma \in \{\pm 1\}$ by

$$\sigma_\alpha^2 \cdot S = \epsilon_\alpha\, S$$

$$\sigma_\beta^2 \cdot S = \epsilon_\beta\, S$$

$$\sigma_\gamma^2 \cdot S = \epsilon_\gamma\, S$$

so that

$$\epsilon_\alpha = \#\{\alpha_j : [Z_{\alpha_j},\, Z_\alpha] \neq 0\}$$

$$\epsilon_\beta = \#\{\alpha_j : [Z_{\alpha_j},\, Z_\beta] \neq 0\}$$

$$\epsilon_\gamma = \#\{\alpha_j : [Z_{\alpha_j},\, Z_\gamma] \neq 0\}.$$

Because $\epsilon_\alpha\, \epsilon_\beta\, \epsilon_\gamma = 1$,[29] we can assume w.l.o.g that either $\sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = \sigma_\gamma^2 \cdot S = +\,S$ or $\sigma_\alpha^2 \cdot S = -\sigma_\beta^2 \cdot S = -\sigma_\gamma^2 \cdot S = +\,S$.
We discuss the two cases separately. The trick will be once again to perform a reduction to small rank cases.

The case $\epsilon_\alpha = \epsilon_\beta = \epsilon_\gamma = +1$

Without loss of generality, we can assume that one of the following subcases holds:

1. $\sigma_\alpha \cdot S = \sigma_\beta \cdot S = +S$
2. $\sigma_\alpha \cdot S = +S, \quad \sigma_\beta \cdot +S \neq S, \quad \sigma_\beta^2 \cdot S = +S$
3. $\sigma_\alpha \cdot S = -S, \quad \sigma_\beta^2 \cdot S = +S, \quad \alpha \in \{\alpha_1, \ldots, \alpha_r\}$
4. $\sigma_\alpha \cdot S = -S, \quad \sigma_\beta^2 \cdot S = +S, \quad \alpha \perp \{\alpha_1, \ldots, \alpha_r\}$

[29] Indeed the elements $m_\alpha = \sigma_\alpha^2$, $m_\beta = \sigma_\beta^2$, $m_\gamma = \sigma_\gamma^2$ of $SO(3)$ satisfy the relations $m_\alpha m_\beta m_\gamma = \mathbb{I}$.

5. $\sigma_\alpha \cdot S \neq \pm S, \quad \sigma_\beta \cdot S \neq \pm S, \quad \sigma_\gamma \cdot S \neq \pm S, \quad \sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = +S.$

We analyze each of these subcases separately. In all of them we reduce computations to a construction for $SL(k)$ for some $k \leqslant 6$ which is explicitly carried out in the next chapter.

⋆ ⋆ ⋆⋆

Subcase $\#1$

Because σ_α, σ_β and σ_γ all act by $(+1)$ on the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$, each root α_j is orthogonal to the triple $\{\alpha, \beta, \gamma\}$. So computations are reduced to the case in which S is an element of $F_{(3)}$, with (3) the trivial representation of $\mathcal{S}_3$.

⋆ ⋆ ⋆⋆

Subcase $\#2$

We assume that the string S satisfies $\begin{cases} \sigma_\alpha \cdot S = +S \\ \sigma_\beta \cdot S \neq +S \\ \sigma_\beta^2 \cdot S = +S. \end{cases}$

Because $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ is a $(+1)$-eigenvector for σ_α, the root α is orthogonal to the set $\{\alpha_1, \dots, \alpha_r\}$, and since we are dealing with a root system of type $\mathcal{A}$, the roots β and γ must have the same property.[30]

Therefore, computations are reduced to the case in which S is an element of a Weyl group representation of $\mathcal{S}_3$ (not containing the sign).

⋆ ⋆ ⋆⋆

[30] Suppose that β is not orthogonal to the entire set $\{\alpha_1, \dots, \alpha_r\}$. Then, since $\epsilon_\beta = 1$ and Δ is of type $\mathcal{A}$, there are exactly two roots α_{i_1} and α_{i_2} to which β is not orthogonal. So β is of the form $\varepsilon_{a_1} - \varepsilon_{a_2}$, with a_j an index appearing in α_{i_j} but not in α (because α is orthogonal to α_{i_j}) for all $j = 1, 2$. It follows that β is orthogonal to α and we reach a contradiction.

Subcase #3

We assume that the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ satisfies: $\begin{cases} \sigma_\alpha \cdot S = -S \\ \sigma_\beta^2 \cdot S = +S \end{cases}$

and that $\boxed{\alpha \in \{\alpha_1, \dots, \alpha_r\}}$.

Without loss of generality we can assume that $\alpha = \alpha_1$. Because $\sigma_\beta^2 \cdot S = +S$, there exists also another root, say α_2, which is not orthogonal to β. We can write $S = Z_\alpha Z_{\alpha_2} T$, and ignore the string T which only involves roots orthogonal to $\{\alpha, \beta, \gamma\}$. We must show that:

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\,Z_\alpha Z_{\alpha_2} v\,]$. By construction, the roots $\{\alpha, \beta, \gamma, \alpha_2\}$ give rise to a root system of type $\mathcal{A}_3$, so we basically need to do a calculation for $SL(4)$. We notice that the vector v is a simultaneous (-1)-eigenvector for the two orthogonal roots $\alpha,\ \alpha_2$, therefore it can only live in (a direct sum of copies of) $F_{(2,2)}$.[31]

It all comes down to showing that bracket relations hold when $\rho = (2, 2)$ is the two-dimensional irreducible representation of $\mathcal{S}_4$.

$\star \star \star\star$

Subcase #4

We assume that the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ satisfies: $\begin{cases} \sigma_\alpha \cdot S = -S \\ \sigma_\beta^2 \cdot S = +S \end{cases}$

and that $\boxed{\alpha \perp \{\alpha_1, \dots, \alpha_r\}}$.

By assumption, the root α is orthogonal to the set $\{\alpha_1, \dots, \alpha_r\}$. Since we are dealing

[31] Recall that an element of the Specth module corresponding to a partition $(n-k, k)$ can be a simultaneous (-1)-eigenvector of at most k disjoint transpositions.

with a root system of type $\mathcal{A}$, the roots β and γ must have the same property. It follows that we can reduce computations to the case in which S is an element of $F_{(2,1)}$, with $(2,1)$ the standard representation of $\mathcal{S}_3$.

$\star \star \star\star$

Subcase #5

We assume that S satisfies $\begin{cases} \sigma_\alpha \cdot S \neq \pm S, \quad \sigma_\beta \cdot S \neq \pm S, \quad \sigma_\gamma \cdot S \neq \pm S \\ \sigma_\alpha^2 \cdot S = \sigma_\beta^2 \cdot S = +S. \end{cases}$

We notice that $\{\alpha,\, \beta,\, \gamma\} \cap \{\alpha_1, \ldots, \alpha_r\} = \varnothing$ (because the string S is not a (-1)-eigenvector of σ_ζ for $\zeta = \alpha,\, \beta,\, \gamma$), and that the number of α_j s not orthogonal to α (or β, or γ) is even, and hence equal to either zero or two. There are basically two possibilities:

(*i*) $\{\alpha,\, \beta,\, \gamma\} \perp \{\alpha_1, \ldots, \alpha_r\}$

(*ii*) $\{\alpha,\, \beta,\, \gamma\} \perp \{\alpha_4, \ldots, \alpha_r\}$ and $\begin{cases} \alpha \not\perp \alpha_1,\, \alpha \not\perp \alpha_2,\, \alpha \perp \alpha_3 \\ \beta \perp \alpha_1,\, \beta \not\perp \alpha_2,\, \beta \not\perp \alpha_3 \\ \gamma \not\perp \alpha_1,\, \gamma \not\perp \alpha_2,\, \gamma \perp \alpha_3. \end{cases}$

We discuss these two possibilities separately.

The first one is easy, because we reduce computations to the case in which S is an element of a Weyl group representation of $\mathcal{S}_3$ (not containing the sign).

In the second case, we write $S = Z_{\alpha_1} Z_{\alpha_2} Z_{\alpha_3} T$, and ignore the string T which only involves roots orthogonal to $\{\alpha,\, \beta,\, \gamma\}$. We must show that:

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z_{\alpha_1} Z_{\alpha_2} Z_{\alpha_3} v \,]$. By construction, the roots

$$\{\alpha,\, \beta,\, \gamma,\, \alpha_1,\, \alpha_2,\, \alpha_3\}$$

give rise to a root system of type $\mathcal{A}_5$, so we basically need to do a calculation for $SL(6)$. We notice that the vector v is a simultaneous (-1)-eigenvector for the three (orthogonal) root reflections σ_{α_1}, σ_{α_2}, , σ_{α_3}, therefore it can only live in (a direct sum of copies of) $F_{(3,3)}$. It all comes down to showing that bracket relations hold when ρ is the Specth module $S^{(3,3)}$ of $\mathcal{S}_6$.

The case $\epsilon_\alpha = -\,\epsilon_\beta = -\,\epsilon_\gamma = +1$

We now discuss the case $\sigma_\alpha^2 \cdot S = -\,\sigma_\beta^2 \cdot S = -\,\sigma_\gamma^2 \cdot S = +S$.

Without loss of generality, we can assume that one of the following subcases holds:

1. $\sigma_\alpha \cdot S = +S, \quad \sigma_\beta^2 \cdot S = -S$

2. $\sigma_\alpha \cdot S = -S, \quad \sigma_\beta^2 \cdot S = -S, \quad \boxed{\alpha \in \{\alpha_1, \ldots, \alpha_r\}}.$

3. $\sigma_\alpha \cdot S = -S, \quad \sigma_\beta^2 \cdot S = -S, \quad \boxed{\alpha \perp \{\alpha_1, \ldots, \alpha_r\}}.$

4. $\sigma_\alpha \cdot S \neq \pm S, \quad \sigma_\alpha^2 \cdot S = +S = -\sigma_\beta^2 \cdot S.$

We analyze each of these subcases separately.

$\star\star\star\star$

Subcase $\#1$

We assume that the string S satisfies $\begin{cases} \sigma_\alpha \cdot S = +S \\ \sigma_\beta^2 \cdot S = -S. \end{cases}$

W.l.o.g. we can also assume that

$$\begin{aligned}
&\alpha \perp \alpha_j \quad \text{for all } j = 1 \dots r \\
&\beta \not\perp \alpha_1 \quad \beta \perp \alpha_j \quad \text{for all } j \neq 1 \\
&\gamma \not\perp \alpha_1 \quad \gamma \perp \alpha_j \quad \text{for all } j \neq 1.
\end{aligned}$$

If we write $S = Z_{\alpha_1} T$, we can ignore the string T because it only involves roots that are orthogonal to α, β, γ. Then, we must show that

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z_{\alpha_1} v \,]$. By construction, the roots

$$\{\alpha,\, \beta,\, \gamma,\, \alpha_1\}$$

give rise to a root system of type $\mathcal{A}_3$, so we basically need to do a calculation for $SL(4)$. We notice that the two (orthogonal) root reflections σ_α, σ_{α_1} act on v by $(+1)$ and (-1) respectively. Therefore,[32] v can only live in (a direct sum of copies of) $F_{(3,1)}$. It all comes down to showing that bracket relations hold when ρ is the Specth module $S^{(3,1)}$ of $\mathcal{S}_4$.

⋆ ⋆ ⋆⋆

Subcase $\#2$

We assume that the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ satisfies: $\begin{cases} \sigma_\alpha \cdot S = -S \\ \sigma_\beta^2 \cdot S = -S \end{cases}$

and that $\boxed{\alpha \in \{\alpha_1, \dots, \alpha_r\}}$.

W.l.o.g. we can suppose that $\alpha = \alpha_1$, so that α, β, γ are orthogonal to α_j for all $j > 1$. If we write $S = Z_\alpha T$, we can ignore the string T because it only involves roots

[32] In $F_{(2,2)}$, disjoint transpositions have the same (-1) eigenspace.

that are orthogonal to α, β, γ, and we must show that

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z\alpha v \,]$. This is a calculation for $SL(3)$. Because v is a (-1)-eigenvector for σ_α, v can only live in (a direct sum of copies of) $F_{(2,1)}$.

It all comes down to showing that bracket relations hold when ρ is the Specth module $S^{(2,1)}$ of $\mathcal{S}_3$.

⋆ ⋆ ⋆⋆

Subcase #3

We assume that the string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ satisfies: $\begin{cases} \sigma_\alpha \cdot S = -S \\ \sigma_\beta^2 \cdot S = -S \end{cases}$

and that $\boxed{\alpha \perp \{\alpha_1, \dots, \alpha_r\}}$. W.l.o.g. we can also assume that

$$\begin{array}{lll} \beta \not\perp \alpha_1 & \beta \perp \alpha_j & \text{for all } j \neq 1 \\ \gamma \not\perp \alpha_1 & \gamma \perp \alpha_j & \text{for all } j \neq 1. \end{array}$$

If we write $S = Z_{\alpha_1} T$, we can ignore the string T because it only involves roots that are orthogonal to α, β, γ. Then, we must show that

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z_{\alpha_1} v \,]$. By construction, the roots

$$\{\alpha, \beta, \gamma, \alpha_1\}$$

give rise to a root system of type $\mathcal{A}_3$, so we basically need to do a calculation for

$SL(4)$. We notice that v is a simultaneous (-1)-eigenvector for the two orthogonal root reflections σ_α and σ_{α_1}. Therefore, v can only live in (a direct sum of copies of) $F_{(2,2)}$. It all comes down to showing that bracket relations hold when ρ is the Specth module $S^{(2,2)}$ of $\mathcal{S}_4$.

⋆ ⋆ ⋆⋆

Subcase $\#4$

We assume that the string S satisfies $\begin{cases} \sigma_\alpha \cdot S \neq \pm S \\ \sigma_\alpha^2 \cdot S = -\sigma_\beta^2 \cdot S = +S. \end{cases}$

Because $\sigma_\alpha^2 \cdot S = +S$, but $\sigma_\alpha \cdot S \neq -S$ the root α is either orthogonal to the set $\{\alpha_1, \ldots, \alpha_r\}$ or not orthogonal to exactly two roots in this set. We need to distinguish between these two cases.

(i) $\alpha \perp \{\alpha_1, \ldots, \alpha_r\}$.

Let α_1 be the (only) root to which β and γ are not orthogonal. As usual, we write $S = Z_{\alpha_1}T$, and we ignore the string T because it only involves roots that are orthogonal to α, β, γ. We must show that

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z_{\alpha_1} v\,]$. By construction, the roots

$$\{\alpha,\, \beta,\, \gamma,\, \alpha_1\}$$

give rise to a root system of type $\mathcal{A}_3$, so we basically need to do a calculation for $SL(4)$. The vector v is an element of a Weyl group representation of $\mathcal{S}_4$ (not containing the sign, nor the trivial representation).[33]

It all comes down to showing that bracket relations hold when ρ is the Specth modules

[33]Because $\sigma_{\alpha_1} \cdot v = -v$

$S^{(2,2)}$ or $S^{(3,1)}$ of $\mathcal{S}_4$.

(ii) $\alpha \not\perp \{\alpha_1, \alpha_2\}$.

We can assume that β is orthogonal to α_2, but not to α_1, and that γ is orthogonal to α_1, but not to α_2. In any case, the roots α, β, γ are orthogonal to α_j for all $j > 2$. We write $S = Z_{\alpha_1} Z_{\alpha_2} T$, and we ignore the string T because it only involves roots that are orthogonal to α, β, γ. We must show that

$$Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x$$

for $x = [\, Z_{\alpha_1} Z_{\alpha_2} v\,]$. By construction, the roots

$$\{\alpha,\, \beta,\, \gamma,\, \alpha_1,\, \alpha_2\}$$

give rise to a root system of type $\mathcal{A}_4$, so we basically need to do a calculation for $SL(5)$. We notice that the v is a simultaneous (-1)-eigenvector for the two (orthogonal) root reflections σ_{α_1} and σ_{α_2}. Therefore, v can only live in (a direct sum of copies of) $F_{(3,2)}$. It all comes down to showing that bracket relations hold when ρ is the Specth module $S^{(3,2)}$ of $\mathcal{S}_5$.

9.5.3 Checking bracket relations-PART 2

Let α and β be **mutually orthogonal** positive roots. We want to verify that

$$\boxed{Z_\beta \cdot (Z_\alpha \cdot x) - Z_\alpha \cdot (Z_\beta \cdot x) = Z_\gamma \cdot x}$$

for all x in $F_{\tilde{\rho}'}$. It is of course enough to consider the case in which x is either an element of the original space F_ρ or the equivalence class of a string:

$$x = [\, S\,] = [\, Z_{\alpha_1} \ldots Z_{\alpha_r} v\,]$$

for some mutually orthogonal positive roots $\alpha_1, \ldots, \alpha_r$ and a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_r}$.

When x belongs to F_ρ, we decompose F_ρ in isotypic components of irreducible representation of the $\mathcal{S}_4$ determined by α and β

$$\rho = \left(S^{(4)}\right)^{m_0} \oplus \left(S^{(3,1)}\right)^{m_1} \oplus \left(S^{(2,2)}\right)^{m_2}$$

and we write $x = x_0 + x_1 + x_2$ for the corresponding decomposition of x. We just need to check that the bracket relations hold for the x_js, and this problem is equivalent to showing that our construction produces a well defined Lie algebra representation when ρ is a Weyl group representation of $\mathcal{S}_4$ not containing the sign. We will do this in the next chapter, in the section dedicated to the examples.

We now discuss the case in which $x = [\,S\,] = [\ \ z\,]$ is the equivalence class of a string. The trick is again to perform a reduction to a calculation for a small Weyl group, namely some symmetric group $\mathcal{S}_k$, with $k \leqslant 8$ depending on the number of α_j s not orthogonal to α and β. We distinguish several subcases:

1. When α and β are both orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$, we reduce computations to the case in which ρ is a representation of $\mathcal{S}_4$ not containing the sign.

2. When α is orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$ and β belongs to this set, we reduce computations to the case in which ρ is a sum of copies of the representations $S^{(3,1)}$ and $S^{(2,2)}$ of $\mathcal{S}_4$.

3. When α is orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$ and there is exactly one root in this set which is not orthogonal to β (and different from β), we reduce computations to the case in which ρ is a a sum of copies of the representations $S^{(3,2)}$ and $S^{(4,1)}$ of $\mathcal{S}_5$.

4. When α is orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$ and there are two roots in this set that

are not orthogonal to β, we reduce computations to the case in which ρ is a sum of copies of the representations $S^{(4,2)}$ and $S^{(3,3)}$ of $\mathcal{S}_6$.

5. When both α and β belong to the set $\{\alpha_1, \dots, \alpha_r\}$, we reduce computations to the case in which ρ is the representation $S^{(2,2)}$ of $\mathcal{S}_4$.

6. When only α belongs to the set $\{\alpha_1, \dots, \alpha_r\}$, and there is one root in this set not orthogonal to β, we reduce computations to the case in which ρ is the representation $S^{(3,2)}$ of $\mathcal{S}_5$.

7. When only α belongs to the set $\{\alpha_1, \dots, \alpha_r\}$, and there are two roots in this set not orthogonal to β, we reduce computations to the case in which ρ is the representation $S^{(3,3)}$ of $\mathcal{S}_6$.

8. When neither α nor β belong to the set $\{\alpha_1, \dots, \alpha_r\}$, and they are both not orthogonal a specific root, we reduce computations to the case in which ρ is a sum of copies of the representations $S^{3,1)}$ and $S^{(2,2)}$ of $\mathcal{S}_4$.

9. When neither α nor β belong to the set $\{\alpha_1, \dots, \alpha_r\}$, and there are two roots α_{j_1}, α_{j_2} such that α is not orthogonal to a root α_{j_1} and β is not orthogonal to α_{j_2}, we reduce computations to the case in which ρ is a sum of copies of the representations $S^{(4,2)}$ and $S^{(3,3)}$ of $\mathcal{S}_6$.

10. When neither α nor β belong to the set $\{\alpha_1, \dots, \alpha_r\}$ and there are two roots α_{j_1}, α_{j_2} such that α is not orthogonal to α_{j_1}, and β is not orthogonal to α_{j_1} and α_{j_2}, is not orthogonal to one of the roots, we reduce computations to the case in which ρ is a sum of copies of the representations $S^{(4,2)}$ and $S^{(3,3)}$ of $\mathcal{S}_6$.

11. When neither α nor β belong to the set $\{\alpha_1, \dots, \alpha_r\}$, and there are three roots α_{j_1}, α_{j_2}, α_{j_3} such that α is not orthogonal to α_{j_1}, and β is not orthogonal to α_{j_2}, α_{j_3}, we reduce computations to the case in which ρ is a sum of copies of the representation $S^{(4,3)}$ of $\mathcal{S}_7$.

12. When there are two roots to which neither α nor β are orthogonal, we reduce computations to the case in which ρ is the representation $S^{(2,2)}$ of $\mathcal{S}_4$.

13. When there are three roots α_{j_1}, α_{j_2}, α_{j_3} such that α is not orthogonal to α_{j_1}, α_{j_2}, and β is not orthogonal to α_{j_2}, α_{j_3}, we reduce computations to the case in which ρ is the representation $S^{(3,3)}$ of $\mathcal{S}_6$.

14. When there are two distinct pairs of roots to which α and β are not orthogonal, we reduce computations to the case in which ρ is the representation $S^{(4,4)}$ of $\mathcal{S}_8$.

Most of the examples we refer to are explicitly worked out in the next chapter. The only exceptions are $S^{(4,2)}$, $S^{(4,3)}$, and $S^{(4,4)}$, but the computations in these cases are very similar to the ones done for $S^{(3,3)}$.

Claim 7. *Alternatively, we can check the bracket relations using the definition of the the action of $Lie(K)$ on $F_{\tilde{\rho'}}$, with a case-by-case analysis.*

9.5.4 Checking the eigenvalue condition

The purpose of this subsection is to show that, for all positive roots β, the eigenvalues of $\tilde{\rho'}(Z_\beta)$ lie in the set $\{0, \pm i, \pm 2i\}$.

Because the action of Z_β on $F_{\tilde{\rho'}}$ strongly depends on how the corresponding element σ_β of M' acts on the same space, it is convenient to decompose $F_{\tilde{\rho'}}$ in σ_β-stable subspaces and to study the eigenvalues of Z_β separately on each of these subspaces.

We start by decomposing $F_{\rho'} = F_\rho \oplus \mathrm{Span}(\,strings\,)$ in σ_β-stable subspaces.[34]For brevity of notations, let $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ be an arbitrary string of orthogonal roots. Then:

[34]The existence of this decomposition follows from the fact that for any string S, $\sigma_\beta^2 \cdot S = \pm S$. Once again we identify a string with its equivalence class modulo the commutativity and linearity relations, so that$F_{\rho'}$ is identified with $F_\rho \oplus \mathrm{Span}(\,strings\,)$.

$F_{\rho'} = F_\rho \oplus \mathrm{Span}(\,strings\,)$

$$= F_\rho \oplus \underbrace{\mathrm{Span}\{S\colon \sigma_\beta \cdot S = +S)\}}_{U_{\mathrm{I}}} \oplus \underbrace{\mathrm{Span}\{S\colon \sigma_\beta \cdot S = -S\}}_{U_{\mathrm{II}}} \oplus$$

$$\oplus \underbrace{\mathrm{Span}\{S\colon \sigma_\beta \cdot S \neq \pm S,\, \sigma_\beta^2 \cdot S = +S,\, \beta \not\perp \{\alpha_1, \ldots, \alpha_r\}\}}_{U_{\mathrm{III}}} \oplus$$

$$\oplus \underbrace{\mathrm{Span}\{S\colon\; \sigma_\beta^2 \cdot S = -S\}}_{U_{\mathrm{IV}}}. \qquad (\clubsuit)$$

Let us explain why we have imposed the extra condition "$\beta \not\perp \{\alpha_1, \ldots, \alpha_r\}$" on the strings generating U_{III}. If β is orthogonal to $\alpha_1, \ldots, \alpha_r$ and v is an element of F_ρ such that $\sigma_\beta \cdot v = \pm v$, then using the linearity conditions[35] we can write:

$$S = Z_{\alpha_1} \ldots Z_{\alpha_r} v = \underbrace{\frac{1}{2}\, Z_{\alpha_1} \ldots Z_{\alpha_r}(v + \sigma_\beta \cdot v)}_{\text{in } U_{\mathrm{I}}} + \underbrace{\frac{1}{2}\, Z_{\alpha_1} \ldots Z_{\alpha_r}(v - \sigma_\beta \cdot v)}_{\text{in } U_{\mathrm{II}}}$$

so the string S satisfies $\sigma_\beta \cdot S \neq \pm S$ and $\sigma_\beta^2 \cdot S = S$, but it already lies in $U_{\mathrm{I}} \oplus U_{\mathrm{II}}$.

We now give a detailed description of each summand of the decomposition ($\clubsuit$):

- U_{I} is spanned by those strings on which σ_β acts as scalar multiplication by $+1$.[36] It is stable under the action of σ_β, and it is included in the $(+1)$-eigenspace of σ_β.

- U_{II} is spanned by those strings on which σ_β acts as scalar multiplication by -1. Recall that a string $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ is a (-1)-eigenvectors of σ_β if and only if

[35] If v is a (-1)-eigenvector of $\alpha_1, \ldots, \alpha_r$ in F_ρ and β is orthogonal to $\alpha_1, \ldots, \alpha_r$, then the vectors $w^\pm = \frac{1}{2}(v \pm \sigma_\beta \cdot v)$ are also simultaneous (-1)-eigenvector of $\alpha_1, \ldots, \alpha_r$. Therefore, the following "linearity condition" holds:

$$Z_{\alpha_1} \ldots Z_{\alpha_r} v = (Z_{\alpha_1} \ldots Z_{\alpha_r} w^+) + (Z_{\alpha_1} \ldots Z_{\alpha_r} w^-).$$

[36] A string $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ is a $(+1)$-eigenvector of σ_β if and only if $\beta \perp \{\alpha_1, \ldots, \alpha_n\}$ and $\sigma_\beta \cdot v = +v$.

one of the following two conditions occur:

1. $\beta \perp \{\alpha_1, \dots, \alpha_n\}$ and $\sigma_\beta \cdot v = -v$
2. $\beta \in \{\alpha_1, \dots, \alpha_n\}$ and $\sigma_\beta \cdot v = -v$.

So we can write: $U_{\mathrm{II}} = \mathrm{Span}\{S = Z_{\alpha_1} \dots Z_{\alpha_r} v : \ \sigma_\beta \cdot S = -S\} =$

$$= \underbrace{\mathrm{Span}\{S : \ \sigma_\beta \cdot S = -S; \, \beta \notin \{\alpha_1, \dots, \alpha_n\}\}}_{U_{\mathrm{II}_1}} \oplus$$

$$\oplus \underbrace{\mathrm{Span}\{S : \ \sigma_\beta \cdot S = -S; \, \beta \in \{\alpha_1, \dots, \alpha_n\}\}}_{U_{\mathrm{II}_2}}.$$

Both subspaces are included in the (-1)-eigenspace of σ_β, hence they are stable under σ_β.

- U_{III} is also stable under σ_β. Indeed for each string $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ in U_{III} we have

$$\sigma_\beta \cdot (\sigma_\beta \cdot S) = S \neq \pm (\sigma_\beta \cdot S) \qquad \sigma_\beta^2 \cdot (\sigma_\beta \cdot S) = \sigma_\beta \cdot (\sigma_\beta^2(S)) = +(\sigma_\beta \cdot S).$$

Therefore, we can decompose U_{III} as the direct sum of (two-dimensional) subspaces of the form

$$< S, \ (\sigma_\beta \cdot S) >_{\mathbb{C}}$$

with $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ a string of orthogonal roots such that $(\sigma_\beta \cdot S) \neq \pm S$ and $\sigma_\beta^2 \cdot S = +S$. Because σ_β acts on each of these subspaces by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the vector $(S + \sigma_\beta \cdot S)$ is a $(+1)$-eigenvector of σ_β, and $(S - \sigma_\beta \cdot S)$ is a (-1)-eigenvector. We can write:

$$U_{\mathrm{III}} = \mathrm{Span}\{S = Z_{\alpha_1} \dots Z_{\alpha_r} v : \ \sigma_\beta \cdot S \neq \pm S, \ \sigma_\beta^2 \cdot S = +S, \ \beta \not\perp \{\alpha_1, \dots, \alpha_r\}\} =$$

$$= \underbrace{\left(\bigoplus_{\substack{S :\ \sigma_\beta S \neq \pm S \\ \sigma_\beta^2 S = +S \\ \beta \not\perp \{\alpha_1, \dots, \alpha_r\}}} \mathbb{C}(S + \sigma_\beta \cdot S) \right)}_{U_{\mathrm{III}_1}} \bigoplus \underbrace{\left(\bigoplus_{\substack{S :\ \sigma_\beta S \neq \pm S \\ \sigma_\beta^2 S = +S \\ \beta \not\perp \{\alpha_1, \dots, \alpha_r\}}} \mathbb{C}(S - \sigma_\beta \cdot S) \right)}_{U_{\mathrm{III}_2}} .$$

Both subspaces are stable under σ_β, and they are included in the $(+1)$-eigenspace and (-1)-eigenspace of σ_β respectively.

- U_{IV} admits a similar decomposition:

$$U_{\mathrm{IV}} = \mathrm{Span}\{S = Z_{\alpha_1} \dots Z_{\alpha_r} v \colon \sigma_\beta \cdot S \neq \pm S, \, \sigma_\beta^2 \cdot S = -S\} =$$

$$= \underbrace{\left(\bigoplus_{S :\ \sigma_\beta^2 S = -S} \mathbb{C}(S - i\, \sigma_\beta \cdot S) \right)}_{U_{\mathrm{IV}_1}} \bigoplus \underbrace{\left(\bigoplus_{S :\ \sigma_\beta^2 S = -S} \mathbb{C}(S + i\, \sigma_\beta \cdot S) \right)}_{U_{\mathrm{IV}_2}} .$$

Both subspaces are stable under σ_β. Indeed they are the $(+i)$-eigenspace and $(-i)$-eigenspace of σ_β respectively.

The description of the decomposition of $F_{\rho'}$ in σ_β-stable subspaces is now complete.

Question. *What happens when we impose the $(\star)$-relations (i.e. we pass to the quotient $F_{\tilde{\rho'}} = F_{\rho'} / R$)?*

Claim 8. *The subspaces U_{II_2} and U_{III_2} have the same image under the projection $\pi \colon F_{\rho'} \to F_{\tilde{\rho'}} = F_{\rho'} / R$.*

Proof. Suppose that $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ is a string of orthogonal roots that lies in U_{III}.

By definition, we have:

$$\sigma_\beta \cdot S \neq \pm S \qquad \sigma_\beta^2 \cdot S = S \qquad \beta \not\perp \{\alpha_1, \ldots, \alpha_r\}.$$

We notice that β does not belong to the set $\{\alpha_1, \ldots, \alpha_r\}$[37], and that there are exactly two roots in this set, say α_{i_1} and α_{i_2}, to which β is not orthogonal. W.l.o.g. we can assume that $i_1 = 1$ and $i_2 = 2$. Hence $\beta \not\perp \alpha_1$, $\beta \not\perp \alpha_2$, but $\beta \perp \alpha_3, \ldots, \alpha_r$.
Let β_1 be a positive root such that the set $\{\pm\,\alpha_1,\, \pm\,\beta,\, \pm\,\beta_1\}$ is a root system of type $\mathcal{A}_2$. Because

$$\sigma_{\alpha_1} \cdot S = -S \qquad \sigma_\beta^2 \cdot S = +S$$

the following relation holds in the quotient space: $[\,S\,] = [\,\sigma_\beta \cdot S,] + [\,\sigma_{\beta_1} \cdot S\,]$.
Therefore we can write:

$$[\,S - \sigma_\beta \cdot S\,] = [\,\sigma_{\beta_1} \cdot S\,] = [\,\sigma_{\beta_1} \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v)\,] =$$

$$= [\pm\ \underbrace{Z_\beta(\mathrm{Ad}(\sigma_{\beta_1}) \cdot Z_{\alpha_2}) Z_{\alpha_3} \cdots Z_{\alpha_r}\, (\sigma_{\beta_1} \cdot v)}_{\text{in } U_{\mathrm{II}_2}}\,].$$

This proves the claim. $\square$

As a consequence, we obtain that:

$$F_{\tilde{\rho'}} = F_\rho \bigoplus \frac{\mathrm{Span}(\,\mathit{strings}\,)}{R} = F_\rho \oplus \underbrace{\mathrm{Span}\{[\,S\,]:\ \sigma_\beta \cdot S = +S\}}_{[\,U_{\mathrm{I}}\,] \subseteq (+1)\text{-eigenspace of } \sigma_\beta} \bigoplus$$

$$\bigoplus \underbrace{\mathrm{Span}\{\,[\,S\,]:\ \sigma_\beta \cdot S = -S;\ \beta \notin \{\alpha_1, \ldots, \alpha_n\}\}}_{[\,U_{\mathrm{II}_1}\,] \subseteq (-1)\text{-eigenspace of } \sigma_\beta} \bigoplus$$

[37] Or $\sigma_\beta \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v) = -Z_{\alpha_1} \ldots Z_{\alpha_r} v$.

$$\bigoplus \underbrace{\operatorname{Span}\{\,[\,S\,]:\ \sigma_\beta \cdot S = -S;\ \beta \in \{\alpha_1, \ldots, \alpha_n\}\}}_{[\,U_{\mathrm{II}_2}\,] \subseteq (-1)\text{-eigenspace of } \sigma_\beta} \bigoplus$$

$$\bigoplus \underbrace{\left(\bigoplus_{S:\ \sigma_\beta^2 S = -S} \mathbb{C}(S - i\,\sigma_\beta \cdot S) \right)}_{[\,U_{\mathrm{IV}_1}\,]} \bigoplus \underbrace{\left(\bigoplus_{S:\ \sigma_\beta^2 S = -S} \mathbb{C}(S + i\,\sigma_\beta \cdot S) \right)}_{[\,U_{\mathrm{IV}_2}\,]} \bigoplus$$

$$\bigoplus \underbrace{\left(\bigoplus_{\substack{S:\ \sigma_\beta S \neq \pm S \\ \sigma_\beta^2 S = +S \\ \beta \not\in \{\alpha_1, \ldots, \alpha_r\}}} \mathbb{C}(S + \sigma_\beta \cdot S) \right)}_{[\,U_{\mathrm{III}_1}\,]}.$$

Recall that, by definition, Z_β acts by 0, $+i$ and $-i$ on the $(+1)$, $(+i)$ and $(-i)$-eigenspace of σ_β. Therefore:

- Z_β only admits the eigenvalue $(+1)$ on $\{v \in F_\rho\colon\ \sigma_\beta \cdot v = +v\} \cup [\,U_{\mathrm{I}}\,] \cup [\,U_{\mathrm{III}_1}\,]$
- Z_β only admits the eigenvalue $(+i)$ on the subspace $[\,U_{\mathrm{IV}_1}\,]$ of $F_{\tilde{\rho}'}$
- Z_β only admits the eigenvalue $(-i)$ on the subspace $[\,U_{\mathrm{IV}_2}\,]$ of $F_{\tilde{\rho}'}$.

We still have to compute the eigenvalues of Z_β on the set

$$\{v \in F_\rho\colon\ \sigma_\beta \cdot v = -v\} \cup [\,U_{\mathrm{II}}\,].$$

Identifying the elements of F_ρ with strings of length zero, we can decompose this set as a direct sum of two-dimensional subspaces of the form

$$<[\,Z_{\alpha_1} \ldots Z_{\alpha_r} v\,],\ [\,Z_\beta Z_{\alpha_1} \ldots Z_{\alpha_r} v]>_{\mathbb{C}}$$

where $\alpha_1, \ldots, \alpha_r$ are mutually orthogonal roots (all orthogonal to β), and $v \in F_\rho$ is a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_r}, \sigma_\beta$.

It is easy to check that Z_β acts on each of these subspace by $\begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$. Hence it has eigenvalues $\pm 2i$.

We conclude that Z_β on $F_{\tilde{\rho}'}$ with eigenvalues $\{0, \pm i, \pm 2i\}$.

9.5.5 Lifting $\tilde{\rho}'$ to K

The purpose of this subsection is to show that the representation $(\tilde{\rho}', F_{\tilde{\rho}'})$ of $\mathrm{Lie}(K) = \mathfrak{so}(n, \mathbb{R})$ can be lifted to a representation of $K = SO(n)$.

We notice that $\tilde{\rho}'$ might be reducible, and that a lifting exists if and only if the highest weight of *each* irreducible summand of $\tilde{\rho}'$ is analytically integral.

Of course, the odd and the even cases should be treated separately. We cover only the even case, the odd one being similar.

Following the notations of chapter 8, we set $\mathfrak{k} = \mathfrak{so}(2n, \mathbb{C})$ and we let $\mathfrak{h}$ be the Cartan subalgebra

$$\mathfrak{h} = H_{\theta_1 \ldots \theta_n} \left\{ \left(\begin{array}{cc|c|cc} 0 & \theta_1 & \ldots & 0 & 0 \\ -\theta_1 & 0 & \ldots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \ldots & 0 & \theta_n \\ 0 & 0 & \ldots & -\theta_n & 0 \end{array} \right) : \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{C} \right\}.$$

For all $j = 1 \ldots n$, we denote by ψ_j the linear functional $\mathfrak{h}\colon H_{\theta_1 \ldots \theta_n} \mapsto -i\,\theta_j$, so that

$$\Psi = \{\pm(\psi_i \pm \psi_{n_j})\}_{i,\, j=1,\ldots,\, n,\, i<j}$$

is the root system of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$, and

$$\Delta^+ = \{\psi_i \pm \psi_j\}_{i,\,j=1,\dots,\,n,\,i<j}$$

is a choice of positive roots.

We say that an element $\lambda = a_1\psi_1 + \cdots + a_{n-1}\psi_{n-1} + a_n\psi_n$ of $\mathfrak{h}^*$ is

- *algebraically integral* if $a_j \in \mathbb{Z}$, $\forall\, j = 1 \dots n$, or $a_j \in \left(\mathbb{Z} + \frac{1}{2}\right)$, $\forall\, j$
- *analytically integral* if $a_1, \dots, a_{n-1}, a_n \in \mathbb{Z}$
- *dominant* if $a_1 \geqslant \cdots \geqslant a_{n-1} \geqslant \mid a_n \mid$.

Let $\upsilon = b_1\psi_1 + \cdots + b_n\psi_n$ be the highest weight of an irreducible summand μ_υ of $\tilde{\rho'}$, and let y_υ be a weight vector of weight υ. We notice that, for all $j = 1 \dots n$, the element

$$Z^j \equiv Z_{\psi_{2j-1}-\psi_{2j}} = E_{2j-1,2j} - E_{2j,2j-1}$$

belongs to the Cartan subalgebra $\mathfrak{h}$, and

$$\mu_\upsilon(Z^j) \cdot y_\upsilon = \left(\textstyle\sum_{k=1}^n b_k\psi_k(Z^j)\right) y_\upsilon = (b_j\psi_j(Z^j))y_\upsilon = (-ib_j)y_\upsilon.$$

So there exists a positive root $\beta_j = \psi_{2j-1} - \psi_{2j}$ such that $-ib_j$ is an eigenvalue of $\mu_\upsilon(Z_{\beta_j})$. Because the eigenvalues of Z_β on $F_{\tilde{\rho'}}$ lie in the set $\{0, \pm i, \pm 2i\}$, we conclude that b_j belongs to $\{0, \pm 1, \pm 2\}$. Therefore, υ has integer coefficients, and it is analytically integral.

9.5.6 It is an extension of ρ

Let (Θ, F_Θ) be the representation of K whose differential is $\tilde{\rho'}$. Then $F_\Theta = F_{\tilde{\rho'}}$ contains the space F_ρ on which the original Weyl group representation ρ is defined.[38] Let us verify that

$$Res^K_{M'}\Theta \mid_{F_\rho} = \rho.$$

[38] Because all the relations are imposed at the level of strings of *positive* length.

Because M' is generated by the elements $\{\sigma_\alpha\}_{\alpha\in\Delta^+}$, it is enough to prove that

$$\Theta(\sigma_\alpha) = \exp\left(\frac{\pi}{2}\,\tilde{\rho}'(Z_\alpha)\right)(v) = \rho(\sigma_\alpha)(v) \qquad (\circledast)$$

for all positive roots α and all v in F_ρ.

For any fixed α, we just need to check that $(\circledast)$ holds for v in a basis of F_ρ and because σ_α acts semi-simply on F_ρ, we can pick a basis consisting of $(+1)$- and(-1)-eigenvectors of $\rho(\sigma_\alpha)$:

$$\mathcal{B} = \{\underbrace{u_1,\, u_2, \dots,\, u_t}_{+1\, eig.s},\, \underbrace{v_1,\, v_2, \dots,\, v_s}_{-1\, eig.s}\}.$$

Then, we are reduced to show that $\Theta(\sigma_\alpha)(u_i) = +\,u_i$ and $\Theta(\sigma_\alpha)(v_j) = -v_j$, for all $i = 1\dots t$ and all $j = 1\dots r$.

Let V be the subspace of $F_{\tilde{\rho}'}$ defined by:

$$V = \mathrm{Span}(u_1,\, u_2, \dots,\, u_t,\, v_1,\, v_2, \dots,\, v_s,\, Z_\alpha v_1,\, Z_\alpha v_2, \dots,\, Z_\alpha v_s).$$

The strings $Z_\alpha v_1 \dots Z_\alpha v_s$ are well defined and they are linearly independent[39], so V has dimension $2s + t$. Because, by definition,

- $Z_\alpha \cdot u_i = 0 \quad \forall\, i = 1\dots t$
- $Z_\alpha \cdot v_j = Z_\alpha v_j \quad \forall\, j = 1\dots s$
- $Z_\alpha \cdot (Z_\alpha v_j) = -4v_j \quad \forall\, j = 1\dots s$

[39]There is no relation among them.

the action of Z_α on V is described by the matrix:

$$\tilde{\rho}'(Z_\alpha)\mid_V \rightsquigarrow \left(\begin{array}{c|c|c} O_{r\times r} & O_{s\times s} & O_{s\times s} \\ \hline O_{s\times r} & O_{s\times s} & -4I_{s\times s} \\ \hline O_{s\times r} & I_{s\times s} & O_{s\times s} \end{array}\right)$$

with the first two blocks corresponding to F_ρ. We obtain:

$$\Theta(\sigma_\alpha) = \exp\left(\frac{\pi}{2}\tilde{\rho}'(Z_\alpha)\right)\mid_V \rightsquigarrow \left(\begin{array}{c|c|c} I_{r\times r} & O_{s\times s} & O_{s\times s} \\ \hline O_{s\times r} & -I_{s\times s} & O_{s\times s} \\ \hline O_{s\times r} & O_{s\times s} & -I_{s\times s} \end{array}\right).$$

In particular, $\Theta(\sigma_\alpha)$ acts on F_ρ via:

$$\Theta(\sigma_\alpha)\mid_{F_\rho} \rightsquigarrow \left(\begin{array}{c|c} I_{r\times r} & O_{s\times s} \\ \hline O_{s\times r} & -I_{s\times s} \end{array}\right)$$

and this shows that $\Theta(\sigma_\alpha)\mid_{F_\rho} = \rho$.

9.6 Conclusions

For every representation (ρ, F_ρ) of the Weyl group of $SL(n, \mathbb{R})$ whose restriction to any $W(SL(3))$ does not contain the sign representation, we have constructed a finite-dimensional representation $(\tilde{\rho}', F_{\tilde{\rho}'})$ of $SO(n)$ with the following properties:

- for each root α of $SL(n)$, the subgroup $K^\alpha \simeq SO(2)$ only acts with characters zero, plus or minus one, plus or minus two (i.e. $\tilde{\rho}'$ is *petite*)

- the space of M-fixed vectors in $\tilde{\rho}'$ is non-zero (i.e. $\tilde{\rho}'$ is *spherical*)
- ρ is a submodule of the representation of $W(SL(n))$ on the space of M-fixed vectors of $\tilde{\rho}'$.

Some natural questions arise:

◇ Is $\tilde{\rho}'$ irreducible?

◇ Is $(F_{\tilde{\rho}'})^M = \rho$?

◇ What is the highest weight decomposition of $\tilde{\rho}'$?
In particular, how does $\tilde{\rho}'$ fit in the classification of petite spherical K-types described in chapter 8 (for n even)?

We will answer these questions in the next chapter.

Chapter 10

Further results and examples

In the first section, we descibe the petite representation $(\tilde{\rho},\, F_{\tilde{\rho'}})$ of $SO(2n,\mathbb{R})$ that arises from an irreducible representation $(\rho,\, F_\rho)$ of the Weyl group through the construction described in the previous chapter.

As usual, we identify the Weyl group of $SL(2n)$ with the symmetric group in $2n$ letters, and the irreducible representations of W with partitions of $2n$. The main results are the following:

1. For every representation ρ of $\mathcal{S}_{2n}$, we compute the highest weight decomposition of the petite K-type $F_{\tilde{\rho'}}$:

   ◇ if $\rho = (2n)$, then $F_{\tilde{\rho'}} = L(0\psi_1 + 0\psi_2 + \cdots + 0\psi_n)$

   ◇ if $\rho = (2n-k, k)$, then $F_{\tilde{\rho'}} = L(2\psi_1 + 2\psi_2 + \cdots + 2\psi_k)$ for all $0 < k < n$

   ◇ if $\rho = (n, n)$, then $F_{\tilde{\rho'}} = L(2\psi_1 + \cdots + 2\psi_{n-1} - 2\psi_n) \oplus L(2\psi_1 + \cdots + 2\psi_{n-1} + 2\psi_n)$.

2. For every representation ρ of $\mathcal{S}_{2n}$, we compute the Weyl group representation on the the space of M-invariants of $F_{\tilde{\rho'}}$:

   ◇ if $\rho = (2n)$, then $\left(F_{\tilde{\rho'}}\right)^M = (2n)$

◇ if $\rho = (n-k, k)$, then $\left(F_{\tilde{\rho'}}\right)^M = (2n-k, k)$
for all $0 < k < n$

◇ if $\rho = (n, n)$, then $\left(F_{\tilde{\rho'}}\right)^M = (n, n) \oplus (n, n)$.

The result for $SO(2n+1)$ is even simpler, because the representation $F_{\tilde{\rho'}}$ is always irreducible:

◇ if $\rho = (2n+1)$, then $F_{\tilde{\rho'}} = L(0\psi_1 + 0\psi_2 + \cdots + 0\psi_n)$

◇ if $\rho = (2n+1-k, k)$, then $F_{\tilde{\rho'}} = L(2\psi_1 + 2\psi_2 + \cdots + 2\psi_k)$
for all $0 < k \leq n$

and

◇ if $\rho = (2n+1)$, then $\left(F_{\tilde{\rho'}}\right)^M = (2n+1)$

◇ if $\rho = (2n+1-k, k)$, then $\left(F_{\tilde{\rho'}}\right)^M = (2n+1-k, k)$
for all $0 < k \leq n$.

In the second section we provide a number of examples. The construction is explicitly carried out for partitions (in at most two parts) of $n = 3$, $n = 4$, $n = 5$ and $n = 6$. These examples give a useful insight on the general case. In the last section we construct the extension for the non-spherical representations of $M' \subseteq SL(3)$.

10.1 The highest weight decomposition of $F_{\tilde{\rho'}}$

Throughout the section $W = \mathcal{S}_{2n}$ is the Weyl group of $SL(2n, \mathbb{R})$, ρ is an irreducible representation of W (i.e. a partition of $2n$) and $F_{\tilde{\rho'}}$ is its extension to a petite K-type. We want to determine the highest weight decomposition of $F_{\tilde{\rho'}}$.

The case $\rho = (2n)$

When ρ is the trivial representation of W, i.e. $\rho = (2n)$, a root-reflection does not have any (-1)-eigenvector. So the extension $F_\rho \rightsquigarrow F_{\tilde{\rho'}}$ is trivial: $F_{\tilde{\rho'}} = F_\rho$. It is easy to see that $\tilde{\rho}$ is the trivial representation of $SO(2n)$, and it has highest weight $0\psi_1 + 0\psi_2 + \cdots + 0\psi_n$:

$$\boxed{\rho = (2n) \Rightarrow F_{\tilde{\rho'}} = L(0\psi_1 + 0\psi_2 + \cdots + 0\psi_n)}.$$

The case $\rho = (2n-k, k)$, for $1 \leq k < n$

We will show that $F_{\tilde{\rho'}}$ is irreducible, and equal to the irreducible representation of $SO(2n)$ of highest weight $2\psi_1 + 2\psi + \cdots + 2\psi_k$:

$$\boxed{\rho = (2n-k, k) \Rightarrow F_{\tilde{\rho'}} = L(2\psi_1 + 2\psi + \cdots + 2\psi_k)}.$$

For clarity, we split the argument it in several steps:

Lemma 10. *For $\rho = (2n-k, k)$, with $k = 1, \ldots, n-1$, the representation of the Weyl group on the space of M-invariants in $F_{\tilde{\rho'}}$ is irreducible and equal to $(2n-k, k)$:*

$$\rho = (2n-k) \Rightarrow \left(F_{\tilde{\rho'}}\right)^M = F_{(2n-k,k)}.$$

Lemma 11. *If $\rho = (2n-k, k)$, with $k = 1, \ldots, n-1$, then $F_{\tilde{\rho'}} = L(2\psi_1 + 2\psi_2 + \cdots + 2\psi_k)$.*

Corollary 13. *If $\rho = (2n-k, k)$, with $k = 1, \ldots, n-1$, then $F_{\tilde{\rho'}}$ contains a weight vector of weight $2\psi_1 + 2\psi_2 + \cdots + 2\psi_k$.*

$\star\star\star$

Proof of lemma 10. We set $\rho = (2n-k, k)$ and we look at the the space of M-invariants in $F_{\rho'}$, which is the vector space generated by F_ρ and all the strings of orthogonal roots.[1] The first step is to observe that every string defines a one-dimensional representation of M. Because M is generated by the the elements m_βs, with β varying in the set of positive roots, it is sufficient to prove that each m_β acts on a string by a scalar. This is easy to do.

For all roots α and β (of $SL(2n)$), we have:

$$\mathrm{Ad}(m_\beta)(Z_\alpha) = \epsilon_{\alpha,\beta} Z_\alpha$$

with $\epsilon_{\alpha,\beta} = +1$ if $\alpha = \beta$ or $\alpha \perp \beta$, and $\epsilon_{\alpha,\beta} = -1$ otherwise.[2] Hence

$$m_\beta \cdot (Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v) = (\epsilon_{\alpha_1,\beta} Z_{\alpha_1})(\epsilon_{\alpha_2,\beta} Z_{\alpha_2}) \dots (\epsilon_{\alpha_r,\beta} Z_{\alpha_r})(m_\beta \cdot v) =$$

$$= \epsilon_{\alpha_1,\beta}\epsilon_{\alpha_2,\beta} \cdots \epsilon_{\alpha_r,\beta}(Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v) = \pm(Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v).$$

The second step is to understand whether the one-dimensional representation of M spanned by a root is the trivial M-type.

We write $\alpha_l = \varepsilon_{i_l} - \varepsilon_{j_l}$, for all $l = 1 \dots r$ and notice that being $r \leq k < n$, the set $I = \{i_l, \dots, i_r, j_l, \dots, j_r\}$ is properly contained in $\{1, 2, \dots, 2n\}$. Hence we can pick an index s in $[1, 2n] - I$. By construction, $\beta = \pm(\varepsilon_{i_1} - \varepsilon_s)$ is orthogonal to $\alpha_2, \dots, \alpha_r$; it is different from α_1 and not orthogonal to it. Therefore

$$m_\beta \cdot (Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v) = -(Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v).$$

This shows that for all $r = 1 \dots k$, $\mathbb{C}(Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v)$ is *not* the trivial M-type. Therefore

$$\rho = (2n - k) \;\Rightarrow\; (F_{\rho'})^M = F_{(2n-k,k)}.$$

[1] To simplify notations, we are identifying a string with its equivalence class modulo the linearity and orthogonality relations. At the stage, there are no $(\star)$-relations.

[2] $m_\beta = \sigma_\beta^2$ is the square a representative in M' for the root reflection β.

Passing to the quotient modulo the ($\star$)-relations does not introduce additional trivial M-types, so $(F_{\tilde{\rho'}})^M$ also coincides with F_ρ. We conclude that the representation of the Weyl group on $(F_{\tilde{\rho'}})^M$ is irreducible, and equal to $(2n-k,k)$. $\square$

$\star\star\star$

Proof of lemma 11. We set $\rho = (2n-k,k)$ and we look at the decomposition of $F_{\tilde{\rho'}}$ in irreducible summands. Because $F_{\tilde{\rho'}}$ is a petite representation of $SO(2n)$, this decomposition can only include the K-types with highest weight[3]

- $0\psi_1 + \cdots + 0\psi_n$
- $1\psi_1 + \cdots + 1\psi_s \quad 0 < s < n$
- $1\psi_1 + \cdots + 1\psi_{n-1} \pm 1\psi_n$
- $2\psi_1 + \cdots + 2\psi_s \quad 0 < s < n$
- $2\psi_1 + \cdots + 2\psi_a + 1\psi_{a+1} + \cdots + 1\psi_b \quad 0 < a,\, b < n$
- $2\psi_1 + \cdots + 2\psi_k + 1\psi_{k+1} + \cdots + 1\psi_{n-1} \pm 1\psi_n \quad 0 < k < n$
- $2\psi_1 + \cdots + 2\psi_{n-1} \pm 2\psi_n$.

Therefore, we can write

$$F_{\tilde{\rho'}} = L(0)^{\oplus m_0} \oplus \left(\bigoplus_{s=0}^{n-1} L(2\psi_1 + \cdots + 2\psi_s)^{\oplus m_s} \right) \oplus L(2\psi_1 + \cdots + 2\psi_{n-1} + 2\psi_n)^{\oplus m_n^+} \oplus$$

$$\oplus L(2\psi_1 + \cdots + 2\psi_{n-1} - 2\psi_n)^{\oplus m_n^-} \oplus \text{ some non spherical } K\text{-types.}$$

We decompose $(F_{\tilde{\rho'}})^M$ accordingly:

$$(F_{\tilde{\rho'}})^M = (2n)^{\oplus m_0} \oplus \left(\bigoplus_{s=1}^{n-1} (2n-s,s)^{\oplus m_s} \right) \oplus (n,n)^{\oplus (m_n^+ + m_n^-)}.$$

[3]Please, refer to chapter 8 for a classification of the petite spherical representations of $SO(2n)$, and an analysis of the corresponding representation of the Weyl group on the space of M-fixed vectors.

By the previous lemma $(F_{\tilde{\rho'}})^M = (2n-k,k)$, so

$$m_n^+ = m_n^- = m_j = 0 \quad \forall j \in \{0,\, 1, \dots n-1\} - \{k\}$$

and $m_k = 1$. We obtain:

$$F_{\tilde{\rho'}} = L(2\psi_1 + \cdots + 2\psi_k) \oplus \text{ some non spherical } K\text{-types.}$$

The next step is to show that $F_{\tilde{\rho'}}$ does not contain any non-spherical K-type, and to do this we need a better understanding of the structure of $F_{\tilde{\rho'}}$.
$F_{\tilde{\rho'}}$ is spanned by F_ρ and all the strings of the form

$$[Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v]$$

where $\alpha_1, \dots, \alpha_r$ are mutually orthogonal positive roots and v is a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_r}$ in F_ρ. Because

$$[Z_{\alpha_1} Z_{\alpha_2} \dots Z_{\alpha_r} v] = Z_{\alpha_1} \cdot (Z_{\alpha_2} \cdot (\dots (Z_{\alpha_r} \cdot [v]) \cdots))$$

we conclude that, as a representation of $\mathrm{Lie}(K)$, $F_{\tilde{\rho'}}$ is generated by the elements of F_ρ. We write $F_{\tilde{\rho'}} = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ for the decomposition of $F_{\tilde{\rho'}}$ in isotypic components of K-types (with $V_1 = L(2\psi_1 + \cdots + 2\psi_k)$), and we identify each V_j with the image of the projection

$$\pi_j \colon F_{\tilde{\rho'}} \to \frac{F_{\tilde{\rho'}}}{\bigoplus_{i \in \{1,\dots,t\} - \{j\}} V_i}.$$

By construction, V_j is generated by

$$\pi_j(F_\rho) = \frac{F_\rho}{F_\rho \cap \left(\bigoplus_{i\in\{1,\ldots,t\}-\{j\}} V_i\right)}$$

and because V_i is non-spherical for all $i \neq 1$, we get:

- $F_\rho \cap \left(\bigoplus_{i\in\{1,\ldots,t\}-\{j\}} V_i\right) = \{0\} \quad$ if $j = 1$
- $F_\rho \cap \left(\bigoplus_{i\in\{1,\ldots,t\}-\{j\}} V_i\right) = F_\rho \quad$ if $j \neq 1$.

Therefore $\pi_j(F_\rho) = F_\rho$ if $j = 1$, and $\pi_j(F_\rho) = \{0\}$ if $j \neq 1$.

It follows that $V_j = \{0\}$ for all $j = 2, \ldots, t$, hence $F_{\tilde{\rho'}} = L(2\psi_1 + \cdots + 2\psi_k)$. □

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Proof of corollary 13. Set $\rho = (2n-k, k)$. By the previous lemma, $F_{\tilde{\rho'}}$ contains a weight vector z of weight $2\psi_1 + 2\psi_2 + \cdots + 2\psi_k$. The following argument is meant to provide an explicit construction for z.

Let $v = \underline{e}_T$ be the standard polytabloid associated to the standard tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \begin{array}{c} \cdots \\ \cdots \end{array} \begin{array}{|c|c|c|c|c|}\hline \text{2k-3} & \text{2k-1} & \text{2k+1} & \text{2k+2} & \text{2k+3} \\ \hline \text{2k-2} & \text{2k} \\ \cline{1-2} \end{array} \cdots \begin{array}{|c|c|}\hline \text{2n-1} & \text{2n} \\ \hline \end{array}\,.$$

We notice that v is a simultaneous (-1)-eigenvector in F_ρ for the transpositions

$$(1,2) \quad (3,4) \quad (5,6) \ldots (2k-3, 2k-2) \quad (2k-1, 2k)$$

and a simultaneous $(+1)$-eigenvector for the transpositions

$$(2k+1, 2k+2) \quad (2k+3, 2k+4) \ldots (2n-1, 2n).$$

The string $Z_{(\varepsilon_1-\varepsilon_2)}Z_{(\varepsilon_3-\varepsilon_4)}\cdots Z_{(\varepsilon_{2k-1}-\varepsilon_{2k})}v$ is therefore well defined. So is any string of the form

$$Z_{(\varepsilon_{2j_1-1}-\varepsilon_{2j_1})}Z_{(\varepsilon_{2j_2-1}-\varepsilon_{2j_2})}\cdots Z_{(\varepsilon_{2j_s-1}-\varepsilon_{2j_s})}v$$

for $1 \le j_1 < j_2 < \cdots < j_s \le k$. For brevity of notations, we set $Z_{(\varepsilon_{2m-1}-\varepsilon_{2m})} = Z_m$, for all integers $m = 1 \ldots n$. We also set[4]

$$z = \sum_{s=0}^{k} (-2i)^{k-s} \left(\sum_{1\le j_1<j_2<\cdots<j_s\le k} Z_{j_1}Z_{j_2}\cdots Z_{j_s}v \right).$$

With abuse of notations, we denote the equivalence class of z in the quotient space $F_{\tilde{\rho}'}$ with the same symbol. We claim that $z \in F_{\tilde{\rho}'}$ is a weight vector for the Cartan subalgebra

$$\mathfrak{h} = \left\{ \left(\begin{array}{cc|c|cc} 0 & \theta_1 & \ldots & 0 & 0 \\ -\theta_1 & 0 & \ldots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \ldots & 0 & \theta_n \\ 0 & 0 & \ldots & -\theta_n & 0 \end{array} \right) : \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{C} \right\}$$

of $= \mathfrak{so}(2n, \mathbb{C})$, of weight $2\psi_1 + 2\psi_2 + \cdots + 2\psi_k$. Indeed, the element $H_m = E_{2m-1,2m} - E_{2m,2m-1}$ of $\mathfrak{h}$ acts on z by $-2i$ if $m \le k$, and by 0 if $k < m \le n$. Because $H_m = Z_{\varepsilon_{2m-1}-\varepsilon_{2m}}$, to compute the action of H_m on a string $Z_{j_1}Z_{j_2}\cdots Z_{j_s}v$ we must first understand how the reflection $\sigma_{(\varepsilon_{2m-1}-\varepsilon_{2m})}$ acts on it. This is easy[5]:

$$\sigma_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot (Z_{j_1}Z_{j_2}\cdots Z_{j_s}v) = Z_{j_1}Z_{j_2}\cdots Z_{j_s}\left(\sigma_{(\varepsilon_{2m-1}-\varepsilon_{2m})}\cdot v\right) =$$

[4] A string $Z_{j_1}Z_{j_2}\cdots Z_{j_s}v$ is intended to be equal to v when $s = 0$.

[5] Because $\mathrm{Ad}(\sigma_{\varepsilon_{2m-1}-\varepsilon_{2m}})(Z_j) = \mathrm{Ad}(\sigma_{\varepsilon_{2m-1}-\varepsilon_{2m}})(Z_{(\varepsilon_{2j-1}-\varepsilon_{2j})}) = Z_{(\varepsilon_{2j-1}-\varepsilon_{2j})} = Z_j$ for all j.

$$= \begin{cases} -Z_{j_1} Z_{j_2} \cdots Z_{j_s} v & \text{if } m \leq k \\ \\ +Z_{j_1} Z_{j_2} \cdots Z_{j_s} v & \text{if } k < m \leq n. \end{cases}$$

It follows that:

$$Z_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot (Z_{j_1} \cdots Z_{j_r} \cdots Z_{j_s} v) = \begin{cases} Z_m Z_{j_1} \cdots Z_{j_r} \cdots Z_{j_s} v & \text{if } m \leq k \text{ and } m \notin \{j_1 \ldots j_s\} \\ \\ -4 Z_{j_1} \cdots \check{Z}_{j_r} \cdots Z_{j_s} v & \text{if } m \leq k \text{ and } m = j_r \\ \\ 0 & \text{if } m > k\,. \end{cases}$$

As a result, we obtain that $Z_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot z = 0$ for all $m > k$, and that the vectors

$$u_{m,0} \equiv (-2i)v + Z_m v$$

$$u_{m,s,\underline{j}} \equiv (-2i) Z_{j_1} Z_{j_2} \cdots Z_{j_{s-1}} v + Z_m Z_{j_1} Z_{j_2} \cdots Z_{j_{s-1}} v$$

are $(-2i)$-eigenvectors of $Z_{\varepsilon_{2m-1}-\varepsilon_{2m}}$, for all $m \leq k$, for all $1 \leq s < k$ and for every multi-index $\underline{j}$ of length $s-1$ with entries in $\{1, \ldots, k\} - m$.

Because

$$z = (-2i)^{k-1} u_{m,0} + \sum_{s=1}^{k-1} (-2i)^{k-s} \left(\sum_{\underline{j}} u_{m,s,\underline{j}} \right)$$

also z is a $(-2i)$-eigenvector of $Z_{\varepsilon_{2m-1}-\varepsilon_{2m}}$, for all $m \leq k$. This ends the proof. $\square$

The case $\rho = (n,n)$

When $\rho = (n,n)$, the representation $F_{\tilde{\rho'}}$ fails to be irreducible:

$$\boxed{\rho = (n,n) \;\Rightarrow\; F_{\tilde{\rho'}} = L(2\psi_1 + \cdots + 2\psi_{n-1} - 2\psi_n) \oplus L(2\psi_1 + \cdots + 2\psi_{n-1} + 2\psi_n)}\,.$$

As a representation of the Weyl group, $(F_{\tilde{\rho}'})^M$ consists of two copies of (n,n).

As usual, it is convenient to split the proof in several parts.

Lemma 12. *$F_{\tilde{\rho}'}$ contains a weight vector of weight $2\psi_1+\cdots+2\psi_{n-1}+2\psi_n$, and also a weight vector of weight $2\psi_1+\cdots+2\psi_{n-1}-2\psi_n$.*

Lemma 13. *$F_{\tilde{\rho}'} \supseteq L(2\psi_1+\cdots+2\psi_{n-1}-2\psi_n)\oplus L(2\psi_1+\cdots+2\psi_{n-1}+2\psi_n)$.*

Lemma 14. *As a representation of the Weyl group, $\left(F_{\tilde{\rho}'}\right)^M = (n,n)\oplus(n,n)$.*

Corollary 14. *$F_{\tilde{\rho}'} \equiv L(2\psi_1+\cdots+2\psi_{n-1}-2\psi_n)\oplus L(2\psi_1+\cdots+2\psi_{n-1}+2\psi_n)$.*

Proof of lemma 12. The existence of a weight vector z of weight $2\psi_1+2\psi_2+\cdots+2\psi_{n-1}+2\psi_n$ is proven just like in the case $\rho=(2n-k,k)$ for $k<n$. The vector z is explicitly given by:

$$z=\sum_{s=0}^{n}(-2i)^{n-s}\left(\sum_{1\le j_1<j_2<\cdots<j_s\le n} Z_{j_1}Z_{j_2}\cdots Z_{j_s}v\right)$$

with $v=\underline{e}_T$ the standard polytabloid associated to the standard tableau

$$T=\begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 5 & \cdots & \text{2n-3} & \text{2n-1}\\ \hline 2 & 4 & 6 & \cdots & \text{2n-2} & \text{2n}\\ \hline\end{array}$$

and with $Z_m=Z_{(\varepsilon_{2m-1}-\varepsilon_{2m})}$, for all integers $m=1\ldots n$.[6]

We now construct a weight vector of weight $2\psi_1+2\psi_2+\cdots+2\psi_{n-1}-2\psi_n$. By the same argument used before, the vector

$$x=\sum_{s=0}^{n-1}(-2i)^{n-1-s}\left(\sum_{1\le j_1<j_2<\cdots<j_s\le n-1} Z_{j_1}Z_{j_2}\cdots Z_{j_s}v\right)$$

[6]As usual, a string $Z_{j_1}Z_{j_2}\cdots Z_{j_s}v$ is intended to be equal to v when $s=0$.

is a simultaneous $(-2i)$-eigenvector of

$$H_m = E_{2m-1,2m} - E_{2m,2m-1} = Z_{\varepsilon_{2m-1}-\varepsilon_{2m}}$$

for all $m = 1, \ldots, n-1$. We also introduce the element

$$y = Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot x + (2i)x$$

which is an eigenvector of $H_n = Z_{\varepsilon_{2n-1}-\varepsilon_{2n}}$ of eigenvalue $(+2i)$:[7]

$$Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot y = -4x + (2i)Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot x = (2i)\left(Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot x + (2i)x\right).$$

We notice that y is also a $(-2i)$-eigenvector of $H_1, H_2, \ldots, H_{n-1}$:

$$Z_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot y = Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot \left(Z_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot x\right) + (2i)\left(Z_{\varepsilon_{2m-1}-\varepsilon_{2m}} \cdot x\right) =$$

$$= (-2i)[\, Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot x \, + 2i\, x] = (-2i)y \qquad \forall\, m = 1, \ldots, n-1.$$

Therefore, y is a weight vector for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2n, \mathbb{C})$, of weight $2\psi_1 + 2\varepsilon_2 + \cdots + 2\psi_{n-1} - 2\psi_n$. □

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Proof of lemma 13. Let $\rho = (n, n)$. As seen in the previous lemma, $\tilde{\rho}'$ contains both

[7]Notice that x only involves strings in the roots $\alpha_1 = \psi_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_3 - \varepsilon_4 \ldots \alpha_{n-1} = \varepsilon_{2n-3} - \varepsilon_{2n-2}$. Being all these roots orthogonal to $\varepsilon_{2n-1} - \varepsilon_{2n}$, we have

$$Z_{\varepsilon_{2n-1}-\varepsilon_{2n}} \cdot x = \sum_{s=0}^{n-1} (-2i)^{n-1-s} \left(\sum_{1 \le j_1 < j_2 < \cdots < j_s \le n-1} Z_m Z_{j_1} Z_{j_2} \cdots Z_{j_s} v \right)$$

.

weights

$$\upsilon_n^- = 2\psi_1 + 2\varepsilon_2 + \cdots + 2\psi_{n-1} - 2\psi_n$$

$$\upsilon_n^+ = 2\psi_1 + 2\varepsilon_2 + \cdots + 2\psi_{n-1} + 2\psi_n.$$

The purpose of this lemma is to show that $F_{\tilde{\rho}'}$ actually contains the irreducible representation with highest weight υ_n^- and υ_n^+. More precisely, we will prove that if an irreducible submodule ϕ of $F_{\tilde{\rho}'}$ contains the weight $\upsilon_n^\pm$, then $\phi \equiv L(\upsilon_n^\pm)$.

We start with υ_n^+. We assume that $\phi \subseteq F_{\tilde{\rho}'}$ is irreducible and contains the weight υ_n^+, and we write

$$\lambda = b_1\psi_1 + b_2\varepsilon_2 + \cdots + b_n\psi_n$$

for the highest weight of ϕ.[8] Then

- $b_1 \geqq 2$, because the coefficient of ψ_1 in

  $$\lambda - \upsilon_n^+ = (b_1 - 2)\psi_1 + (b_2 - 2)\psi_2 + \cdots + (b_n - 2)\psi_n$$

  must be non negative.[9]

- $b_1 \leq 2$, because $\phi \subseteq F_{\tilde{\rho}'}$ is petite, hence the coefficients of any weight of ϕ cannot exceed 2.[10]

We deduce that $b_1 = 2$, and that

$$\lambda - \upsilon_n^+ = (b_2 - 2)\psi_2 + (b_3 - 2)\psi_3 + \cdots + (b_n - 2)\psi_n$$

is a sum of positive roots. Iterate this argument to show that $b_j = 2$ for all $j = 1, \ldots, n-1$. Finally, we obtain that $(b_n - 2)\psi_n$ is a sum of positive roots, and of

[8] Of course $b_1 \geq b_2 \cdots \geq b_{n-1} \geq | b_n |$.

[9] Since $\upsilon_k \preceq \lambda$, the difference $\lambda - \upsilon_n^+$ is a sum of positive roots. As usual, we choose the positive system: $\{\psi_i \pm \psi_j\}_{\{}i < j\}$, so the first non-zero coefficient in a sum of positive roots must be positive.

[10] By construction, $-ib_1$ is an eigenvalue of $H_1 = Z_{\psi_1 - \psi_2}$.

course this can only happen if $b_n = 2$, so $\lambda = \upsilon_n^+$.

The argument for υ_n^- is similar. Indeed,

$$\lambda - \upsilon_n^- = (b_1 - 2)\psi_1 + (b_2 - 2)\psi_2 + \cdots + (b_{n-1} - 2)\psi_{n-1} + (b_n + 2)\psi_n$$

can only be a sum of positive roots if $b_j = 2$ for all $j = 1, \ldots, n-1$ and $b_n = -2$. $\square$

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Proof of lemma 14. We fix $\rho = (n, n)$ and show that the representation of the Weyl group on $(F_{\tilde{\rho}'})^M$ is equal to the direct sum of two copies of (n, n).

We apply the same argument used in the proof of lemma 10, and show that the equivalence class of a string of orthogonal roots of length $1 < r < n-1$ spans a one-dimensional irreducible representation of M which is *not* equal to the trivial representation. Hence, we just need to consider strings of length n.

If β is an arbitrary root and $Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_n} v$ is a string of orthogonal roots (of length n), two possibilities can occur:

1. $\beta = \alpha_j$ for some $j = 1, \ldots, n$, and $\beta \perp \alpha_i$ for all $i \neq j$
2. there are exactly two roots, say α_1 and α_2, to which β is not orthogonal (so $\beta \perp \alpha_i$ for all $i > 2$.

In the first case, m_β acts trivially on all the Z_{α_i}s, hence

$$m_\beta \cdot (Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_n} v) = +Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_n} v.$$

In the second case we have

$$\mathrm{Ad}(m_\beta)\,(Z_{\alpha_1}) = -\,(Z_{\alpha_1}) \qquad \mathrm{Ad}(m_\beta)\,(Z_{\alpha_2}) = -\,(Z_{\alpha_2})$$

and $\mathrm{Ad}(m_\beta)\,(Z_{\alpha_i}) = +\,(Z_{\alpha_i})$ for all $i = 3, \ldots, n$.

Once again

$$m_\beta \cdot (Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_n} v) = +Z_{\alpha_1} Z_{\alpha_2} \ldots Z_{\alpha_n} v.$$

This proves that M acts trivially on every string of length n. Therefore

$$\rho = (n,n) \;\Rightarrow\; \left(F_{\rho'}\right)^M = F_\rho \oplus \mathrm{Span}\{n\text{-}strings\}.$$

Passing to the quotient does not introduce additional trivial M-types, so

$$\left(F_{\tilde{\rho'}}\right)^M = F_\rho \oplus \mathrm{Span}\{[\,n\text{-}strings\,]\}.$$

To conclude the proof we must show that the representation of the Weyl group on the set of linear combinations of equivalence classes of n-strings is isomorphic to (n,n). We will make use of the following theorem:[11]

Theorem 18. *Let S^λ be the Specht module associated to a partition λ. Then S^λ is the vector space with generators the polytabloids $\underline{e}_T$, as T varies over all the tableaux of shape λ, and relations of the form $\underline{e}_T - \sum_S \underline{e}_S$, where the sum is over all S obtained from T by exchanging the top k elements of one column with any k elements of the preceding column, maintaining the vertical order of each set exchanged.*
There is one such relation for each tableau T, each choice of adjacent columns, and each k at least equal to the length of the shorter column.

We intend to apply these remarks to the Specht module $S^{(n,n)}$, which is equal to F_ρ. A set of generators for $S^{(n,n)}$ consists of the polytabloids $\underline{e}_T$ associated to *all* the tableaux of the form

$$T = \begin{array}{|c|c|c|c|c|} \hline j_1 & j_3 & \cdots & j_{2n-3} & j_{2n-1} \\ \hline j_2 & j_4 & \cdots & j_{2n-2} & j_{2n} \\ \hline \end{array} \quad \text{with } \{j_1 \ldots j_{2n}\} = \{1 \ldots 2n\}.$$

[11] See [13], §7.4.

Let us explore the relations among these polytabloids. When $k = 1$ and the adjacent columns under consideration are the first and second one, we can only take

- $S = \begin{array}{|c|c|c|c|c|}\hline j_3 & j_1 & \cdots & j_{2n-3} & j_{2n-1} \\ \hline j_2 & j_4 & \cdots & j_{2n-2} & j_{2n} \\ \hline\end{array} = \sigma_{\varepsilon_{j_1}-\varepsilon_{j_3}} \cdot T$, or

- $S = \begin{array}{|c|c|c|c|c|}\hline j_1 & j_2 & \cdots & j_{2n-3} & j_{2n-1} \\ \hline j_3 & j_4 & \cdots & j_{2n-2} & j_{2n} \\ \hline\end{array} = \sigma_{\varepsilon_{j_2}-\varepsilon_{j_3}} \cdot T.$

We obtain the relation:

$$\underline{e}_{\,T} = \underline{e}_{\,[\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_3})}\cdot T\,]} + \underline{e}_{\,[\sigma_{(\varepsilon_{j_2}-\varepsilon_{j_3})}\cdot T\,]}$$

which we can rewrite as:[12]

$$\left(\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_2})} \cdot \underline{e}_T\right) + \left(\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_3})} \cdot \underline{e}_T\right) + \left(\sigma_{(\varepsilon_{j_2}-\varepsilon_{j_3})} \cdot \underline{e}_T\right) = 0.$$

Please notice the analogy with the $(\star)$-relation

$$\left(\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_2})} \cdot S\right) + \left(\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_3})} \cdot S\right) + \left(\sigma_{(\varepsilon_{j_2}-\varepsilon_{j_3})} \cdot S\right) = 0$$

where S is the string $Z_{\varepsilon_{j_1}-\varepsilon_{j_2}} Z_{\varepsilon_{j_3}-\varepsilon_{j_4}} \cdots Z_{\varepsilon_{j_{2n-1}}-\varepsilon_{j_{2n}}} v$.[13]

Similarly, when $k = 1$ and we pick the i^{th} and $(i+1)^{th}$ column, we find the relation:

$$\left(\sigma_{(\varepsilon_{j_{2i-1}}-\varepsilon_{j_{2i}})}\right) \cdot \underline{e}_T) + \left(\sigma_{(\varepsilon_{j_{2i-1}}-\varepsilon_{j_{2i+1}})} \cdot \underline{e}_T\right) + \left(\sigma_{(\varepsilon_{j_{2i}}-\varepsilon_{j_{2i+1}})} \cdot \underline{e}_T\right) = 0.$$

When $k = 2$ and the adjacent columns under considerations are the first and second

[12] Because $\underline{e}_T = -\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_2})} \cdot \underline{e}_T$.

[13] This relation holds because the string S satisfies: $\sigma_{(\varepsilon_{j_1}-\varepsilon_{j_2})} \cdot S = -S$ and $\sigma^2_{(\varepsilon_{j_1}-\varepsilon_{j_3})} \cdot S = +S$.

one, we can only take

- $S = \begin{array}{|c|c|c|c|c|}\hline j_3 & j_1 & \cdots & j_{2n-3} & j_{2n-1} \\ \hline j_4 & j_2 & \cdots & j_{2n-2} & j_{2n} \\ \hline\end{array}$ (switch 1^{st} and 2^{nd} columns of T).

Therefore, we obtain the relation $\underline{e}_T = \underline{e}_S$, which is analogous to the commutativity relation:

$$\underbrace{Z_{\varepsilon_{j_1}-\varepsilon_{j_2}} Z_{\varepsilon_{j_3}-\varepsilon_{j_4}}}_{\text{switch}} Z_{\varepsilon_{j_5}-\varepsilon_{j_6}} \cdots Z_{\varepsilon_{j_{2n-1}}-\varepsilon_{j_{2n}}} v =$$

$$= Z_{\varepsilon_{j_3}-\varepsilon_{j_4}} Z_{\varepsilon_{j_1}-\varepsilon_{j_2}} Z_{\varepsilon_{j_5}-\varepsilon_{j_6}} \cdots Z_{\varepsilon_{j_{2n-1}}-\varepsilon_{j_{2n}}} v.$$

Finally, when $k = 2$ and we pick the i^{th} and $(i+1)^{th}$ column, we find the relation $\underline{e}_T = \underline{e}_S$, where S is the tableau obtained from T by switching the i^{th} and the $(i+1)^{th}$ columns.

Consider that the mapping

$$< \{ \text{ polytabloids } \} >_{\mathbb{C}} \quad \rightarrow \quad < \{ \text{ n-strings } \} >_{\mathbb{C}}$$

that carries the polytabloid $\underline{e}_T$ associated to a tableau

$$T = \begin{array}{|c|c|c|c|c|}\hline j_1 & j_3 & \cdots & j_{2n-3} & j_{2n-1} \\ \hline j_2 & j_4 & \cdots & j_{2n-2} & j_{2n} \\ \hline\end{array}$$

into the n-string of orthogonal roots

$$Z_{\varepsilon_{j_1}-\varepsilon_{j_2}} Z_{\varepsilon_{j_3}-\varepsilon_{j_4}} \cdots Z_{\varepsilon_{j_{2n-1}}-\varepsilon_{j_{2n}}} \underline{e}_T.$$

It follows from the previous remarks that this mapping descends to an isomorphism between $F_\rho = S^{(n,n)}$ and the set of linear combinations of equivalence classes of n-strings. Hence

$$\boxed{\left(F_{\tilde{\rho'}}\right)^M = (n,n) \oplus (n,n)}\,.$$

$\square$

10.2 The construction of $\tilde{\rho'}$: some examples

10.2.1 Construction for the trivial representation of $\mathcal{S}_3$

When $(\rho,\, F_\rho)$ is the trivial representation of $\mathcal{S}_3$, each root reflection σ_α acts trivially on $F_\rho \simeq \mathbb{C}$. In particular, there are no (-1)-eigenvectors, so $F_{\tilde{\rho'}} \equiv F_\rho^{'} \equiv F_\rho$.
By definition, $Z_\alpha \cdot v = 0$, for all $v \in F_\rho$ and all $\alpha \in \Delta$. Hence, $\tilde{\rho'}$ is the trivial representation of $\mathfrak{so}(3)$, and it lifts to the trivial representation of $SO(3)$.

In general, if $(\rho,\, F_\rho)$ is the trivial representation of $W(SL(n))$, then $(\tilde{\rho'}, F_{\tilde{\rho'}})$ is the trivial representations of $SO(n)$.

10.2.2 Construction for the standard representation of $\mathcal{S}_3$

Let $(\rho,\, F_\rho)$ be the two-dimensional irreducible representations of $\mathcal{S}_3$.[14] We fix a basis $\{v,\, u\}$ of $F_\rho = \mathbb{C}^2$ that consists of a $(+1)$ and a (-1) eigenvector of σ_{12}. With respect to this basis $\{v, u\}$ we can write:

$$\sigma_{12} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad u\colon (-1) \text{ eigenvector of } \sigma_{12}$$

$$\sigma_{13} \rightsquigarrow \begin{pmatrix} -1/2 & -1/2 \\ -3/2 & 1/2 \end{pmatrix} \qquad u+v\colon (-1) \text{ eigenvector of } \sigma_{13}$$

$$\sigma_{23} \rightsquigarrow \begin{pmatrix} -1/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad v-u\colon (-1) \text{ eigenvector of } \sigma_{23}$$

[14] It can be realized as the action of $\mathcal{S}_3$ on the hyperplane $\{x_1 + x_2 + x_3 = 0\}$ of $\mathbb{C}^3$. Then $v = (1,\, 1,\, -2)$ and $u = (1,\, -1,\, 0)$.

Since there is no pair of distinct orthogonal roots, we only have strings of length at most one. Therefore $F_{\rho'} = \mathbb{C}v \oplus \mathbb{C}u \oplus \mathbb{C}Z_{12}(u) \oplus \mathbb{C}Z_{13}(u+v) \oplus \mathbb{C}Z_{23}(v-u)$.

We notice that $\begin{cases} \sigma_{12}(Z_{12}u) = -Z_{12}u \\ \sigma_{13}^2(Z_{12}u) = -Z_{12}u \\ \sigma_{23}^2(Z_{12}u) = -Z_{12}u \end{cases}$, so there are no relations. We get:

$$F_{\tilde{\rho'}} = F_{\rho'} = \mathbb{C}v \oplus \mathbb{C}u \oplus \mathbb{C}Z_{12}(u) \oplus \mathbb{C}Z_{13}(u+v) \oplus \mathbb{C}Z_{23}(v-u).$$

Let us compute the action of the element Z_{12} of $\mathfrak{so}(3)$ on $F_{\tilde{\rho'}}$:

$\diamond$ $Z_{12} \cdot v = 0$, because $\sigma_{12}v = v$

$\diamond$ $Z_{12} \cdot u = Z_{12}u$, because $\sigma_{12}u = -u$

$\diamond$ $Z_{12} \cdot (Z_{12}u) = -4u$

$\diamond$ $Z_{12} \cdot (Z_{13}(u+v)) = \sigma_{12} \cdot (Z_{13}(u+v)) = -Z_{23}(v-u)$

because $\sigma_{12}^2(Z_{13}(u+v)) = -Z_{13}(u+v)$

$\diamond$ $Z_{12} \cdot (Z_{23}(v-u)) = \sigma_{12} \cdot (Z_{23}(v-u)) = Z_{13}(u+v)$

because $\sigma_{12}^2(Z_{23}(v-u)) = -Z_{23}(v-u)$.

Hence,

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the basis $\{v,\, u,\, Z_{12}(u),\, Z_{13}(u+v),\, Z_{23}(v-u)\}$ of $F_{\tilde{\rho}'}$.

The action of Z_{13} is given by:

$\diamond$ $Z_{13} \cdot v = \frac{1}{2} Z_{13}(v - \sigma_{13} v) = \frac{3}{4} Z_{13}(v+u)$

because $\sigma_{13} v \neq \pm v$ and $\sigma_{13}^2 v = v$

$\diamond$ $Z_{13} \cdot u = \frac{1}{2} Z_{13}(u - \sigma_{13} u) = \frac{1}{4} Z_{13}(u+v)$

because $\sigma_{13} u \neq \pm u$ and $\sigma_{13}^2 u = u$

$\diamond$ $Z_{13} \cdot (Z_{12} u) = \sigma_{13} \cdot (Z_{12} u) = -\frac{1}{2} Z_{23}(v-u)$

because $\sigma_{13}^2 (Z_{12} u) = -Z_{12} u$

$\diamond$ $Z_{13} \cdot (Z_{13}(u+v)) = -4(u+v)$

$\diamond$ $Z_{13} \cdot (Z_{23}(v-u)) = \sigma_{13} \cdot (Z_{23}(v-u)) = 2 Z_{12} u$

because $\sigma_{13}^2 (Z_{23}(v-u)) = -Z_{23}(v-u)$.

Hence,

$$Z_{13} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \end{pmatrix}$$

w.r.t. $\{v,\, u,\, Z_{12}(u),\, Z_{13}(u+v),\, Z_{23}(v-u)\}$.

Finally, we compute the action of Z_{23}:

$\diamond$ $Z_{23} \cdot v = \frac{1}{2} Z_{23} \cdot (v - \sigma_{23} v) = \frac{3}{4} Z_{23}(v-u)$

$\diamond$ $Z_{23} \cdot u = \frac{1}{2} Z_{23} \cdot (u - \sigma_{23} u) = -\frac{1}{4} Z_{23}(v-u)$

$\diamond$ $Z_{23} \cdot (Z_{12} u) = \sigma_{23}(Z_{12} u) = -\frac{1}{2} Z_{13}(u+v)$

$\diamond$ $Z_{23} \cdot (Z_{13}(u+v)) = \sigma_{23}(Z_{13}(u+v)) = 2 Z_{12} u$

$\diamond$ $Z_{23} \cdot (Z_{13}(v-u)) = -4(v-u)$.

And hence we have

$$Z_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 3/4 & -1/4 & 0 & 0 & 0 \end{pmatrix}.$$

We notice that $\mathbb{C}Z_{12}$ is a Cartan subalgebra of $\mathfrak{so}(3)$. Therefore, to identify the representation, we just have to look at the eigenvalues of Z_{12}. An easy computation shows that they are given by 0, $\pm i$, $\pm 2i$.

We conclude that $\tilde{\rho}'$ is an irreducible representation of $SO(3)$, isomorphic to $\mathcal{H}_2$.

10.2.3 Construction for the standard representation of $\mathcal{S}_4$

The standard representation of $\mathcal{S}_4$ can be realized as the action of the symmetric group in four letters on the hyperplane

$$F_\rho = \{\, \underline{z} = (z_1,\, z_2,\, z_3,\, z_4) \in \mathbb{C}^4 \colon z_1 + z_2 + z_3 + z_4 = 0 \,\}$$

of $\mathbb{C}^4$. The action is explicitly given by:

$$\sigma \cdot (z_1,\, z_2,\, z_3,\, z_4) = \big(z_{\sigma^{-1}(1)},\, z_{\sigma^{-1}(2)},\, z_{\sigma^{-1}(3)},\, z_{\sigma^{-1}(4)}\big)$$

for all $\sigma \in \mathcal{S}_4$ and all $\underline{z} \in F_\rho$. We choose the basis:

$$v_1 = (1,\, -1,\, 0,\, 0) \qquad v_2 = (0,\, 1,\, -1,\, 0) \qquad v_3 = (0,\, 0,\, 1,\, -1)$$

of F_ρ, and write down the matrix of each root reflection w.r.t. this basis.

$$\sigma_{12} \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & v_3 \\ 1 & \Rightarrow & v_1 + 2v_2 \\ -1 & \Rightarrow & v_1 \end{cases}$$

$$\sigma_{13} \rightsquigarrow \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & v_1 + v_3 \\ 1 & \Rightarrow & v_2 + v_3 \\ -1 & \Rightarrow & v_1 + v_2 \end{cases}$$

$$\sigma_{23} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & 2v_1 + v_2 \\ 1 & \Rightarrow & v_3 - v_1 \\ -1 & \Rightarrow & v_2 \end{cases}$$

$$\sigma_{14} \rightsquigarrow \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & v_2 \\ 1 & \Rightarrow & v_1 - v_3 \\ -1 & \Rightarrow & v_1 + v_2 + v_3 \end{cases}$$

$$\sigma_{24} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & v_1 + v_2 \\ 1 & \Rightarrow & v_1 + v_3 \\ -1 & \Rightarrow & v_2 + v_3 \end{cases}$$

$$\sigma_{34} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{with eigenvalues} \begin{cases} 1 & \Rightarrow & 2v_2 + v_3 \\ 1 & \Rightarrow & v_1 \\ -1 & \Rightarrow & v_3 \end{cases}$$

We notice that there is no pair of mutually orthogonal roots with a simultaneous (-1)-eigenvector, so there are no strings of length bigger than one. We can write:

$$F_{\rho'} = \mathbb{C}\, v_1 \,\oplus \mathbb{C}\, v_2 \,\oplus \mathbb{C}\, v_3 \,\oplus \mathbb{C}\, Z_{12}(v_1) \,\oplus \mathbb{C}\, Z_{13}(v_1+v_2) \,\oplus \mathbb{C}\, Z_{23}(v_2) \oplus$$

$$\oplus \mathbb{C}\, Z_{14}(v_1+v_2+v_3) \,\oplus \mathbb{C}\, Z_{24}(v_2+v_3) \,\oplus \mathbb{C}\, Z_{34}(v_3).$$

Because $\begin{cases} \sigma_{12}(Z_{12}v_1) = -Z_{12}v_1 \\ \sigma_{13}(Z_{12}v_1) \neq \pm Z_{12}v_1 \\ \sigma_{13}^2(Z_1 2v_1) = -Z_{12}v_1 \end{cases}$, no relations are imposed on the strings $Z_{12}v_1$. The situation for the other strings is similar, so we conclude that there are no relations and $F_{\tilde{\rho'}} = F_{\rho'}$.

We now compute the action of $\mathfrak{so}(4)$ on $F_{\tilde{\rho'}}$ w.r.t. to the basis

$$\{v_1,\, ,v_2,\, ,v_3,\, ,Z_{12}(v_1),\, ,Z_{13}(v_1+v_2),\, Z_{23}(v_2),\, Z_{14}(v_1+v_2+v_3),\, Z_{24}(v_2+v_3),\, Z_{34}(v_3)\},$$

starting with the element Z_{12}.

$\diamond$ $Z_{12}\cdot v_1 = Z_{12}v_1$

$\diamond$ $Z_{12}\cdot v_2 = \frac{1}{2}Z_{12}\cdot(v_2 - \sigma_{12}v_2) = -\frac{1}{2}Z_{12}v_1$

$\diamond$ $Z_{12}\cdot v_3 = 0$

$\diamond$ $Z_{12}\cdot(Z_{12}v_1) = -4v_1$

$\diamond$ $Z_{12} \cdot (Z_{13}(v_1 + v_2)) = \sigma_{12} \cdot (Z_{13}(v_1 + v_2)) = -Z_{23}v_2$

$\diamond$ $Z_{12} \cdot (Z_{23}v_2) = \sigma_{12} \cdot (Z_{23}v_2) = Z_{13}(v_1 + v_2)$

$\diamond$ $Z_{12} \cdot (Z_{14}(v_1 + v_2 + v_3)) = \sigma_{12} \cdot (Z_{14}(v_1 + v_2 + v_3)) = -Z_{24}(v_2 + v_3)$

$\diamond$ $Z_{12} \cdot (Z_{24}(v_2 + v_3)) = \sigma_{12} \cdot (Z_{24}(v_2 + v_3)) = Z_{14}(v_1 + v_2 + v_3)$

$\diamond$ $Z_{12} \cdot (Z_{34}v_3) = 0.$

Hence we have

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In particular, we notice that Z_{12} acts on $F_{\tilde{\rho}'}$ with eigenvalues

$$0,\ 0,\ 0,\ -i,\ -i,\ +i,\ +i,\ -2i,\ +2i.$$

The action of Z_{34} is given by:

$\diamond$ $Z_{34} \cdot v_1 = 0$

$\diamond$ $Z_{34} \cdot v_2 = \frac{1}{2} Z_{34} \cdot (v_2 - \sigma_{34} v_2) = -\frac{1}{2} Z_{34} v_3$

$\diamond$ $Z_{34} \cdot v_3 = Z_{34} v_3$

$\diamond$ $Z_{34} \cdot (Z_{12} v_1) = 0$

$\diamond$ $Z_{34} \cdot (Z_{13}(v_1 + v_2)) = -Z_{14}(v_1 + v_2 + v_3)$

$\diamond$ $Z_{34} \cdot (Z_{23} v_2) = \sigma_{34} \cdot (Z_{23} v_2) = -Z_{24}(v_2 + v_3)$

$\diamond$ $Z_{34} \cdot (Z_{14}(v_1 + v_2 + v_3)) = \sigma_{34} \cdot (Z_{14}(v_1 + v_2 + v_3)) = Z_{13}(v_1 + v_2)$

$\diamond$ $Z_{34} \cdot (Z_{24}(v_2 + v_3)) = Z_{23}(v_2)$

$\diamond$ $Z_{34} \cdot (Z_{34} v_3) = -4 v_3$

Hence

$$Z_{34} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and again the eigenvalues are

$$0,\ 0,\ 0,\ -i,\ -i,\ +i,\ +i,\ -2i,\ +2i.$$

If we denote by ξ_1, ξ_2 the linear functionals on $\mathfrak{h}$

$\xi_1 \colon a\, Z_{12} + b\, Z_{34} \mapsto -i\, a$

$\xi_2 \colon a\, Z_{12} + b\, Z_{34} \mapsto -i\, b,$

then we can write

$$0 \qquad \pm 2\xi_1 \qquad \pm 2\xi_2 \qquad \pm(\xi_1 + \xi_2) \qquad \pm(\xi_1 - \xi_2)$$

for the list of weights of $\tilde{\rho}'$.

We notice that $\tilde{\rho}'$ has highest weight $2\xi_1$, so we compare it with the irreducible representation $\rho_{2\xi_1}$ of $SO(4)$ of highest weight $2\xi_1$. Because also $\rho_{2\xi_1}$ has dimension equal to 9, there is no doubt that $\rho \equiv \rho_{2\xi_1}$.

10.2.4 Construction for the representation $S^{2,2}$ of $\mathcal{S}_4$

We can choose a basis $\{v, u\}$ of $F_\rho = \mathbb{C}^2$ that consists of a $(+1)$ and a (-1) eigenvector of σ_{12}. With respect to this basis, we have:

$$\sigma_{12},\, \sigma_{34} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad u\colon (-1) \text{ eigenvector of } \sigma_{12},\, \sigma_{34}$$

$$\sigma_{13},\, \sigma_{24} \rightsquigarrow \begin{pmatrix} -1/2 & -1/2 \\ -3/2 & 1/2 \end{pmatrix} \qquad\qquad u+v\colon (-1) \text{ eigenvector of } \sigma_{13},\, \sigma_{24}$$

$$\sigma_{23},\, \sigma_{14} \rightsquigarrow \begin{pmatrix} -1/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad\qquad v-u\colon (-1) \text{ eigenvector of } \sigma_{23},\, \sigma_{14}.$$

Therefore, we can write:

$$F_{\rho'} = \mathbb{C}u \,\oplus\, \mathbb{C}v \,\oplus\, \mathbb{C}Z_{12}u \,\oplus\, \mathbb{C}Z_{34}u \,\oplus\, \mathbb{C}Z_{13}(u+v) \,\oplus\, \mathbb{C}Z_{24}(u+v) \oplus$$

$$\oplus\, \mathbb{C}Z_{23}(v-u) \,\oplus\, \mathbb{C}Z_{14}(v-u) \,\oplus\, \mathbb{C}Z_{12}Z_{34}u \,\oplus\, \mathbb{C}Z_{13}Z_{24}(v+u) \,\oplus\, \mathbb{C}Z_{23}Z_{14}(v-u).$$

Claim 9. *There are no relations among strings of length one.*

Proof. We prove that there are no relations involving the string $Z_{12}u$. The other cases are similar.

The string $Z_{12}u$ is a (-1)-eigenvector for the root reflections σ_{12} and σ_{34}. We notice that there are two $\mathfrak{so}(3)$s containing Z_{12}, and they are generated by Z_{12}, Z_{13}, Z_{23} and

Z_{12}, Z_{14}, Z_{24} respectively. Because

$$\sigma_{13}^2(Z_{12}u) = \sigma_{14}^2(Z_{12}u) = -Z_{12}u$$

none of them gives rise to a relation. Similarly, the two $\mathfrak{so}(3)$s containing Z_{34} are generated by Z_{34}, Z_{13}, Z_{14} and Z_{34}, Z_{23}, Z_{24}, and because

$$\sigma_{13}^2(Z_{12}u) = \sigma_{23}^2(Z_{12}u) = -Z_{12}u$$

none of them gives rise to a relation. $\square$

Claim 10. *There is only one relation among strings of length two, namely:*

$$Z_{12}Z_{34}u = -\frac{1}{2}Z_{23}Z_{14}(v-u) - \frac{1}{2}Z_{13}Z_{24}(u+v) \qquad (*)\,.$$

Proof. We look for possible relations involving the string $Z_{12}Z_{34}u$. Since $Z_{12}Z_{34}u$ is a (-1) eigenvector for σ_{12} and σ_{34}, we must consider the $\mathfrak{so}(3)$s containing either Z_{12} or Z_{34}. There are 4 of them, and they are generated by

1. Z_{12}, Z_{13}, Z_{23}
2. Z_{12}, Z_{14}, Z_{24}
3. Z_{34}, Z_{13}, Z_{14}
4. Z_{34}, Z_{23}, Z_{24}

We start with (1). Because

$$\sigma_{13}(Z_{12}Z_{34}u) \neq \pm(Z_{12}Z_{34}u) \qquad \sigma_{23}(Z_{12}Z_{34}u) \neq \pm(Z_{12}Z_{34}u)$$

but

$$\sigma_{13}^2(Z_{12}Z_{34}u) = \sigma_{23}^2(Z_{12}Z_{34}u) = +Z_{12}Z_{34}u$$

the following relation holds:

$$Z_{12}Z_{34}u = \sigma_{13}(Z_{12}Z_{34}u) + \sigma_{23}(Z_{12}Z_{34}u).$$

Being

$\sigma_{13} \cdot (Z_{12}Z_{34}u) = -\frac{1}{2}Z_{23}Z_{14}(v-u)$, and

$\sigma_{23} \cdot (Z_{12}Z_{34}u) = -\frac{1}{2}Z_{13}Z_{24}(u+v)$

our relation becomes:

$$Z_{12}Z_{34}u = -\frac{1}{2}Z_{23}Z_{14}(v-u) - \frac{1}{2}Z_{13}Z_{24}(u+v). \qquad (*)$$

Let us now look at (2), which is the $\mathfrak{so}(3)$ generated by Z_{12}, Z_{14}, Z_{24}. Because

$$\sigma_{14}(Z_{12}Z_{34}u) \neq \pm Z_{12}Z_{34}u \qquad \sigma_{24}(Z_{12}Z_{34}u) \neq \pm Z_{12}Z_{34}u$$

but

$$\sigma_{14}^2(Z_{12}Z_{34}u) = \sigma_{24}^2(Z_{12}Z_{34}u) = Z_{12}Z_{34}u$$

we have the relation:

$$Z_{12}Z_{34}u = \sigma_{14}(Z_{12}Z_{34}u) + \sigma_{24}(Z_{12}Z_{34}u).$$

Because σ_{14} acts like σ_{23}, and σ_{24} acts like σ_{13}, this relation is the same as (*).

Similarly, (3) gives rise to the relation

$$Z_{12}Z_{34}u = \sigma_{14} \cdot (Z_{12}Z_{34}u) + \sigma_{13} \cdot (Z_{12}Z_{34}u)$$

which is equivalent to

$$Z_{12}Z_{34}u = \sigma_{14}(Z_{12}Z_{34}u) + \sigma_{24}(Z_{12}Z_{34}u)$$

because σ_{13} acts as σ_{24}, and (4) gives

$$Z_{12}Z_{34}u = \sigma_{23}(Z_{12}Z_{34}u) + \sigma_{24}(Z_{12}Z_{34}u)$$

which is equivalent to

$$Z_{12}Z_{34}u = \sigma_{14} \cdot (Z_{12}Z_{34}u) + \sigma_{13} \cdot (Z_{12}Z_{34}u).$$

In both cases we obtain the relation (*). □

This proves that there is a unique relation among the 11 generators of F'_ρ. In other words, the quotient space $F_{\tilde{\rho}'}$ is the vector space with generators

$$v,\ u,\ Z_{12}u,\ Z_{34}u,\ Z_{23}(v-u),\ Z_{14}(v-u),\ Z_{13}(v+u)\ldots$$

$$\ldots Z_{24}(v+u),\ Z_{12}Z_{34}u,\ Z_{23}Z_{14}(v-u),\ Z_{13}Z_{24}(v+u)$$

and one relation:

$$Z_{12}Z_{34}u = -\frac{1}{2}Z_{23}Z_{14}(v-u) - \frac{1}{2}Z_{13}Z_{24}(u+v) \qquad (*)\,.$$

It is clear that $F_{\tilde{\rho}'}$ has dimension 10. We pick the first ten elements in the above list as a basis, and we use (*) to express $Z_{13}Z_{24}(v+u)$ as a linear combination of basis vectors.

We now compute the action of $\mathfrak{so}(4)$ on $F_{\tilde{\rho}'}$, starting with Z_{12}.

- ◇ $Z_{12} \cdot v = 0$
- ◇ $Z_{12} \cdot u = Z_{12}u$
- ◇ $Z_{12} \cdot (Z_{12}u) = -4u$
- ◇ $Z_{12} \cdot (Z_{34}u) = Z_{12}Z_{34}u$
- ◇ $Z_{12} \cdot (Z_{13}(u+v)) = \sigma_{12} \cdot (Z_{13}(u+v)) = -Z_{23}(v-u)$
- ◇ $Z_{12} \cdot (Z_{24}(u+v)) = \sigma_{12} \cdot (Z_{24}(u+v)) = Z_{14}(v-u)$
- ◇ $Z_{12} \cdot (Z_{23}(v-u)) = \sigma_{12} \cdot (Z_{23}(v-u)) = Z_{13}(v+u)$
- ◇ $Z_{12} \cdot (Z_{14}(v-u)) = \sigma_{12} \cdot (Z_{14}(v-u)) = -Z_{24}(v+u)$
- ◇ $Z_{12} \cdot (Z_{12}Z_{34}u) = -4Z_{34}u$
- ◇ $Z_{12} \cdot (Z_{23}Z_{14}(v-u)) = 4Z_{34}u.$

Only the last equation requires a comment. Using the relation (*) we obtain:

$$Z_{12} \cdot (Z_{23}Z_{14}(v-u)) = \tfrac{1}{2}Z_{12} \cdot (Z_{23}Z_{14}(v-u) - \sigma_{12} \cdot (Z_{23}Z_{14}(v-u))) =$$

$= \frac{1}{2} Z_{12} \cdot (Z_{23} Z_{14}(v-u) + Z_{13} Z_{24}(v+u)) =$

$= \frac{1}{2} Z_{12} \cdot (-2\, Z_{12} Z_{34} u) = 4 Z_{34} u \ \checkmark.$

Hence:

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We notice, in particular that Z_{12} acts with eigenvalues

$$0,\, 0,\, i,\, i,\, -i,\, -i,\, 2i,\, 2i,\, -2i,\, -2i.$$

Let us compute the action of Z_{34}.

- $\diamond$ $Z_{34} \cdot v = 0$
- $\diamond$ $Z_{34} \cdot u = Z_{34} u$
- $\diamond$ $Z_{34} \cdot (Z_{12} u) = Z_{12} Z_{34} u$
- $\diamond$ $Z_{34} \cdot (Z_{34} u) = -4u$

$\diamond$ $Z_{34} \cdot (Z_{13}(u+v)) = \sigma_{34} \cdot (Z_{13}(u+v)) = -Z_{14}(v-u)$

$\diamond$ $Z_{34} \cdot (Z_{24}(u+v)) = \sigma_{34} \cdot (Z_{24}(u+v)) = Z_{23}(v-u)$

$\diamond$ $Z_{34} \cdot (Z_{23}(v-u)) = \sigma_{34} \cdot (Z_{23}(v-u)) = -Z_{24}(v+u)$

$\diamond$ $Z_{34} \cdot (Z_{14}(v-u)) = \sigma_{34} \cdot (Z_{14}(v-u)) = Z_{13}(v+u)$

$\diamond$ $Z_{34} \cdot (Z_{12}Z_{34}u) = -4Z_{12}u$

$\diamond$ $Z_{34} \cdot (Z_{23}Z_{14}(v-u)) = 4Z_{12}u.$

A comment on the last equation:

$$Z_{34} \cdot (Z_{23}Z_{14}(v-u)) = \tfrac{1}{2}Z_{34} \cdot (Z_{23}Z_{14}(v-u) - \sigma_{34} \cdot (Z_{23}Z_{14}(v-u)) =$$

$$= \tfrac{1}{2}Z_{34} \cdot (Z_{23}Z_{14}(v-u) + Z_{24}Z_{13}(v+u)) =$$

$$= Z_{34} \cdot (-Z_{12}Z_{34}u) = +4Z_{12}u. \checkmark$$

Therefore:

$$Z_{34} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and it acts with eigenvalues

$$0,\, 0,\, i,\, i,\, -i,\, -i,\, 2i,\, 2i,\, -2i,\, -2i.$$

To identify the representation, we compute the list of eigenvalues of $\tilde{\rho}'(b\, Z_{12} + a\, Z_{34})$:

$$0 \quad 0 \quad \pm\, 2i(a+b) \quad \pm\, 2i(a-b) \quad i(a+b) \quad \pm\, i(a-b).$$

We conclude that $\tilde{\rho}'$ has weights:

$$0 \quad 0 \quad \pm\,(2\xi_1 + 2\xi_2) \quad (2\xi_1 - 2\xi_2) \quad \pm\,(\xi_1 + \xi_2) \quad \pm\,(\xi_1 - \xi_2),$$

therefore

$$\tilde{\rho}' = \rho_{2\varepsilon_1 + 2\varepsilon_2} \,\oplus\, \rho_{2\varepsilon_1 - 2\varepsilon_2}.$$

10.2.5 Construction for the standard representation of $\mathcal{S}_5$

The standard representation of $\mathcal{S}_5$ can be realized as the action of the symmetric group in four letters on the hyperplane

$$F_\rho = \{\, \underline{z} = (z_1,\, z_2,\, z_3,\, z_4,\, z_5) \in \mathbb{C}^5 \colon z_1 + z_2 + z_3 + z_4 + z_5 = 0\,\}$$

of $\mathbb{C}^5$. The action is explicitly given by:

$$\sigma \cdot (z_1,\, z_2,\, z_3,\, z_4,\, z_5) = \big(z_{\sigma^{-1}(1)},\, z_{\sigma^{-1}(2)},\, z_{\sigma^{-1}(3)},\, z_{\sigma^{-1}(4)},\, z_{\sigma^{-1}(5)}\big)$$

for all $\sigma \in \mathcal{S}_5$ and all $\underline{z} \in F_\rho$. We choose the basis:

$$v_1 = (1,\, -1,\, 0,\, 0,\, 0) \qquad v_2 = (0,\, 1,\, -1,\, 0,\, 0)$$

$$v_3 = (0,\, 0,\, 1,\, -1,\, 0) \qquad v_4 = (0,\, 0,\, 0,\, 1,\, -1)$$

of F_ρ, and notice that each transpositions acts on F_ρ with eigenvalues $+1,\, +1,\, +1,\, -1$. For each transposition, we list the (-1)-eigenvector, and the corresponding string:

$$\sigma_{12} \rightsquigarrow v_1 \rightsquigarrow Z_{12}(v_1)$$

$$\sigma_{13} \rightsquigarrow v_1 + v_2 \rightsquigarrow Z_{13}(v_1 + v_2)$$

$$\sigma_{14} \rightsquigarrow v_1 + v_2 + v_3 \rightsquigarrow Z_{14}(v_1 + v_2 + v_3)$$

$$\sigma_{15} \rightsquigarrow v_1 + v_2 + v_3 + v_4 \rightsquigarrow Z_{15}(v_1 + v_2 + v_3 + v_4)$$

$$\sigma_{23} \rightsquigarrow v_2 \rightsquigarrow Z_{23}(v_2)$$

$\sigma_{24} \rightsquigarrow v_2 + v_3 \rightsquigarrow Z_{24}(v_2 + v_3)$

$\sigma_{25} \rightsquigarrow v_2 + v_3 + v_4 \rightsquigarrow Z_{25}(v_2 + v_3 + v_4)$

$\sigma_{34} \rightsquigarrow v_3 \rightsquigarrow Z_{34}(v_3)$

$\sigma_{35} \rightsquigarrow v_3 + v_4 \rightsquigarrow Z_{35}(v_3 + v_4)$

$\sigma_{45} \rightsquigarrow v_4 \rightsquigarrow Z_{45}(v_4)$.

We notice that there is no pair of mutually orthogonal roots with a simultaneous (-1)-eigenvector, so there are no strings of length bigger than one and the extension $F_{\rho'}$ has dimension equal to 14. Since there are no relations among strings, $F_{\tilde{\rho'}} = F_{\rho'}$.

If we choose in $\mathfrak{so}(4)$ the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\, Z_{12} \oplus \mathbb{C}\, Z_{34}$, we can obtain the the weights of $\tilde{\rho'}$ simply by computing the eigenvalues of the generic element $(aZ_{12} + bZ_{34})$ of $\mathfrak{h}$ on $F_{\tilde{\rho'}}$.

We start by computing the action of Z_{12} on $F_{\tilde{\rho'}}$

$\diamond$ $Z_{12} \cdot v_1 = Z_{12}v_1$

$\diamond$ $Z_{12} \cdot v_2 = \frac{1}{2} Z_{12} \cdot (v_2 - \sigma_{12}v_2) = -\frac{1}{2} Z_{12}v_1$

$\diamond$ $Z_{12} \cdot v_3 = 0$

$\diamond$ $Z_{12} \cdot v_4 = 0$

$\diamond$ $Z_{12} \cdot (Z_{12}v_1) = -4v_1$

$\diamond$ $Z_{12} \cdot (Z_{13}(v_1 + v_2)) = \sigma_{12} \cdot (Z_{13}(v_1 + v_2)) = -Z_{23}v_2$

$\diamond$ $Z_{12} \cdot (Z_{14}(v_1 + v_2 + v_3)) = \sigma_{12} \cdot (Z_{14}(v_1 + v_2 + v_3)) = -Z_{24}(v_2 + v_3)$

$\diamond$ $Z_{12} \cdot (Z_{15}(v_1 + v_2 + v_3 + v_4)) = \sigma_{12} \cdot (Z_{15}(v_1 + v_2 + v_3)) = -Z_{25}(v_2 + v_3 + v_4)$

$\diamond$ $Z_{12} \cdot (Z_{23}v_2) = \sigma_{12} \cdot (Z_{23}v_2) = Z_{13}(v_1 + v_2)$

$\diamond$ $Z_{12} \cdot (Z_{24}(v_2 + v_3)) = \sigma_{12} \cdot (Z_{24}(v_2 + v_3)) = Z_{14}(v_1 + v_2 + v_3)$

$\diamond$ $Z_{12} \cdot (Z_{25}(v_2 + v_3 + v_4)) = \sigma_{12} \cdot (Z_{25}(v_2 + v_3 + v_4)) = Z_{15}(v_1 + v_2 + v_3 + v_4)$

$\diamond$ $Z_{12} \cdot (Z_{34}v_3) = 0$

$\diamond$ $Z_{12} \cdot (Z_{35}(v_3 + v_4)) = 0$

$\diamond$ $Z_{12} \cdot (Z_{45}v_4) = 0.$

Hence we have

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In particular, we notice that Z_{12} acts on $F_{\tilde{\rho}'}$ with eigenvalues

$$0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -i,\ -i,\ -i,\ +i,\ +i,\ +i,\ -2i,\ +2i.$$

The action of the other $Z_\alpha s$ is computed similarly.

We find that $\tilde{\rho}'$ has highest weight $2\xi_1$, so we compare it with the irreducible representation $\rho_{2\xi_1}$ of $SO(5)$ of highest weight $2\xi_1$. Because $\rho_{2\xi_1}$ has also dimension 14, there is no doubt that $\rho \equiv \rho_{2\xi_1}$.

10.2.6 Construction for the representation $S^{(3,3)}$ of $\mathcal{S}_6$

We start by giving an explicit description of the representation of $\mathcal{S}_6$ associated to the partition $(3,3)$.

Let F be the Specht module $S^{(3,3)}$. It follows from the Hook formula that F has dimension five: filling up each box of the Young diagram $T_{3,3}$ with the corresponding hook length we obtain:

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 3 & 2 & 1 \\ \hline \end{array} \quad \text{hence } \dim(F) = \frac{6!}{4\cdot3\cdot2\cdot1\cdot3\cdot2\cdot1} = 5.$$

We pick a basis of F consisting of *standard* polytabloids:[15]

$$v_1 = P\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}\right) \qquad v_2 = P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}\right)$$

$$v_3 = P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}\right) \qquad v_4 = P\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}\right)$$

$$v_5 = P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}\right)$$

To describe the representation of $\mathcal{S}_6$ on F, it is sufficient to explain how the transposi-

[15]Recall that for any tableau $T = \begin{array}{|c|c|c|}\hline \text{a1} & \text{a2} & \text{a3} \\ \hline \text{a4} & \text{a5} & \text{a6} \\ \hline \end{array}$ of shape $(3,3)$, the corresponding polytabloid $P(T)$ is a linear combination of tabloids. More precisely:

$$P(T) = \frac{\overline{a1 \quad a2 \quad a3}}{\underline{a4 \quad a5 \quad a6}} - \frac{\overline{a4 \quad a2 \quad a3}}{\underline{a1 \quad a5 \quad a6}} - \frac{\overline{a1 \quad a5 \quad a3}}{\underline{a4 \quad a2 \quad a6}} - \frac{\overline{a1 \quad a2 \quad a6}}{\underline{a4 \quad a5 \quad a3}}$$

$$+ \frac{\overline{a4 \quad a5 \quad a3}}{\underline{a1 \quad a2 \quad a6}} + \frac{\overline{a1 \quad a5 \quad a6}}{\underline{a4 \quad a2 \quad a3}} + \frac{\overline{a4 \quad a2 \quad a6}}{\underline{a1 \quad a5 \quad a3}} - \frac{\overline{a4 \quad a5 \quad a6}}{\underline{a1 \quad a2 \quad a3}}.$$

We draw a tabloid using bars instead of boxes because the element of each row can be permuted without changing the tabloid. A polytabloid $P(T)$ is called standard if the tableau T is standard, i.e. if the rows and the columns of T are both increasing. The standard polytabloids of shape $(3,3)$ form a basis of the Specht module $S^{(3,3)}$.
Also recall that the for any σ in $S^{(3,3)}$, and all tableaux T, we define $\sigma \cdot P(T) = P(\sigma \cdot T)$, and the tableau $\sigma \cdot T$ is obtained from T by applying the permutation σ to its entries.

tions $(1\,2)$, $(1\,3)$, $(1\,4)$, $(1\,5)$, $(1\,6)$ act on F. However, for the purpose of constructing the extension $F_{\tilde{\rho}'}$, it is convenient to have an explicit expression for the action of *each* transposition σ of $\mathcal{S}_6$.

For all σ, we give the matrix (w.r.t. the basis $\{v_1, v_2, v_3, v_4, v_5\}$) of the action of σ on F, and we list the corresponding (-1)-eigenvectors:

$$(1\,2) \Rightarrow \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} (-1)\text{-eigenvectors:} \\ v_1,\ v_4 \end{matrix}$$

$$(1\,3) \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{matrix} (-1)\text{-eigenvectors:} \\ v_2,\ v_3 \end{matrix}$$

$$(1\,4) \Rightarrow \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix} \qquad \begin{matrix} (-1)\text{-eigenvectors:} \\ v_5,\ v_1 - v_2 \end{matrix}$$

$$(1\,5) \Rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_1 - v_3 + v_5$

$v_3 - v_4$

$$(1\,6) \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_1 - v_2 + v_3 - v_4$

$v_3 - v_5$

$$(2\,3) \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(-1)-eigenvectors:

$v_3 - v_4,\ v_1 - v_2$

$$(2\,4) \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_2,\ v_1 - v_4 + v_5$

$$(2\,5) \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

(-1)-eigenvectors:

$v_3,\ v_5$

$$(2\,6) \Rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_1 - v_2 + v_5,\ v_2 - v_3$

$$(3\,4) \Rightarrow \begin{pmatrix} -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_3 - v_5,\ v_1$

$$(3\,5) \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(-1)-eigenvectors:

$v_1 - v_2 + v_5,\ v_4$

$$
(3\,6) \Rightarrow \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \end{pmatrix} \qquad \begin{array}{c} (-1)\text{-eigenvectors:} \\ v_1 - v_4,\ v_5 \end{array}
$$

$$
(4\,5) \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{c} (-1)\text{-eigenvectors:} \\ v_1 - v_4,\ v_2 - v_3 \end{array}
$$

$$
(4\,6) \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{c} (-1)\text{-eigenvectors:} \\ v_3,\ v_4 \end{array}
$$

$$
(5\,6) \Rightarrow \begin{pmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{c} (-1)\text{-eigenvectors:} \\ v_1,\ v_2 \end{array}
$$

Now that the structure of $F = S^{(3,3)}$ is understood, we proceed to the construction of the extension $F_{\tilde{\rho}'}$. The first step is to add all the strings of length one. Since we have 15 transpositions and each of them has two (-1)-eigenvectors, there is a total of 30 strings of length one. They are:

$$
\begin{array}{lll}
Z_{1\,2}v_1 & Z_{1\,3}v_2 & Z_{1\,4}v_5 \\[2em]
Z_{1\,2}v_4 & Z_{1\,3}v_3 & Z_{1\,4}(v_1 - v_2) \\[2em]
Z_{2\,3}(v_1 - v_2) & Z_{1\,6}(v_3 - v_5) & Z_{1\,5}(v_1 - v_3 + v_5) \\[2em]
Z_{1\,5}(v_3 - v_4) & Z_{2\,3}(v_3 - v_4) & Z_{1\,6}(v_1 - v_2 + v_3 - v_4) \\[2em]
Z_{2\,5}v_3 & Z_{2\,6}(v_2 - v_3) & Z_{2\,4}(v_1 - v_4 + v_5) \\[2em]
Z_{2\,4}v_2 & Z_{2\,5}v_5 & Z_{2\,6}(v_1 - v_2 + v_5) \\[2em]
Z_{3\,4}v_1 & Z_{3\,6}(v_1 - v_4) & Z_{3\,5}(v_1 - v_2 + v_5) \\[2em]
Z_{3\,5}v_4 & Z_{3\,4}(v_3 - v_5) & Z_{3\,6}v_5 \\[2em]
Z_{4\,6}v_3 & Z_{4\,6}v_4 & Z_{4\,5}(v_1 - v_4) \\[2em]
Z_{5\,6}v_2 & Z_{5\,6}v_1 & Z_{4\,5}(v_1 - v_4).
\end{array}
$$

Claim 11. *The 30 strings of length one are linearly independent..*

Proof. This is very easy, because a string $S = Z_\mu v$ (of length one) is included in a relation only if there exists a triple of roots $\{\alpha, \beta, \gamma\}$, forming an $\mathcal{A}_2$, such that

$$\sigma_\alpha \cdot Z_\mu v = -Z_\mu v \qquad \sigma_\beta^2 \cdot Z_\mu v = +Z_\mu v \qquad \sigma_\gamma^2 \cdot Z_\mu v = +Z_\mu v$$

and $\sigma_\beta \cdot Z_\mu v \neq \pm(Z_\mu v)$, $\sigma_\gamma \cdot Z_\mu v \neq \pm(Z_\mu v)$. Suppose that all these assumption are satisfied by a triple α, β, γ. Then, because $Z_\mu v$ is a (-1)-eigenvector of σ_α, the root α is either equal to μ or orthogonal to it. If $\alpha = \mu$, then β does not commute with μ, and we reach a contradiction:

$$\sigma_\beta^2 \cdot (Z_\mu v) = (-Z_\mu)(+v) = -Z_\mu v.$$

If α is orthogonal to μ and β is not orthogonal to μ, then again we reach a contradiction. Finally we discuss the case in which α, β (and of course also γ) are orthogonal to μ. Because

$$\sigma_\beta \cdot (Z_\mu v) = Z_\mu(\sigma_\beta \cdot v) \qquad \sigma_\gamma \cdot (Z_\mu v) = Z_\mu(\sigma_\gamma \cdot v)$$

the relation "$Z_\mu v + \sigma_\beta \cdot (Z_\mu v) + \sigma_\gamma \cdot (Z_\mu v) = 0$" is just linearity. Again, we do not obtain a linear dependence condition on the string $Z_\mu v$. $\square$

The first step of the construction of $F_{\tilde{\rho}'}$ is now complete. So far we have obtained an extension of dimension 35. The second step consists of adding strings of pairs of orthogonal roots "$Z_\alpha Z_\beta v$" (with $\alpha \perp \beta$ and $\sigma_\alpha \cdot v = \sigma_\beta \cdot v = -v$). Writing $\alpha = \epsilon_i - \epsilon_j$ and $\beta = \epsilon_k - \epsilon_l$, the problem is reduced to finding simultaneous (-1)-eigenvectors for the disjoint transpositions $(i\,j)$, $(k\,l)$.

These are all the possible simultaneous (-1)-eigenvectors for disjoint transpositions:[16]

- $v_1 \overset{-1\, eigenvector}{\text{- - - - - - - >}} (1\,2),\ (3\,4),\ (5\,6)$

[16]The arrow joins a vector v to a group of disjoint transpositions that acts by (-1) on v.

- $v_2 \; \text{- - - - - - - >} \; (1\,3),\ (2\,4),\ (5\,6)$
- $v_3 \; \text{- - - - - - - >} \; (1\,3),\ (2\,5),\ (4\,6)$
- $v_4 \; \text{- - - - - - - >} \; (1\,2),\ (4\,6),\ (3\,5)$
- $v_5 \; \text{- - - - - - - >} \; (1\,4),\ (2\,5),\ (3\,6)$
- $(v_1 - v_2) \; \overset{-1\,eigenvector}{\text{- - - - - - - >}} \; (2\,3),\ (1\,4),\ (5\,6)$
- $(v_3 - v_4) \; \text{- - - - - - - >} \; (2\,3),\ (1\,5),\ (4\,6)$
- $(v_3 - v_5) \; \text{- - - - - - - >} \; (1\,6),\ (3\,4),\ (2\,5)$
- $(v_1 - v_4) \; \text{- - - - - - - >} \; (1\,2),\ (3\,6),\ (4\,5)$
- $(v_2 - v_3) \; \text{- - - - - - - >} \; (2\,6),\ (1\,3),\ (4\,5)$
- $(v_1 - v_3 + v_5) \; \overset{-1\,eigenvector}{\text{- - - - - - - >}} \; (1\,5),\ (3\,4),\ (2\,6)$
- $(v_1 - v_4 + v_5) \; \text{- - - - - - - >} \; (2\,4),\ (3\,6),\ (1\,5)$
- $(v_1 - v_2 + v_5) \; \text{- - - - - - - >} \; (1\,4),\ (2\,6),\ (3\,5)$
- $(v_1 - v_2 + v_3 - v_4) \; \overset{-1\,eigenvector}{\text{- - - - - - - >}} \; (1\,6),\ (2\,3),\ (4\,5)$
- $(v_1 - v_2 - v_4 + v_5) \; \text{- - - - - - - >} \; (1\,6),\ (2\,4),\ (3\,5)$

We obtain a total of 45 strings of length two:

$Z_{1\,2}Z_{3\,4}v_1$	$Z_{1\,2}Z_{5\,6}v_1$	$Z_{3\,4}Z_{5\,6}v_1$
$Z_{1\,3}Z_{2\,4}v_2$	$Z_{1\,3}Z_{5\,6}v_2$	$Z_{2\,4}Z_{5\,6}v_2$
$Z_{1\,3}Z_{2\,5}v_3$	$Z_{1\,3}Z_{4\,6}v_3$	$Z_{2\,5}Z_{4\,6}v_3$

$Z_{12}Z_{46}v_4$ $\quad Z_{12}Z_{35}v_4$ $\quad Z_{35}Z_{46}v_4$

$Z_{14}Z_{25}v_5$ $\quad Z_{14}Z_{36}v_5$ $\quad Z_{25}Z_{36}v_5$

$Z_{14}Z_{23}(v_1 - v_2)$ $\quad Z_{14}Z_{56}(v_1 - v_2)$ $\quad Z_{23}Z_{56}(v_1 - v_2)$

$Z_{15}Z_{23}(v_3 - v_4)$ $\quad Z_{15}Z_{46}(v_3 - v_4)$ $\quad Z_{23}Z_{46}(v_3 - v_4)$

$Z_{16}Z_{34}(v_3 - v_5)$ $\quad Z_{16}Z_{25}(v_3 - v_5)$ $\quad Z_{34}Z_{25}(v_3 - v_5)$

$Z_{12}Z_{36}(v_1 - v_4)$ $\quad Z_{12}Z_{45}(v_1 - v_4)$ $\quad Z_{36}Z_{45}(v_1 - v_4)$

$Z_{13}Z_{26}(v_2 - v_3)$ $\quad Z_{13}Z_{45}(v_2 - v_3)$ $\quad Z_{26}Z_{45}(v_2 - v_3)$

$Z_{15}Z_{26}(v_1 - v_3 + v_5)$ $\quad Z_{15}Z_{34}(v_1 - v_3 + v_5)$ $\quad Z_{26}Z_{34}(v_1 - v_3 + v_5)$

$Z_{15}Z_{24}(v_1 - v_4 + v_5)$ $\quad Z_{15}Z_{36}(v_1 - v_4 + v_5)$ $\quad Z_{24}Z_{36}(v_1 - v_4 + v_5)$

$Z_{14}Z_{26}(v_1 - v_2 + v_5)$ $\quad Z_{14}Z_{35}(v_1 - v_2 + v_5)$ $\quad Z_{26}Z_{35}(v_1 - v_2 + v_5)$

$Z_{16}Z_{23}(v_1 - v_2 + v_3 - v_4)$ $\quad Z_{16}Z_{45}(v_1 - v_2 + v_3 - v_4)$

$Z_{23}Z_{45}(v_1 - v_2 + v_3 - v_4)$ $\quad Z_{16}Z_{24}(v_1 - v_2 - v_4 + v_5)$

$Z_{16}Z_{35}(v_1 - v_2 - v_4 + v_5)$ $\quad Z_{24}Z_{35}(v_1 - v_2 - v_4 + v_5)$.

Claim 12. *There are* 15 *linearly independent relations among strings of length two.*

Hence, there are 30 *linearly independent strings of length two.*

Proof. A string $S = Z_{\mu_1} Z_{\mu_2} v$ (of length two) is included in a relation

$$S = \sigma_\beta \cdot S + \sigma_\gamma \cdot S \qquad (*)$$

only if there exists a triple of roots $\{\alpha, \beta, \gamma\}$, forming an $\mathcal{A}_2$, such that

$$\sigma_\alpha \cdot S = -S \qquad \sigma_\beta^2 \cdot S = +S \qquad \sigma_\gamma^2 \cdot S = +S$$

and $\sigma_\beta \cdot S \neq \pm S$, $\sigma_\gamma \cdot S \neq \pm S$. Suppose that all these assumption are satisfied by a triple α, β, γ. Then, because $S = Z_{\mu_1} Z_{\mu_1} v$ is a (-1)-eigenvector of σ_α, the root α either belongs to the set $\{\mu_1, \mu_2\}$ or it is orthogonal to it. We discuss the two cases separately.

If $\alpha = \mu_1$, then β cannot be orthogonal to μ_2 (or we would have

$$\sigma_\beta^2 \cdot S = \sigma_\beta^2 \cdot (Z_\alpha Z_{\mu_2} v) = (-Z_\alpha)(+Z_{\mu_2} v)(+v) = -S,$$

which is a contradiction). Therefore there exists a subset $I = \{i_1, i_2, i_3, i_4\}$ of $\{1, \ldots, 6\}$ such that the roots $\alpha = \mu_1$, β, γ, μ_2 can all be written in the form $\epsilon_k - \epsilon_l$, with k, l in I. It follows that the relation $(*)$ is of the type

$$a\, Z_{\epsilon_{i_1} - \epsilon_{i_2}} Z_{\epsilon_{i_3} - \epsilon_{i_4}} + b\, Z_{\epsilon_{i_1} - \epsilon_{i_3}} Z_{\epsilon_{i_2} - \epsilon_{i_4}} + c\, Z_{\epsilon_{i_1} - \epsilon_{i_4}} Z_{\epsilon_{i_2} - \epsilon_{i_3}}.$$

We have one such relation for each subset $I = \{i_1, i_2, i_3, i_4\}$ of $\{1, \ldots, 6\}$. It is clear that different Is give rise to linearly independent relations, so we get a total of $15 = \binom{6}{4}$ relations of this type.

We now discuss the case in which α is orthogonal to both μ_1 and μ_2, and show that there are no other relations. Because $\sigma_\beta^2 \cdot S = +S$, the root β is either orthogonal to both μ_1 and μ_2, or is not orthogonal to any of them. The latter case gives rise to a contradiction, because it requires β to be of the form $\pm(\epsilon_k - \epsilon_l)$ with k an index of μ_1 and l an index of μ_2, making β orthogonal to α. Therefore we must assume that α, β and γ are all orthogonal to the set $\{\mu_1, \mu_2\}$. In this case, the relation $(*)$ becomes

$$Z_{\mu_1} Z_{\mu_2}(v) = Z_{\mu_1} Z_{\mu_2}(\sigma_\beta \cdot v) + Z_{\mu_1} Z_{\mu_2}(\sigma_\gamma \cdot v)$$

and it is just linearity. This concludes the proof. □

The second step of the construction is now complete, and we have introduce 30 additional generators, increasing the dimension of the extension to 65. The next step is to look at strings of length three (this will be of course the last step, because three is the maximum number of disjoint transpositions in $\mathcal{S}_6$).

We have already listed the possible simultaneous eigenvectors of disjoint transpositions, so writing down the strings of length three is easy:

$$Z_{1\,2} Z_{3\,4} Z_{5\,6} v_1 \qquad Z_{1\,3} Z_{2\,4} Z_{5\,6} v_2$$

$$Z_{1\,3} Z_{2\,5} Z_{4\,6} v_3 \qquad Z_{1\,2} Z_{3\,5} Z_{4\,6} v_4$$

$$Z_{1\,4} Z_{2\,5} Z_{3\,6} v_5 \qquad Z_{1\,4} Z_{2\,3} Z_{5\,6}(v_1 - v_2)$$

$$Z_{1\,5} Z_{2\,3} Z_{4\,6}(v_3 - v_4) \qquad Z_{1\,6} Z_{3\,4} Z_{2\,5}(v_3 - v_5)$$

$$Z_{1\,3} Z_{2\,6} Z_{4\,5}(v_2 - v_3) \qquad Z_{1\,5} Z_{2\,6} Z_{3\,4}(v_1 - v_3 + v_5)$$

$$Z_{1\,5} Z_{2\,4} Z_{3\,6}(v_1 - v_4 + v_5) \qquad Z_{1\,4} Z_{2\,6} Z_{3\,5}(v_1 - v_2 + v_5)$$

$Z_{16}Z_{23}Z_{45}(v_1 - v_2 + v_3 - v_4) \quad Z_{16}Z_{24}Z_{35}(v_1 - v_2 - v_4 + v_5)$

$Z_{12}Z_{36}Z_{45}(v_1 - v_4)$.

Claim 13. *There are* 15 *relations among the strings of length three, but only* 10 *of them are linearly independent. Hence there is a total of* 5 *linearly independent strings of length three.*

Proof. We list the 15 relations, to prove that only 10 of them are linearly independent is just a linear algebra exercise.

- $Z_{12}Z_{34}Z_{56}v_1 = +Z_{23}Z_{14}Z_{56}(v_1 - v_2) - Z_{13}Z_{24}Z_{56}v_2$

- $Z_{12}Z_{34}Z_{56}v_1 = +Z_{12}Z_{45}Z_{36}(v_1 - v_4) - Z_{12}Z_{35}Z_{46}v_4$

- $Z_{12}Z_{34}Z_{56}v_1 = +Z_{25}Z_{34}Z_{16}(v_3 - v_5) - Z_{15}Z_{34}Z_{26}(v_1 - v_3 + v_5)$

- $Z_{12}Z_{35}Z_{46}v_4 = +Z_{15}Z_{23}Z_{46}(v_3 - v_4) - Z_{13}Z_{25}Z_{46}v_3$

- $Z_{12}Z_{35}Z_{46}v_4 = +Z_{16}Z_{35}Z_{24}(v_1 - v_2 - v_4 + v_5) - Z_{14}Z_{26}Z_{35}(v_1 - v_2 + v_5)$

- $Z_{13}Z_{24}Z_{56}v_2 = +Z_{13}Z_{26}Z_{45}(v_2 - v_3) - Z_{13}Z_{25}Z_{46}v_3$

- $Z_{13}Z_{24}Z_{56}v_2 = +Z_{16}Z_{24}Z_{35}(v_1 - v_2 - v_4 + v_5) - Z_{15}Z_{24}Z_{36}(v_1 - v_3 + v_5)$

- $Z_{13}Z_{25}Z_{46}v_3 = +Z_{16}Z_{25}Z_{34}(v_3 - v_5) - Z_{14}Z_{25}Z_{36}v_5$

- $Z_{14}Z_{23}Z_{56}(v_1 - v_2) = +Z_{23}Z_{45}Z_{16}(v_1 - v_2 + v_3 - v_4) - Z_{15}Z_{23}Z_{46}(v_3 - v_4)$

- $Z_{14}Z_{23}Z_{56}(v_1 - v_2) = +Z_{14}Z_{35}Z_{26}(v_1 - v_2 + v_5) - Z_{14}Z_{25}Z_{36}v_5$

- $Z_{15}Z_{23}Z_{46}(v_3 - v_4) = +Z_{15}Z_{34}Z_{26}(v_1 - v_3 + v_5) - Z_{15}Z_{24}Z_{36}(v_1 - v_4 + v_5)$

- $Z_{14}Z_{25}Z_{36}v_5 = +Z_{15}Z_{24}Z_{36}(v_1 - v_4 + v_5) + Z_{12}Z_{45}Z_{36}(v_1 - v_4)$

- $Z_{15}Z_{34}Z_{26}(v_1 - v_3 + v_5) = -Z_{14}Z_{35}Z_{26}(v_1 - v_2 + v_5) + Z_{13}Z_{45}Z_{26}(v_2 - v_3)$

- $Z_{16}Z_{25}Z_{34}(v_3 - v_5) = +Z_{16}Z_{24}Z_{35}(v_1 - v_2 - v_4 + v_5)+$

  $-Z_{16}Z_{23}Z_{45}(v_1 - v_2 + v_3 - v_4)$

- $Z_{16}Z_{23}Z_{46}(v_1 - v_2 + v_3 - v_4) = +Z_{13}Z_{26}Z_{45}(v_2 - v_3) - Z_{12}Z_{36}Z_{45}(v_1 - v_4)$.

Notice that each relation is characterized by the presence of one repeated root. $\square$

We conclude that the extension $F_{\tilde{\rho}'}$ of F has dimension equal to 70.
This is exactly what we expected. Indeed we should obtain

$$F_{\tilde{\rho}'} = L(2\varepsilon_1 + 2\psi_2 + 2\varepsilon_3) \oplus L(2\varepsilon_1 + 2\psi_2 - 2\varepsilon_3)$$

and the irreducible representations of $SO(6)$ of highest weight $2\varepsilon_1 + 2\psi_2 \pm 2\varepsilon_3$ have both dimension equal to 35.

10.2.7 Construction for the representation $S^{(3,2)}$ of $\mathcal{S}_5$

The Specht module $S^{(3,2)}$ of $\mathcal{S}_5$ is simply the restriction of the Specht module $S^{(3,3)}$ of $\mathcal{S}_6$ to $\mathcal{S}_5$. So the construction of the extension of $S^{(3,2)}$ follows directly from the computations done for $S^{(3,3)}$. We find that

- $S^{(3,2)}$ has dimension 5.
- There are 20 strings of length one, and no relations among them.
- There are 15 strings of length two, and 5 relations among them.
- There are no strings of length three or more.
- The extension $F_{\tilde{\rho'}}$ has dimension 35, and highest weight $2\xi_1 + 2\xi_2$.

For dimensional reasons, $\tilde{\rho'}$ coincides with the irreducible representation of $SO(5)$ of highest weight $2\xi_1 + 2\xi_2$.

⋆ ⋆ ⋆⋆

We conclude our list of examples of construction of $F_{\tilde{\rho'}}$ with a remark.

Remark. *The algorithm described in chapter 9 extends every Weyl group representations ρ that does not include the sign of $\mathcal{S}_3$, to a well defined representation of* Lie(K)*.*

If we carry out the same construction for the sign representation of $\mathcal{S}_3$, we obtain a 4-dimensional extension that fails to be a Lie algebra representation, because the bracket relations do not hold.

10.3 Extension of non-spherical representations

It is remarkable that our algorithm to construct petite K-types also works when the representation ρ of M' is not spherical (but still satisfies the requirement of not including the sign).

We will discuss this generalization in the next chapter. As an example, we construct the extension for the two three dimensional irreducible representations ν_1 and ν_2 of $M' \subseteq SO(3)$.[17]

10.3.1 Construction for ν_2

We can find a basis of $F_{\nu_2} \simeq \mathbb{C}^3$ such that the action of M' is given by:

$$\sigma_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad (-1)\text{-eigenvector: } e_2$$

$$\sigma_{13} \rightsquigarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad (-1)\text{-eigenvector: } e_3$$

$$\sigma_{23} \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \Rightarrow \quad (-1)\text{-eigenvector: } e_1.$$

Hence we obtain three strings of length one: $Z_{12}e_2$, $Z_{13}e_3$ and $Z_{23}e_1$. There is

[17]Recall that ν_1 and ν_2 are the two irreducible summands of $Ind_M^{M'}\delta_1$, where δ_1 the character of M that maps a diagonal matrix in $SO(3)$ into its first diagonal entry. Alternatively, we can construct ν_1 and ν_2 by inducing to M' respectively the trivial and the sign representation of the subgroup H_1 of M' defined as follows:
let $\alpha_1 = \psi_2 - \varepsilon_3$ be the only good root for δ_1 (the attribute "good" refers to the fact that $\delta_1(m_{\alpha_1}) =$ Identity), and let W_1 be the subgroup of the Weyl group generated by the root reflection s_{α_1}. Let $\pi\colon M' \to W = M'/M$ be the projection. We define H_1 to be the inverse image of W_1 via π.

one relation among the one-strings, given by:

$$Z_{12}e_2 = \sigma_{13} \cdot (Z_{12}e_2) + \sigma_{23} \cdot (Z_{12}e_2) \quad \Leftrightarrow \quad Z_{12}e_2 = -Z_{23}e_1 + Z_{13}e_3.$$

We also notice that there are no strings of length two, because there is no pair of orthogonal roots in $\mathcal{A}_2$. The extension $F_{\tilde{\rho}'}$ is therefore five-dimensional. With respect to the basis $\{e_1,\, e_2,\, e_3,\, Z_{12}e_2,\, Z_{23}e_1\}$, we can write

- $Z_{12} \cdot e_1 = \sigma_{12} \cdot e_1 = e_3$
- $Z_{12} \cdot e_2 = Z_{12}e_2$
- $Z_{12} \cdot e_3 = \sigma_{12} \cdot e_3 = -e_1$
- $Z_{12} \cdot (Z_{12}e_2) = -4e_2$
- $Z_{12} \cdot (Z_{23}e_1) = \frac{1}{2} Z_{12} \cdot (Z_{23}e_1 - \sigma_{12} \cdot (Z_{23}e_1)) =$

  $$= \tfrac{1}{2} Z_{12} \cdot (Z_{23}e_1 - Z_{13}e_3) = \tfrac{1}{2} Z_{12} \cdot (-Z_{12}e_2) = +2e_2$$
- $Z_{13} \cdot e_1 = \sigma_{13} \cdot e_1 = e_2$
- $Z_{13} \cdot e_2 = \sigma_{13} \cdot e_2 = -e_1$
- $Z_{13} \cdot e_3 = Z_{13}e_3 = Z_{12}e_2 + Z_{23}e_1$
- $Z_{13} \cdot (Z_{12}e_2) = \frac{1}{2} Z_{13} \cdot (Z_{12}e_2 - \sigma_{13} \cdot (Z_{12}e_2)) =$

  $$= \tfrac{1}{2} Z_{13} \cdot (Z_{12}e_2 + Z_{23}e_1) = \tfrac{1}{2} Z_{13} \cdot (Z_{13}e_3) = -2e_3$$
- $Z_{13} \cdot (Z_{23}e_1) = \frac{1}{2} Z_{13} \cdot (Z_{23}e_1 - \sigma_{13} \cdot (Z_{23}e_1)) =$

  $$= \tfrac{1}{2} Z_{13} \cdot (Z_{23}e_1 + Z_{12}e_2) = \tfrac{1}{2} Z_{13} \cdot (Z_{13}e_3) = -2e_3$$

- $Z_{23} \cdot e_1 = Z_{23} e_1$

- $Z_{23} \cdot e_2 = \sigma_{23} \cdot e_2 = -e_3$

- $Z_{23} \cdot e_3 = \sigma_{23} \cdot e_3 = +e_2$

- $Z_{23} \cdot (Z_{12} e_2) = \frac{1}{2} Z_{23} \cdot (Z_{12} e_2 - \sigma_{23} \cdot (Z_{12} e_2)) =$

$$= \frac{1}{2} Z_{23} \cdot (Z_{12} e_2 - Z_{13} e_3) = \frac{1}{2} Z_{13} \cdot (-Z_{23} e_1) = +2e_1$$

- $Z_{23} \cdot (Z_{23} e_1) = -4e_1$.

We obtain the matrices:

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Z_{13} \rightsquigarrow \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Z_{23} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It is easy to check that:

- the bracket relations hold
- each Z_α acts with eigenvalues $0, \pm i, \pm 2i$
- $F_{\tilde{\rho}'}$ has highest weight $2\xi_1$, and for dimensional reasons, it coincides with the irreducible representation of $SO(3)$ with highest weight $2\xi_1$, which is $\mathcal{H}_2$.[18]

10.3.2 Construction for ν_1

The construction for ν_1 is trivial. We can find a basis of $F_{\nu_2} \simeq \mathbb{C}^3$ such that the action of M' is given by:

$$\sigma_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_{13} \rightsquigarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[18] We have denoted by $\mathcal{H}_2$ the representation of $SO(3)$ on the space of homogeneous harmonic polynomials of degree two in three variables.

$$\sigma_{23} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For each positive root α, σ_α acts with eigenvalues 1, $+i$, $-i$. So there are *no* strings at all, and $F_{\tilde{\rho'}} = F_\rho$. By definition, each Z_α acts by 0, $+i$, $-i$ on the eigenspace of σ_α of eigenvalue 1, $+i$, -1, so we find:

$$Z_{12} \rightsquigarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$Z_{13} \rightsquigarrow \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z_{23} \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to check that $F_{\tilde{\rho'}}$ has highest weight ξ_1, and for dimensional reasons, it coincides with the irreducible representation of $SO(3)$ of highest weight ξ_1, which is $\mathcal{H}_1$.

Chapter 11

A generalization to the non-spherical case

Let G be the group $SL(n, \mathbb{R})$, and let (ρ, F_ρ) be any representation of M' that does not contain the sign of $W(SL(3))$.[1] It is possible to extended ρ to a petite K-type using "almost" the same construction outlined in chapter 9.

Delicate Point. *If ρ is reducible, or ρ is not a Weyl group representation, M does not necessarily act by ± 1 on strings.*

Let $S = Z_{\alpha_1} \dots Z_{\alpha_r} v$ be a string, i.e. let $\alpha_1, \dots, \alpha_r$ be mutually orthogonal positive roots and let $v \in F_\rho$ be a simultaneous (-1)-eigenvector of $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_r}$. For every positive root β, we have

$$\sigma_\beta^2 \cdot (Z_{\alpha_1} \dots Z_{\alpha_r} v) = \underbrace{Z_{\alpha_1} \dots Z_{\alpha_r}}_{=\pm Z_{\alpha_1} \dots Z_{\alpha_r}} \underbrace{(\sigma_\beta^2 \cdot v)}_{=?}.$$

Because $(\sigma_\beta^2 \cdot v)$ might not be a multiple of v, $\sigma_\beta^2 \cdot S$ might be different from $\pm S$. How do we overcome this difficulty?

[1] We do not require ρ to be irreducible, nor to be a Weyl group representation.

Consider the elements $v^+ = (v + \sigma_\beta^2 \cdot v)$ and $v^- = (v - \sigma_\beta^2 \cdot v)$. They belong to F_ρ, and they clearly satisfy the condition $\sigma_\beta^2 \cdot (v^\pm) = \pm(v^\pm)$. We claim that v^+ and v^- are again simultaneous (-1)-eigenvectors of $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_r}$.

If β is orthogonal to the α_js, this claim is obvious, because $\sigma_{\alpha_j} \cdot (\sigma_\beta^2 \cdot v) = \sigma_\beta^2 \cdot (\sigma_{\alpha_j} \cdot v) = -\sigma_\beta^2 \cdot v$. Let us show that the same results holds when β is not orthogonal to one of the roots, say to α_j. Let β_j be a positive root such that the set

$$\{\pm\beta,\, \pm\alpha_j,\, \pm\beta_j\}$$

is a root system of type $\mathcal{A}_2$. The elements σ_β^2, $\sigma_{\alpha_j}^2$, $\sigma_{\beta_j}^2$ of M are all of order two, they commute and their product equals the identity. Therefore, we can write:

$$\sigma_{\alpha_j} \cdot (\sigma_\beta^2 \cdot v) = (\sigma_{\alpha_j} \sigma_\beta^2 \sigma_{\alpha_j}^{-1}) \cdot (\sigma_{\alpha_j} \cdot v) = -\sigma_{\beta_j}^{\pm 2} \cdot v =$$

$$= -\sigma_{\beta_j}^2 \cdot v = -(\sigma_\beta^2 \sigma_{\alpha_j}^2 \cdot v) = -\sigma_\beta^2 \cdot v.$$

It follows that $\sigma_{\alpha_j} \cdot (v^\pm) = -(v^\pm)$, for all $j = 1 \ldots r$. Hence, the strings $Z_{\alpha_1} \ldots Z_{\alpha_r} v^+$ and $Z_{\alpha_1} \ldots Z_{\alpha_r} v^-$ are well defined. By linearity:

$$Z_{\alpha_1} \ldots Z_{\alpha_r} v = \frac{1}{2}(Z_{\alpha_1} \ldots Z_{\alpha_r} v^+) + \frac{1}{2}(Z_{\alpha_1} \ldots Z_{\alpha_r} v^-).$$

This shows that any string S that does not satisfy the condition $\sigma_\beta^2 \cdot S = \pm S$, can be decomposed as a linear combination of strings on which σ_β^2 acts by ± 1. Therefore, a generalization of the construction of $\tilde{\rho'}$ is possible, and it is fairly easy.

We follows exactly the same steps. The only elements of difference are the following:

1. When looking for copies of the sign representation of $\mathcal{S}_3$ inside $F_{\rho'}$, we also need to check the case in which a string S does not generate a one-dimensional irreducible representation of $M_{\alpha,\beta,\gamma}$.

2. When defining the action of Z_α on F_ρ, we also need to consider the $(\pm i)$-eigenspace of σ_α.

3. When defining the action of Z_α on a string S, we also need to contemplate the option $\sigma_\alpha^2 \cdot S \neq \pm S$.

4. When checking the bracket relations, there are more cases to be considered.

We discuss each of these points separately.

(1) If $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ does not generate a one-dimensional irreducible representation of $M_{\alpha,\beta,\gamma}$ (i.e. if the subspace $\mathbb{C}v$ of F_ρ is not stable under $M_{\alpha,\beta,\gamma}$), then we consider the vectors:

$$v_0 = v + \sigma_\alpha^2 \cdot v + \sigma_\beta^2 \cdot v + \sigma_\gamma^2 \cdot v$$

$$v_1 = v + \sigma_\alpha^2 \cdot v - \sigma_\beta^2 \cdot v - \sigma_\gamma^2 \cdot v$$

$$v_2 = x - \sigma_\alpha^2 \cdot v + \sigma_\beta^2 \cdot v - \sigma_\gamma^2 \cdot v$$

$$v_3 = x - \sigma_\alpha^2 \cdot v - \sigma_\beta^2 \cdot v + \sigma_\gamma^2 \cdot v,$$

which do have this property.[2] Because v_i is a (-1)-eigenvectors of σ_{α_j}, for all $j = 1, \ldots, r$ and all $i = 1, \ldots, 4$, we obtain a decomposition of $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$ as a linear combination of strings that generate one-dimensional representations of $M_{\alpha,\beta,\gamma}$:

$$Z_{\alpha_1} \ldots Z_{\alpha_r} v = \tfrac{1}{4}(Z_{\alpha_1} \ldots Z_{\alpha_r} v_0) + \tfrac{1}{4}(Z_{\alpha_1} \ldots Z_{\alpha_r} v_1) +$$

$$+ \tfrac{1}{4}(Z_{\alpha_1} \ldots Z_{\alpha_r} v_2) + \tfrac{1}{4}(Z_{\alpha_1} \ldots Z_{\alpha_r} v_3).$$

[2] For instance, we have:

$$\sigma_\alpha \cdot (v_1) = \sigma_\alpha \cdot ((v - \sigma_\beta^2 \cdot v) + \sigma_\alpha^2 \cdot (v - \sigma_\beta^2 \cdot v)) = +v_1$$

$$\sigma_\beta \cdot (v_1) = \sigma_\alpha \cdot ((v + \sigma_\alpha^2 \cdot v) - \sigma_\beta^2 \cdot (v + \sigma_\alpha^2 \cdot v)) = -v_1$$

$$\sigma_\gamma \cdot (v_1) = \sigma_\alpha \cdot ((v + \sigma_\alpha^2 \cdot v) - \sigma_\gamma^2 \cdot (v + \sigma_\alpha^2 \cdot v)) = -v_1.$$

For each of these strings, the same arguments used in chapter 10 apply, and show that the extension ρ' only contains sign representations of type (a), (b) or (c).[3] Hence we can remove *every* copy of the sign representation from ρ' by imposing relations of the form

$$S = \sigma_\beta \cdot S + \sigma_\gamma \cdot S \quad (\star)$$

for all the strings S such that $\sigma_\alpha \cdot S = -S$ and $\sigma_\beta^2 \cdot S = +S$.

(2) We let Z_α act on the $(\pm\, i)$-eigenspace of σ_α by $Z_\alpha \cdot v = \sigma_\alpha \cdot v$.

(3) If σ_α^2 does not act by ± 1 on a string $S = Z_{\alpha_1} \ldots Z_{\alpha_r} v$, we write

$$Z_{\alpha_1} \ldots Z_{\alpha_r} v = \frac{1}{2}(Z_{\alpha_1} \ldots Z_{\alpha_r} v^+) + \frac{1}{2}(Z_{\alpha_1} \ldots Z_{\alpha_r} v^-)$$

with $v^\pm = v \pm \sigma_\alpha^2 \cdot v$, and we observe that since

$$\sigma_\alpha^2 \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v^+) = +(Z_{\alpha_1} \ldots Z_{\alpha_r} v^+)$$

$$\sigma_\alpha^2 \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v^-) = -(Z_{\alpha_1} \ldots Z_{\alpha_r} v^-)$$

the action of Z_α on these two strings is well defined. Then we set

$$Z_\alpha \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v) = Z_\alpha \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v^+) + Z_\alpha \cdot (Z_{\alpha_1} \ldots Z_{\alpha_r} v^-).$$

(4) Basically, we need to check that the bracket relations hold for the extensions of the two irreducible non-spherical representations of $M' \subseteq SL(3)$. Please, refer to chapter 10 for an explicit construction of these extensions.

[3] Please, refer to 9.4.3 for this terminology.

11.1 An inductive argument for the construction

The generalization of the algorithm to non-spherical representations of M' leads to an inductive argument for the construction of petite K-types.

Let ρ be any representation of M' that does not contain the sign of $\mathcal{S}_3$. We can construct the extension $\tilde{\rho'}$ by induction. The first step consists of adding to F_ρ the linear span of all the strings of the form $Z_\nu v$, where ν is a positive root and v is a (-1)-eigenvector of σ_ν.

We define an action of M' on strings by

$$\sigma \cdot (Z_\nu v) = (\mathrm{Ad}(\sigma)(Z_\nu))(\sigma \cdot v)$$

and we impose two kinds of relations:

- $Z_\nu(a_1 v_1 + a_2 v_2) = a_1 Z_\nu v_1 + a_2 Z_\nu v_2$ "*linearity relations*"

  for all a_1, a_2 in $\mathbb{C}$, and all (-1)-eigenvectors v_1, v_2 of σ_ν in F_ρ

- $Z_\nu v = \sigma_\beta \cdot (Z_\nu v) + \sigma_\gamma \cdot (Z_\nu v)$ "*($\star$)-relations*"

  for all triples of positive roots α, β, γ forming an $\mathcal{A}_2$, such that $\sigma_\alpha \cdot (Z_\nu v) = -(Z_\nu v)$ and $\sigma_\beta^2 \cdot (Z_\nu v) = +(Z_\nu v)$.

When α, β and γ are all orthogonal to ν, a ($\star$)-relation is just a linearity relation.[4] So we can as well assume that $\nu \not\perp \{\alpha, \beta, \gamma\}$. Then the condition

$$\sigma_\alpha \cdot (Z_\nu v) = -(Z_\nu v) = -\sigma_\beta^2 \cdot (Z_\nu v)$$

can only occur if

◦ $\alpha = \nu$ or $\alpha \perp \nu$; $\beta \not\perp \nu$ (so automatically $\gamma \not\perp \nu$)

and

◦ $\sigma_\alpha \cdot v = -v$; $\sigma_\beta^2 \cdot v = -v$ (so automatically $\sigma_\gamma^2 \cdot v = -v$).

The representation of M' on $F_\rho \oplus \mathrm{Span}(\textit{strings})$ descends to a representation $\tilde{\rho}_1$ on

[4] Because v, $\sigma_\beta \cdot v$ and $\sigma_\gamma \cdot v$ are all (-1)-eigenvectors of σ_ν.

the quotient space

$$F_{\tilde{\rho}_1} = \frac{F_\rho \oplus \text{Span}(strings)}{\sim}.$$

We notice that, for any positive root ν, the (-1)-eigenvectors of σ_ν in $F_{\tilde{\rho}_1}$ are either (-1)-eigenvectors of σ_ν in F_ρ or equivalence classes of strings of the form $Z_\alpha v$ with $v \in F_\rho$, $\sigma_\nu \cdot v = \sigma_\alpha \cdot v = -v$ and α either equal to ν or orthogonal to it.
By construction, $\tilde{\rho}_1$ does not contain any sign representation of $\mathcal{S}_3$, so we can iterate this construction. It is at this stage that the real induction begins, because all the next steps resemble the second one.

We add to $F_{\tilde{\rho}_1}$ the linear span of all the strings of the form $Z_\nu v$, with v a (-1)-eigenvector of σ_ν in $F_{\tilde{\rho}_1}$,[5] and we extend the action of M' in the obvious way.
At this stage (and in all the following ones), we impose four kinds of relations:

- $Z_\nu(a_1v_1 + a_2v_2) = a_1Z_\nu v_1 + a_2Z_\nu v_2$ "*linearity relations*"

  for all a_1, a_2 in $\mathbb{C}$, and all (-1)-eigenvectors v_1, v_2 of σ_ν in $F_{\tilde{\rho}_1}$;

- $Z_{\nu_1}Z_{\nu_2}v = Z_{\nu_2}Z_{\nu_1}v$ "*commutativity relations*"

  for all mutually orthogonal positive roots ν_1, ν_2 and all simultaneous (-1)-eigenvectors of σ_{ν_1}, σ_{ν_2} in F_ρ;

- $Z_\nu Z_\nu v = -4v$ "*no-repetitions relations*"

  for all positive roots ν and all (-1)-eigenvectors of σ_ν in F_ρ;

- $Z_\nu v = \sigma_\beta \cdot (Z_\nu v) + \sigma_\gamma \cdot (Z_\nu v)$ "*$(\star)$-relations*"

  for all positive roots ν, all (-1)-eigenvectors v of σ_ν in $F_{\tilde{\rho}_1}$, and all triples of positive roots α, β, γ forming an $\mathcal{A}_2$, such that

  - $\alpha = \nu$ or $\alpha \perp \nu$; $\beta \not\perp \nu$ (so automatically $\gamma \not\perp \nu$)

  and

[5] Equivalently, we add to $F_{\tilde{\rho}_1}$ strings of the form $(Z_{\nu_1}Z_{\nu_2}u)$, with u in F_ρ, $\sigma_{\nu_1} \cdot u = \sigma_{\nu_2} \cdot u = -u$ and either $\nu_1 = \nu_2$ or $\nu_1 \perp \nu_2$.

∘ $\sigma_\alpha \cdot v = -v$; $\sigma_\beta^2 \cdot v = -v$ (so automatically $\sigma_\gamma^2 \cdot v = -v$).

The representation of M' on $F_{\tilde{\rho_1}} \oplus \mathrm{Span}(strings)$ descends to a representation $\tilde{\rho_2}$ on the quotient space

$$F_{\tilde{\rho_2}} = \frac{F_{\tilde{\rho_1}} \oplus \mathrm{Span}(strings)}{\sim}$$

We notice that $F_{\tilde{\rho_2}}$ the (-1)-eigenvectors of σ_ν in $F_{\tilde{\rho_2}}$ are either (-1)-eigenvectors of σ_ν in $F_{\tilde{\rho_1}}$ or equivalence classes of strings of the form $Z_\alpha v$ with $v \in F_{\tilde{\rho_1}}$, $\sigma_\nu \cdot v = \sigma_\alpha \cdot v = -v$ and α either equal to ν or orthogonal to it.
By construction, $\tilde{\rho_2}$ does not contain any sign representation of $\mathcal{S}_3$, so we can iterate this construction.

At the rth step, we obtain equivalence classes of strings of the form $Z_{\alpha_1} \dots Z_{\alpha_r} v$ with $v \in F_\rho$, $\sigma_{\alpha_1} \cdot v = \dots = \sigma_{\alpha_r} \cdot v = -v$ and $[Z_{\alpha_i}, Z_{\alpha_j}] = 0$ for all i, $j = 1 \dots r$.[6] Using the "no-repetition" relations, we get rid of all the repetitions, so we are left with strings that only involve mutually orthogonal roots. Hence, the construction can be completed in finitely many steps.

This inductive construction of $\tilde{\rho'}$ has many advantages, for instance we only need to define the action of $\mathrm{Lie}(K)$ on strings of length one

$$\begin{aligned} Z_\alpha \cdot (Z_\nu v) &= 0 && \text{if } \sigma_\alpha \cdot (Z_\nu v) = +(Z_\nu v) \\ Z_\alpha \cdot (Z_\nu v) &= Z_\alpha Z_\nu v && \text{if } \sigma_\alpha \cdot (Z_\nu v) = -(Z_\nu v) \\ Z_\alpha \cdot (Z_\nu v) &= \sigma_\alpha \cdot (Z_\nu v) && \text{if } \sigma_\alpha^2 \cdot (Z_\nu v) = -(Z_\nu v). \end{aligned}$$

Notice that the definition of the action of Z_α on the (-1)-eigenspace of σ_α is simplified by the fact that strings of repeated roots are allowed.[7]

[6] This condition means that either $\alpha_1 = \alpha_2$ or $\alpha_1 \perp \alpha_2$.

[7] This definition is consistent with the one given for Weyl group representations, because by the "no-repetitions" relations $Z_\alpha \cdot (Z_\alpha) = \sigma_\alpha Z_\alpha = -4v$.

11.2 Suggestions for further generalizations

Our method for constructing petite K-types appears to be generalizable to other split semi-simple Lie groups, other than $SL(n)$, whose root system admits one root length. As seen in chapter 9, this assumption guarantees that we can annihilate every copy of the sign representation of $F_{\rho'}$ by imposing the $(\star)$-equivalence relations. Extending this construction to other groups accounts to verifying that the bracket relations hold, and this problem can be solved by working out some small rank examples.

As ρ varies in the set of Weyl group representations that do not contain the sign of $\mathcal{S}_3$, the output will be a list of petite K-types on which the intertwining operator for a spherical principal series can be tested by means of Weyl group computations. In other words, it is a list of K-types on which we should be able to explicitly calculate the signature.

The final result will be a non-unitarity test for a spherical principal series for split groups of type $\mathcal{A}$, $\mathcal{D}$, E_6, E_7 and E_8.

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